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Mathematical tasks: Clarification, classification and choice of suitable tasks for different types of learning and assessment

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In this paper, the differences between mathematical tasks such as problem-solving tasks, investigative tasks, guided-discovery tasks, project work, real-life tasks, problem-posing tasks, open tasks and ill-structured tasks will be contrasted. Such clarification is important because it can affect how and what teachers teach since the diverse types of tasks have different pedagogical uses, and it can also help researchers to define more clearly the tasks that they are investigating on. A framework to characterise the openness of mathematical tasks based on task variables such as the goal, the method, the answer, scaffolding and extension will be described. The tasks are then classified according to their teaching purpose: mathematically-rich tasks, such as analytical tasks and synthesis tasks, can provide students with opportunities to learn new mathematics and to develop mathematical processes such as problem-solving strategies, analytical thinking, metacognition and creativity; and non-mathematically-rich tasks, such as procedural tasks, can only provide students with practice of procedures. Rich assessment tasks that teachers can use to assess students' conceptual understanding, mathematical communication and thinking processes will also be discussed. The clarification of terminologies and the classification of mathematical tasks will help teachers to understand more about the purpose and characteristics of the diverse types of tasks so that they can choose appropriate tasks to develop the different facets of their students' mental structures and to assess the various aspects of their learning.

1. Introduction

Some educators do not distinguish between mathematical investigation and problem solving but others emphasise the differences (Evans, 1987). Pirie (1987) claimed that no fruitful service will be performed by indulging in the 'investigation' versus 'problem solving' debate but Frobisher (1994) believed that this is a crucial issue that will affect how and what teachers teach their students. The Professional Standards for Teaching Mathematics (NCTM, 1991) stated that it is the "central responsibility of teachers … to select and develop worthwhile tasks and materials that create opportunities for students to develop … mathematical understandings, competence, interests and dispositions" (p. 24). If a teacher does not know the differences between the types of mathematical tasks, how is he or she to use them to develop the various aspects of the students' mental structures since different tasks are used to cultivate different types of skills and thinking? If a teacher refers to standard mathematics textbook tasks as 'problems' that the students should 'solve', then he or she may not realise that practising this type of tasks is not mathematical problem solving which is the central theme of many school curricula (e.g. Cockcroft, 1982; NCTM, 1980; Ministry of Education of Singapore, 1990). If a teacher thinks that 'word problems' are mathematical tasks with real-life contexts, then he or she may not search for genuine real-life tasks for the students to solve. Thus it is crucial that teachers understand the differences between these tasks, especially their teaching purpose, so that they can choose more suitable tasks for their students. This is also important for researchers because it is hard to research on different types of mathematical tasks if the latter are defined vaguely. Therefore, the main purpose of this paper is to provide a characterisation of the diverse types of mathematical tasks to help teachers in their choice of appropriate tasks for their students and to help researchers demarcate their area of research more clearly.

This paper will begin with a discussion of what a mathematical problem is (Section 2) and then the differences between various terminologies such as between problems and exercises, between problems and investigative tasks, between problem solving and problem posing, between investigation and guided discovery learning, between investigation and project work, between academic and real-life tasks, between open-ended and open tasks, and between ill-structured and well-structured tasks will be contrasted (Sections 3 to 10). A framework to characterise the different types of openness of mathematical tasks will be developed in Section 9. All these clarifications will lead to a classification of mathematical tasks according to their purpose for different types of learning and assessment (Sections 11 to 12). The paper will then conclude with a discussion of some implications for the teaching and learning of mathematics (Section 13).

2. What exactly is a mathematical problem?

Many teachers use the word ‘problems’ to describe the tasks in a mathematics textbook but are these really problems? In this section, what exactly constitutes a mathematical problem will be clarified. It will begin with Henderson’s and Pingry’s (1953) three necessary conditions for a situation to be a problem for a particular individual:

> “1. The individual has a clearly defined goal of which he is consciously aware and whose attainment he desires.
> 2. Blocking of the path toward the goal occurs, and the individual’s fixed patterns of behavior or habitual responses are not sufficient for removing the block.
> 3. Deliberation takes place. The individual becomes aware of the problem, defines it more or less clearly, identifies various possible hypotheses (solutions), and tests these for feasibility.” (p. 230)

There are three criteria in the first condition. The first criterion is a clearly defined goal which will be discussed later in Section 4. The second criterion is that the person must be consciously aware of the goal. The third criterion is that the person must desire the attainment of the goal. Lester (1980) interpreted the word ‘desires’ to mean that the person must “be *interested* in resolving the situation” (p. 30). So what if a person is not interested? In the classroom, if an indifferent student refuses to try the tasks set by the teacher (i.e. both the first and third conditions are not satisfied), then these tasks are not problems to the student. But what if an uninterested student attempts the tasks (i.e. the third condition is satisfied but the first condition *seems* not to be satisfied) because he or she may feel that this is part and parcel of schooling or the teacher may have encouraged or coerced the student to do the work? If this student is unable to solve the tasks, are they still problems to him or her? This has a serious pedagogical implication. If these tasks are not problems to the student, then the teacher does not have to do anything to help the student. But from the perspective of the student, these tasks are still problems because he or she has a problem solving them. Thus it would seem that interest on the part of the person is not a criterion to determine whether a situation is a problem to him or her. According to Cambridge Dictionaries Online (Cambridge University Press, 2007), the word ‘desires’ means ‘wants’ and a person may want something out of necessity or responsibility rather than out of interest. Another meaning of ‘wants’ is ‘needs’ (ibid.). In the classroom, an uninterested student may want or need to solve the tasks, be it willingly or unwillingly, and so if we interpret the word ‘desires’ as ‘wants’ or ‘needs’ in Henderson’s and Pingry’s (1953) first condition, then a situation will still be a problem to a person if he or she *wants* or *needs* to attain the goal but is unable to do so. In fact, Reys, Lindquist, Lambdin, Smith and Suydam (2004) defined a problem as “a situation in which a person *wants* [emphasis mine] something and does not know immediately what to do to get it" (p. 115).

The second condition is that the person must "be *unable to proceed directly* to a solution" (Lester, 1980, p. 30). Reys et al. (2004) believed that this difficulty must require "some creative effort and higher-level thinking" (p. 115) to resolve. Schoenfeld (1985) also emphasised that the "difficulty should be an intellectual impasse rather than a computational one" (p. 74). He gave the example that inverting a 27×27 matrix would be a tedious task for him but inverting a matrix was not a problem to him. So *tediousness* in applying a computational procedure is not a factor in determining whether a situation is a problem. But what if a student does not know how to invert a 27×27 matrix? Students may have learnt how to invert a 2×2 matrix but not many of them are aware of how to invert a square matrix of higher dimensions. So the problem can still be a *procedural* one. But if it is just a *routine practice* of procedural tasks commonly found in textbooks, then these tasks may not pose a problem to the student (see Section 3).

The third condition is that the person must "make a *deliberate attempt* to find a solution" (Lester, 1980, p. 30). What happens if the person does not attempt the task? For example, in real life, if a person tries to run away from his or her problem instead of trying to resolve it, does it mean that the situation is no longer a problem to the person? Isn't it like sweeping a problem under a carpet and pretending that the problem is not there, but is the problem really not there? Imagine an ostrich which buries its head into the ground when it senses danger. As long as the ostrich does not see the source of the danger, is the danger still a problem to the bird? What is worse is that the danger may even cause the ostrich its life. There are two different perspectives here. From the viewpoint of the person or the ostrich, the situation is not a problem to the person or the ostrich. But from another viewpoint, the problem is still there. Which perspective you take depends on the situation itself. If the situation is serious, then taking the first viewpoint may not be so helpful to the person although it may so happen that the situation just resolves by itself after some time. If the situation does not concern the person, then taking the second viewpoint may not be so helpful. For example, there are many questions on physics that I cannot answer. Taking the second viewpoint that these questions are problems to me will mean that I have a lot of problems in my life and I may feel so burdened by them when these questions do not actually concern me. Thus whichever perspective you take depends on whether the situation concerns you or how serious it is. In the classroom, if a student does not attempt the tasks set by the teacher, then from the first viewpoint, these tasks are not problems to the student. But is this perspective helpful to the student?

We have been looking at what constitutes a problem from the viewpoint of the student. Let us look at it from the perspective of a teacher. When a teacher designs or chooses a task for his or her class, the teacher will usually target the task difficulty at the level of the average students in the class. But if the average students are not interested in doing any work at all, then the task will not be a problem to these students and so does it mean that the teacher can set a very difficult task since it will pose no problem to these students? From the viewpoint of the teacher, in deciding whether a task is suitable for the class, it may be more useful to just use Henderson's and Pingry's (1953) second condition that a task is a problem to a person if he or she is unable to proceed directly to a solution. Whether the students want to do the task or actually make an attempt on it should not be a factor when deciding whether the task will be a problem to the average students in the class. Of course, if the students do not want to do the task, then it will still be a concern for the teacher but it will be an entirely different problem for the teacher.

In this paper, mathematical problems will be viewed from a pedagogical perspective and so a mathematical task is defined to be a problem to a student if he or she is unable to proceed directly to a solution. Whether the student is interested to solve it or actually makes an attempt on it will not affect the teacher's viewpoint that this is a problem to the student. From this perspective, we will discuss whether standard mathematics textbook tasks are really problems or just exercises.

3. Problems or Exercises?

Consider a typical textbook task:

Task 1: Standard Textbook Task
Find the midpoint of a line segment joining the points (4, –4) and (–2, 4).

This task may just be a routine practice of procedural skills that students have learnt earlier in the class and so they may know immediately what to do to solve it. However, this may be a problem to students who have not been taught the procedure or to low-ability students who have just learnt the procedure but do not know how to apply it properly. But with enough practice, this task can become a *routine exercise* to the students. Moreover, the purpose of this type of tasks is to "provide students with practice in using standard mathematical procedures (for example, computational algorithms, algebraic manipulations, and use of formulas" (Lester, 1980, p. 31). Some researchers called this type of tasks "routine problems" (Orton & Frobisher, 1996, p. 27) but these tasks may not be problems to a student. Moreover, for a student who does not practise these 'routine' tasks found in the textbook, then these tasks are not 'routine' to him or her. So we may want to distinguish between 'routine tasks' as 'routine' in the sense that they are commonly found in textbooks for the main purpose of 'routine' practice of procedures, or 'routine' with respect to a student. It may be more helpful if we use the term 'familiar or unfamiliar tasks' when we want to indicate whether the tasks are familiar or unfamiliar to a student. But it may be fine to use the term 'routine practice' since this implies that it is 'routine' to the student practising it.

Let us contrast Task 1 with another task:

Task 2(a): Last Digit
Find the last digit of 3^{2007}.

The main purpose of this task is for the solver to make use of some problem-solving strategies, such as looking for patterns, to solve it. For a student who has been exposed to such tasks before, this task may no longer be a problem to him or her. However, this task is inherently different from the first task in that it requires some problem-solving strategies and not just a direct application of a procedure.

Since both types of tasks may or may not be problems to a student, the phrase 'mathematical task' will be used as the more general term in this paper. For example, the Professional Standards for Teaching Mathematics (NCTM, 1991) used the phrase 'mathematical tasks' instead of 'mathematical problems' (see, for example, p. 25) and Schoenfeld (1985) wrote, "… being a 'problem' is not a property inherent in a *mathematical task* [emphasis mine]" (p.

74). So whether a task is a problem depends on the individual. In this paper, the first type of mathematical tasks will be called ‘procedural tasks’ since they involve practising on procedures, and the second type ‘problem-solving tasks’ since they require the use of some problem-solving strategies to solve. But the phrase ‘problem-solving tasks’ can be misleading because the term ‘problem-solving’ suggests that the task is a problem to the person when it may not be so. Nevertheless, this phrase will still be used to emphasise the problem-solving strategies involved in this type of tasks, even if they may not pose a problem to some people.

Sometimes, the term ‘problem’ may still be used when referring to a task in this paper. When such term is used, it implies that the task is a problem to the particular person. The phrase ‘solving a problem’ may also be used instead of ‘solving a task’ because if the task is not a problem to the person, then there is really no need for him or her to solve it. But what about procedural tasks that are not problems to a student? Teachers usually tell their students to ‘solve’ such ‘problems’ although the term ‘practise procedural tasks’ may be more appropriate, especially if such tasks are not problems to the students and their main purpose is for them to practise procedures learnt in the class earlier.

As with many classifications, one problem is that there are always grey areas. For example, consider the following task:

> **Task 3: Standard Textbook Task**
> A(4, −4), B(9, 6), C(−2, 4) and D are the vertices of a rhombus. Find the coordinates of D.

Students who see this task for the first time may not know how to solve it. They may try to use the formula for the distance between two points or the formula for the gradient of a line segment but these approaches will involve the tedious process of solving two simultaneous equations which may not be linear. Other ‘less tedious’ methods of solution require some creative effort and higher-level thinking on the part of the students: find the midpoint of AC which is also the midpoint of BD; or using position vectors, $\overrightarrow{CD} = \overrightarrow{BA} \Rightarrow \overrightarrow{OD} - \overrightarrow{OC} = \overrightarrow{OA} - \overrightarrow{OB} \Rightarrow \overrightarrow{OD} = \overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OC}$. On one hand, this task is not a direct application of a procedure but it requires some higher-level thinking. On the other hand, with some practice, this task can become a routine exercise and this *is* in fact a standard Singapore mathematics textbook task (e.g. Teh & Loh, 2007). We may argue either way or we may just say that this task falls in the grey area between procedural and problem-solving tasks.

Therefore, in this paper, whenever mathematical tasks are classified into two or more classes, it is assumed that there may be grey areas and that there may be overlaps. For example, “there is little doubt that a great deal of overlap exists between problems and investigations” (Frobisher, 1994, p. 152) which will be discussed in the next section.

4. Problems or Investigative Tasks?

We return to the first criterion in Henderson's and Pingry's (1953) first condition of what a problem is: a clearly defined goal (see Section 2). Consider Task 2(a) above and the following task:

> **Task 2(b): Investigate Powers of 3**
> Investigate powers of 3.

In Task 2(a), the goal is clearly defined: find the last digit. In Task 2(b), the goal is ill-defined: investigate, but investigate what? A student may pose any problem to investigate (Cai & Cifarelli, 2005) or they can just search for some underlying patterns (Height, 1989). Orton and Frobisher (1996) claimed that "very few mathematics educators would classify explorations of this kind as problems" (p. 27) because these tasks have no clear goals. In some countries (e.g. the United Kingdom), Task 2(b) is called a mathematical investigation, but in other countries (e.g. the United States of America), it is called an open problem (ibid.). However, the term 'open problem' is an oxymoron: if a problem must have a well-defined goal and not an open goal, then an investigation is not a problem. But when faced with an investigation which a student does not know what to do, then is not the task still a problem to the student? So if we relax Henderson's and Pingry's (1953) first criterion to include an ill-defined or open goal, then an investigative task can be viewed as an 'open problem' to the person who does not know how to solve it. Since whether a task is a problem depends on the individual, the term 'open tasks' will be used instead of 'open problems'. However, Task 2(b) will be called an 'investigative task' because not all open tasks are investigative ones (see Sections 5 and 9).

The Cambridge Dictionaries Online (Cambridge University Press, 2007) defined the word 'investigate' as 'examine a crime, problem, statement, etc. carefully, especially to discover the truth'. Bastow, Hughes, Kissane and Mortlock (1991) defined a mathematical investigation as the "systematic exploration of open situations that have mathematical features" (p. 1). Other authors stressed that "investigations, by their very nature, demand an open-minded, multifaceted approach" (Lee & Miller, 1997, p. 6). Thus an investigative task is an open task where the goal is open and students can set their own specific goals to investigate anything they want (Orton & Frobisher, 1996).

Some educators do not differentiate between an investigative task and an investigation. For example, Orton and Frobisher (1996) compared the differences between problems and investigations while Evans (1987) contrasted problem solving and investigation. But there is a difference between problems and problem solving. A problem refers to a situation that is problematic to a person, and in the classroom, this usually involves a given task, while problem solving refers to the process of solving the problem. If problem solving is viewed as an activity, then it includes both the problem and the process of solving it. Similarly, there is a difference between an investigative task and the process of doing an investigation (Ernest, 1991). But the process of doing an investigation can be called simply as an 'investigation'. So if investigation is viewed as an activity, then it will include both the investigative task and the process of investigation. However, in mathematics education, there has been a shift in the meaning of the word 'investigation' to refer to the investigative task itself (ibid.). In this paper, the context will decide whether the term 'investigation' refers to the investigative task, or the process of investigation, or both.

Another main difference between investigative and problem-solving tasks is how the task is phrased. If we rephrase the problem-solving Task 2(a) by making the goal more open, such as to use the word 'investigate' as in Task 2(b), then we have changed a problem-solving task into an investigative task but it will widen the scope of the original problem-solving task because a student can investigate anything in Task 2(b). Some educators (e.g. Mason, Burton & Stacey, 1985; Schoenfeld, 1985) prefer to start with a problem-solving task like Task 2(a) and extend the problem to find some other patterns, such as a pattern in the last two digits or the sum of all the digits of the powers of 3. Then it is "nearly always possible to restate [a problem-solving task *with its extension*] in order to make it into an … investigation" (Frobisher, 1994, p. 158) without changing much of the scope. The difference is that starting from Task 2(a) restricts the scope of the investigation at the beginning but it provides more scaffolding for the students than starting from Task 2(b). Moreover, initial findings of a research study that I have conducted reveal that, when given Task 2(b), some students did not look for patterns in the last digit, which is the goal of Task 2(a), but that they went to find some other patterns instead. Therefore, rephrasing a problem-solving task with its extension as an investigative task may change the original goal of the problem-solving task because some students may set completely different goals to investigate when given the investigative task. Frobisher (1994) believed that "a distinction should be made between [problem-solving tasks] which lead to investigations, and … investigations which have their own separate existence" (p. 158).

From the discussion above, there are at least three main differences between investigative and problem-solving tasks:

(i) Investigative tasks have more open goals than problem-solving tasks.

(ii) Investigative tasks lead to investigation which is a divergent activity since different students can set different goals to pursue but problem-solving tasks lead to "problem solving [which] is a convergent activity" (Evans, 1987, p. 27) since there is only one goal to achieve, although problem solving can be extended (see Section 9.5) and thus divergent if we consider its extension.

(iii) Investigative tasks involve both problem posing and problem solving but problem-solving tasks involve mostly problem solving.

The third difference will be elaborated in the next section.

5. Problem Solving or Problem Posing?

Another main difference between problem-solving and investigative tasks is that the latter involves both problem posing and problem solving. When given an investigative task like Task 2(b), the students have to pose their own problems to solve. But when faced with a problem-solving task like Task 2(a), the students just need to solve the problem although the students can still extend the problem by posing more problems if they want to. However, they can choose not to, and so this type of problem posing is not implicit in problem-solving tasks but they are essential in investigate tasks.

Problem "posing can occur before, during, or after the solution of a problem" (Silver, 1994, p. 19). In investigative tasks, problem posing occurs first at the beginning: students need to pose their own problems to solve. Problem posing can also occur when students try to solve the problems they pose, just like in problem-solving tasks. This type of problem posing is a problem-solving strategy when a difficult problem is reformulated as another related problem which can be solved more easily, or as a smaller related problem to solve first before extending to the original problem. Problem posing can also occur after the solution of a problem to generate more problems to solve: this can occur at the end of a problem-solving task or during the middle of an investigative task after the students have solved some problems that they have posed.

The purpose of all the above examples of problem posing is to generate problems to solve, whether they are related or new problems (Brown & Walter, 2005). But there is another type of problem posing where the purpose is to develop or assess students' creativity (Silver, 1994). For example, consider the following task:

> **Task 4: Problem Posing**
> You are given an equation like this: $5x - 2 = 33$. Write a problem … that will lead to the formation of the above equation and then solve the equation to answer your own problem. Let your creativity flows! (Teh, Loh, Yeo & Chow, 2007, p. 60)

In this task, the main purpose is not to pose a problem to solve but to be creative and pose an original, interesting and complex 'word problem' although there is still some analytical thinking involved because the student must ensure that the story matches the equation, and higher marks will usually be awarded to a story that is more complex (Getzels & Jackson, 1962; Yeap, 2002). In Bloom Taxonomy (Bloom, Engelhart, Furst, Hill & Krathwohl, 1956), one of the six educational objectives is 'synthesis' which includes the ability to write a story or an essay, or to compose a poem or a song. This type of tasks used to be outside the domain of mathematics, but in recent years, there appear mathematical tasks such as mathematical journal writing (Waywood, 1992), composing mathematical poems or songs and writing 'word problems' or stories that contain mathematical elements (Teh et al., 2007). Bloom et al. (1956) believed that synthesis is "the category in the cognitive domain which most clearly provides for creative behavior on the part of the learner … [although] this is not completely free creative expression since generally the student is expected to work within the limits set by particular problems" (p. 162). In this paper, the term 'problem-posing tasks' will be used to describe this type of tasks where the main purpose is to pose an original, interesting and complex task rather than to pose a problem to solve. Problem-posing tasks are another example of open tasks and so we cannot equate open tasks with investigative tasks only (see Section 4).

6. Mathematical Investigation or Guided Discovery Learning?

In this section, another confusion about mathematical investigation will be dealt with. Some teachers associate mathematical investigation with guided discovery learning (Jaworski, 1994). In guided discovery learning, students are guided to explore some mathematical ideas in order to discover a formula, a procedure or some mathematical fact which the teacher has in mind (Bruner, 1961). For example, students may explore angles at the circumference and angles at the centre of a circle subtended by the same arc and discover a relation between these two. But is this an investigation? Ernest (1991) described problem solving as "trail-blazing to a desired location" (p. 285) and investigation as the exploration of an unknown land where "the journey, not the destination, is the goal" (Pirie, 1987, p. 2). Similarly, Jaworski (1994) believed that guided discovery learning is more of a trail-blazing to a desired location which the teacher has in mind and so it is different from mathematical investigation. In fact, Ernest (1991) contrasted the difference among three inquiry methods for teaching mathematics, namely, problem solving, guided discovery and investigative approaches, thus implying that these three terms are not exactly the same.

An important pedagogical question will then be, “How do we integrate the content of a school mathematics curriculum into an investigative approach to mathematics teaching because by introducing the content, there is a desired outcome but mathematical investigation has no fixed destination?” One suggestion is to make the guided-discovery task more open. Instead of focusing on the relation between an angle at the circumference and an angle at the centre of a circle subtended by the same arc, we may open up the task by letting students investigate angles at the circumference and angles at the centre of a circle subtended by *any* arc. Then students may discover not just one relation but *many* relations, for example, angle at centre = twice angle at circumference, angle in semicircle = 90°, angles in the same segment are equal, and angles in opposite segments are supplementary. Moreover, some students may make other surprising discoveries that teachers do not foresee and thus are unprepared for. This investigative approach is similar to the open-ended approach proposed by Becker and Shimada (1997) in Japan which will be described later in Section 9.1.

To summarise this section, there are at least three main differences between an investigative approach to mathematics teaching and guided discovery learning:

(i) An investigative approach uses more open tasks than guided discovery learning.

(ii) An investigative approach provides less guidance than guided discovery learning. This is only at the beginning stage. How much to guide during an investigative approach depends on the judgment of the teacher based on his or her experience. If there is too much guidance, then it is the teacher who is doing the investigation (Lerman, 1989). If there is too little guidance, then the students may lose interest when they are stuck for too long (Tanner, 1989; Teong, 2002).

(iii) An investigative approach permits other unexpected discoveries but guided-discovery tasks are not open enough to allow such discoveries which are usually not encouraged as well.

We may also want to clarify what the term ‘exploration’ means. Some researchers (e.g. Cai & Cifarelli, 2005; Orton & Frobisher, 1996) use it to mean ‘investigation’ or something very similar to it, while others (e.g. Brown, 1996) use it to include both investigative and problem-solving tasks. The word ‘exploration’ can also mean ‘exploration of mathematics in the guided discovery approach’ (Yeo, Hon & Cheng, 2006). Since guided discovery learning is different from mathematical investigation, it may be misleading to say that the students are being guided to ‘investigate’ some mathematical concepts. So we may say that they are exploring mathematics which can also be problematic if one subscribes to the view that an exploration is very similar to an investigation. In this paper, the word ‘exploration’ will not be restricted to any one meaning but the context will suggest in which sense it is used.

7. Project Work, Investigational Tasks or Practical Tasks?

In this section, the confusion between mathematical investigation and project work (Cockcroft, 1982) will be dealt with. With the publication of the Cockcroft Report (ibid.), extended coursework or project work was introduced into the Mathematics GCSE (General Certificate in Secondary Education) in the United Kingdom (Jaworski, 1994). This generally fell into two categories: investigational and practical tasks. Wolf (1990) explained that investigational tasks are explorations of more abstract contexts but practical tasks involve more ‘real’ situations. For example, Task 2(b) is an investigational task but Task 5(a) below is a practical task.

Task 5(a): Design Playground
Design a playground for the school.

The problem with some of these practical tasks is that they may not be purely investigative tasks. For example, in Task 5(a), doing some investigation is not enough: the designer needs to be creative to come up with a good design. So we may want to distinguish between purely investigative tasks (e.g. Task 2(b)) and tasks that involve some form of investigation. Moreover, this type of practical tasks is usually multi-disciplinary, unlike investigational tasks. So we may also want to differentiate between mathematical investigative tasks (e.g. Task 2(b)) and interdisciplinary investigative tasks. Another important distinction is that some investigative tasks can be solved purely by thinking without any research (e.g. Task 2(b)) while others can only be solved with some form of research. The latter are usually the more real-life or practical type (e.g. Task 5(a)) where it may not be possible to provide enough information in the task statement to solve it (see Section 9.4).

Thus project work is not the same as investigation (Cockcroft, 1982). Shorter investigation is usually not suitable for project work while longer investigation can be used as project work but they can also be used as normal investigative tasks: the difference is that for project work, the students usually have to do a portfolio and give a formal presentation, but these are not required for an investigative task done in class or over a couple of days. Project work may also not be pure investigation but they can involve creative elements as in the practical tasks described earlier. The distinction made between investigational and practical tasks in project work highlights the difference between academic and real-life tasks which will be discussed in more details in the next section.

8. Academic, Semi-Real or Real-Life Tasks?

Many textbook tasks refer to mathematics only. These are the ‘pure mathematics’ or ‘academic’ tasks with no context at all (Skovsmose, 2002). An example is procedural tasks such as Task 1 above. Some students may find it meaningless to practise pure mathematics problems because these problems are not relevant to them. Therefore, some educators have tried to inject some reality into school mathematics problems by providing some kind of storyline (Masingila, 2002). Skovsmose (2002) called this kind of storyline a reference to a semi-reality: “not a reality that we actually observe, but a reality constructed by, for instance, an author of a mathematical textbook” (p. 119). Christiansen (1997) called this ‘virtual reality’ (p. 20) as opposed to ‘real-life’ reality. An example of a semi-real problem is the following ‘word problem’:

Task 6: Word Problem
“This is the sign in a lift at an official block:

This lift can carry up to 14 people

In the morning rush, 269 people want to go up in this lift. How many times must it go up?” (Cooper, 2001, p. 246)

‘Word problems’ (notice inverted commas are used because they may not be problems to some students) share the same characteristic as pure mathematics or academic procedural tasks: the skills needed to solve the ‘word problems’ will already have been taught and practised (Frobisher, 1994; Moschkovich, 2002a). But the context is supposed to make ‘word problems’ more meaningful to the students than pure mathematics problems. However, in Task 6, a student can still ask the teacher whether the lift must always be full or whether some people will go up the stairs instead (Cooper, 2001). These are valid questions for a real context but students who consider these alternatives will usually be penalised (Skovsmose, 2002). Therefore, some educators (e.g. Kastberg, D’Ambrosio, McDermott & Saada, 2005) questioned the limitations of ‘clean’ contexts which they believed are not useful in developing solution strategies. But research literature on mathematics practice in everyday situations within cultures supports the fact that *meaningfulness*, rather than realism or usefulness, is the key to effective classroom instruction (Carraher & Schliemann, 2002). Therefore, “there is still a role for such [word] problems in a student’s experience” (Orton & Frobisher, 1996, p. 24).

When someone uses the term ‘real problems’ or ‘real-life problems’, we must be clear what the person is referring to. Sometimes, the phrase ‘real problems’ may be used to mean genuine problems and if a student does not know how to find the last digit of 3^{2007}, then this is a genuine problem to the student (see Section 2) although this is a pure mathematics task. At other times, we may use ‘real problems’ to refer to real-life or authentic tasks. But real-life or authentic tasks can also have different meanings. For example, a teacher can use real-life data such as actual recent unemployment figures and ask questions about the increase or decrease of employment at different periods of time and for different countries (Frankenstein, 1989). Real-life tasks can also refer to ‘applied mathematics’ used in the workplace or the ‘real world’, for example, in architecture or engineering. Some educators (e.g. Moschkovich, 2002b) tried to bring this workplace mathematics into the classroom but this type of tasks may still not be real within the students’ realm of experiences (Skovsmose, 2002). So another interpretation of real-life tasks is that they must be real to the students. Frobisher (1994) gave the example of organising an *actual* field trip. This task is only meaningful to the students if they are actually going for the field trip. Organising a hypothetical field trip will be at best a semi-reality. But the problem with such real-life tasks is that they cannot be written down in a textbook for students to use successfully because they will not be real to every student if the task does not arise from the student’s own experience (Orton & Frobisher, 1996). However, letting students play mathematical-rich games, which can be written down in a textbook, is very real for them (van Oers, 1996) because the outcome, whether they win or lose, matters to them (Ainley, 1988, 1990). Examples of such games include Nim (Civil, 2002) and the Tower of Hanoi (Yeo, 2007b) where students can develop mathematical processes such as looking for patterns, inductive reasoning and generalisation (Yeo, 2007a).

Usually, real-life tasks are more open in nature and they entail some sort of investigation or research, but academic tasks such as the investigative Task 2(b) can also be very open in nature. So what does it mean when we say that a task is open? Exactly which aspect of the task is open? This will be discussed in the next section.

9. Open-Ended or Open Tasks?

A closed task is a task where the goal and the answer are closed: the goal is specified in the task statement and there is only one correct answer (Becker & Shimada, 1997). This includes procedural tasks found commonly in textbooks whose main purpose is for students to practise procedural skills (Lester, 1980) but many educators (e.g. Schoenfeld, 1988) believed that these skills are of limited use in new situations. Thus there is a growing support for other types of mathematical tasks such as open-ended and open tasks (Frobisher, 1994). Some researchers (e.g. Boaler, 1998) use the term 'open-ended' and 'open' interchangeably while others distinguish between them (e.g. Orton & Frobisher, 1996). In this section, the openness of a task will be discussed with reference to the following task variables: the goal, the answer, the method, scaffolding and extension.

9.1 Open or Ill-Defined Answer?

Becker and Shimada (1997) described one type of tasks where there are multiple *correct* answers. This does not mean that the task has multiple answers (Chow, 2004). For example, solving a quadratic equation may produce two 'answers' or solutions but this is the *only* correct answer because if only one solution is given when there are two solutions, then the answer will be wrong. Becker and Shimada (1997) gave the following example of a task with multiple correct answers:

> **Task 7: Open-Ended Task**
> "A transparent flask in the shape of a right rectangular prism is partially filled with water. When the flask is placed on a table and tilted, with one edge of its base being fixed, several geometric shapes of various sizes are formed by the cuboid's faces and the surface of the water. The shapes and sizes may vary according to the degree of tilt or inclination. Try to discover as many invariant relations (rules) concerning these shapes and sizes as possible. Write down all your findings." (p. 10).

In this task, there is more than one correct answer and so the answer is open. Since the 'end' of the problem is the answer, Becker and Shimada (1997) called these "open-ended problems" (p. 1). We may also say that the answer is ill-defined because there is no way to specify all the correct answers. If a task has only one correct answer (e.g. Task 2(a)), then we can specify it and so the answer is well-defined. Therefore, in Task 7, the answer is ill-defined because it is open.

There is another meaning to an open answer. Can we judge the design of the playground in Task 5(a) by saying that it is correct or wrong? Of course, if we design a physical fitness park instead of a playground, then we may say that the answer is 'wrong' or invalid. But there is no such thing as the 'correct' answer and so the answer is not well-defined. Even if there are some specifications, e.g. Task 5(b) below, there is still plenty of scope for the designer to be creative and to come up with various designs.

> **Task 5(b): Design Playground with Some Given Specifications**
> Design a playground for the school. The playground must be 20 m by 20 m and it must contain at least one slide and two swings. The budget is $10 000.

So it is possible to have multiple *valid* answers or 'end products' and so the answer is open. In this case, the task is open because it is ill-defined, unlike Task 7 which is ill-defined because it is open. This is a subtle difference between 'ill-defined' and 'open': either one of them can be the cause and the other the effect. Problem-posing tasks (e.g. Task 4) are another type of mathematical tasks where there are multiple valid answers. Since there is no such thing as the correct answer, then we have to use a scoring rubric to evaluate or assess the answer which will be described later in Section 12.

Therefore, there are at least two meanings when we say that the answer is open. An open answer may mean that there are multiple *correct* answers or that there are multiple *valid* answers because there is no such thing as the correct answer. Another viewpoint is to call the first type 'open answer' and the second type 'ill-defined answer'. In this paper, the context will decide whether the term 'open answer' is used to refer to the first type or to both, but the term 'ill-defined answer' will always refer to the second type. There is also a difference in the variety of the answer. In Task 7, there can be many different types of answers, but in Task 5(a), there is only one type of answer, namely a playground design, although there can be many variations within this one type. This subtle difference will affect the goal of the task which will be discussed next.

9.2 Open or Ill-Defined Goal?

Orton and Frobisher (1996) defined "a 'problem' … to be 'open' when no goal is specified" (p. 32), for example, investigative tasks like Task 2(b). But 'investigate' is still a goal, albeit a vague *general* goal, and so a student can choose any *specific* goal to investigate. Thus we may say that the goal is open. It may be tempting to think that the goal is open because there is more than one goal, just like an open answer has multiple correct or valid answers. However, if a task specifies a few goals, then this task has multiple goals but the goal is still closed because a student cannot choose his or her own goal. Therefore, a task has an open goal if there are *no* specific goals and a student can choose any goal to investigate. This implies that there exist multiple specific goals for a student to choose because if a task has only one specific goal, then the goal cannot be open. But what happens if we rephrase Task 2(b) as follows?

> **Task 2(c): Find Patterns in Powers of 3**
> Find as many patterns as possible about the powers of 3.

Task 2(c) is still an investigative task but the general goal is now well-defined because a student will be clear that the goal of the task is to find as many patterns as possible. But there is still a sense that something is open. First, the answer is open because there are multiple correct answers. This also serves to illustrate that the "goal and [the] answer are not … the same" (Orton & Frobisher, 1996, p. 26): although there are multiple correct answers, there is only one general goal. Secondly, a student can still choose different specific goals to pursue because there are multiple correct answers. For example, he or she may wish to find whether there is a pattern in the last digit, or in the last two digits, or in the sum of all the digits of the powers of 3. Thus there are no specific goals in the task and so the goal is open. Therefore, Task 2(c) has a well-defined but open goal. The 'open-ended' Task 7 is another example of a task with a well-defined but open goal. This shows that the terms 'ill-defined' and 'open' have different meanings (see Section 9.1 also). However, there is still the sense that Task 2(c) is less open than Task 2(b) because the goal in Task 2(c) is more defined and so the student will be clearer about the goal. This suggests that openness is a continuum: there are different degrees of openness.

In Task 2(c), the goal is open because there are multiple correct answers. If there is only one correct answer, e.g. Task 2(a), then the goal cannot be open. However, an open answer does not always result in an open goal. For example, in Task 5(a), the goal is well-defined because if you design anything else other than a playground, then it is not the goal of this task. But there is only one specific goal: design a playground. So Task 5(a) has a well-defined and closed goal although the answer, like Task 2(c), is still open. The difference is that there are at least two types of open answer (see Section 9.1). In Task 2(c), there are multiple correct answers and this results in an open goal because a student can pursue different goals to reach different answers. However, in Task 5(a), there is no such thing as the correct answer but there is still only one type of answer, namely a playground design, although there can be many variations within this one type. A designer can pursue different paths that will lead to different designs but all these paths will lead towards the same goal, i.e. the design of a playground, and so the goal is closed. As you may have observed, many confusions can arise if we do not distinguish between the two types of open answer and between the terms ‘ill-defined’ and ‘open’.

9.3 Open or Ill-Defined Method?

A task can also be open when the method of solution is open. Becker and Shimada (1997) described open-ended problems where “students are asked to focus on and develop different methods, ways, or approaches to getting an answer to a given problem and not on finding the answer to the problem” (p. 1). Frobisher (1994) attributed “the openness [of problem-solving tasks] to the method of solution, not to the solution” (p. 158). For example, consider this problem-solving task:

> **Task 8: Handshake Problem**
> At a workshop, each of the 14 participants shakes hand once with each of the other participants. Find the total number of handshakes.

There are at least four different problem-solving strategies to solve this task: (i) use logical deduction to reason that the answer is $13 + 12 + 11 + \ldots + 1$; (ii) simplify the problem by starting with a smaller number of participants and trying to find a pattern; (iii) draw a diagram; and (iv) restate the problem as a combinatorics problem because every different pair of participants will give rise to one distinct handshake and so the answer is ${}^{14}C_2$. So the method is open. But consider another typical textbook task:

> **Task 9: Solve Quadratic Equation**
> Solve the quadratic equation $x^2 + 2x - 3 = 0$.

There are at least four methods of solution: (i) trial and error, (ii) factorisation, (iii) completing the square and (iv) quadratic formula. So do we really want to call this a task with an open method? First, this task has essentially one *efficient* method to solve: by factorisation. The other methods are either too long or not foolproof. Secondly, these methods are more procedural than the problem-solving strategies used to solve Task 8. In Task 8, although some methods may be longer than others, they are generally considered useful problem-solving strategies that stimulate mathematical thinking (Orton & Frobisher, 1996) and it does not really matter which one the students use, unlike Task 9 which has essentially one efficient method. So we may not want to consider Task 9 as a task with an open method.

Therefore, when a task has an open method, not only must there be multiple methods of solution but these methods must involve more problem-solving strategies than procedures. Some researchers call this type of tasks ‘open-middle tasks’ because the method, which is at the ‘middle’ of the problem solving process, is open (Sheffield, Meissner and Foong, 2004). However, if a student just solves Task 8 using only one method, then the method is closed to the student. So Task 8 does *not* have the sense that it is open unless the student is encouraged to find as many different methods as possible to solve the problem. But if that is the case, won’t the answer include the different methods and so what is open is no longer the method but the answer? Technically, this may be true, but we may want to differentiate between this type of open-middle tasks and those open-ended tasks discussed in Section 9.1.

What about investigative tasks, like Task 2(b), where there are multiple correct answers? On one hand, we may say that there are multiple methods of solution because it is usually not possible to use only one method to generate all the correct answers. But on the other hand, for each correct answer, there may be only one method of solution and so is the method still open? (Even if there are multiple methods of solutions for each answer, just like the problem-solving tasks described above, it also depends on whether the student chooses to find other alternative methods.) Therefore, whether the method is open for investigative tasks depends on which perspective we are looking from. But there is really no need to decide whether investigative tasks have a closed or open method because investigative tasks are already open in their goal and in their answer.

There is another meaning of an open method. In Tasks 4 and 5(a), there is no such thing as the correct answer. As a result, there is no method that will guarantee the answer (Frederiksen, 1984; Kilpatrick, 1987; Simon, 1973). This is a common problem with mathematical tasks that have some elements of creativity, for example, problem-posing tasks and designing tasks. You cannot help the student by providing him or her a method that will definitely lead to the answer. Compare this with problem-solving tasks like Task 8. Although there are multiple methods of solution, you can guide the student by teaching him or her a particular method that will surely lead to the answer. Therefore, for tasks with multiple valid answers, especially those that involve creativity, the method is open because there is *no* well-defined method that will guarantee the answer. Thus we may say that the method is ill-defined. This is an *inherent* property of the task, unlike problem-solving tasks with an open method but depending on whether the student wanting to find alternative methods.

Therefore, there are at least two meanings when we say that the method is open. An open method may mean that there are multiple methods of solutions, or that there is no well-defined method that will guarantee the answer. Another viewpoint is to call the first type ‘open method’ and the second type ‘ill-defined method’. In this paper, the context will decide whether the term ‘open method’ is used to refer to the first type or to both, but the term ‘ill-defined method’ will always refer to the second type. Sometimes, we may also structure the method of solution into the task statement to guide students to solve the problem. This is called scaffolding which will be discussed next.

9.4 Scaffolding

Let us consider the following problem-solving task:

Task 10: Make Box with Biggest Volume
Make an open box using a given vanguard sheet so that it has the biggest possible volume.

There is the sense that something is open in this task. But it cannot be the goal because the goal is well-defined and closed: make an open box. It also cannot be the answer because there is only one correct answer. Unlike Tasks 4 and 5(a), this task has a method that will guarantee a solution. There may also be multiple methods of solution, but unlike Task 8 above, there is still the sense that something else is open. Students who see this task for the first time may not know how to begin: the task is too big or complex and there is no scaffolding in the task statement. But it does not mean that every task with no scaffolding is open. For example, consider the following tasks:

Task 11(a): Unguided Word Problem
The product of three consecutive even natural numbers is 260 times their sum. Find the three numbers.

Task 11(b): Guided Word Problem
$x - 2$, x and $x + 2$ are three consecutive even natural numbers whose product is 260 times their sum.

(i) Form an equation in x and show that it simplifies to $x^3 - 784x = 0$.
(ii) Solve the equation and hence find the three numbers.

Task 11(b) has some scaffolding and it is not open. Task 11(a) does not have any scaffolding but it is still not open. The difference between Tasks 10 and 11(a) is that Task 10 is a more complex problem than Task 11(a). So it seems that a more complex task with no scaffolding is more open than a simpler task without any scaffolding. Therefore, a task can be open if it is complex enough and there is no scaffolding. Now, what does it mean by 'complex enough'? Again, complexity is a continuum. At both ends, many educators may agree that the tasks are either very simple or very complex. The problem always lies in the middle and so there will always be grey areas. This type of scaffolding is not an inherent property of the task but a property of the task statement because the task (e.g. Task 10) can always be rephrased to include more guidance as to how to solve it. Thus this type of scaffolding refers more to providing guidance in terms of the method of solution.

There is another type of scaffolding. In Task 5(b), the guidance is in terms of providing more information on the design of the playground, thus making Task 5(b) less open than Task 5(a). But this depends on the type of answer. If the answer is closed, then you really do not have a choice to decide how much information to give because if insufficient information is given, then it will make the task insolvable rather than more open. For example, consider the following tasks in coordinate geometry:

Task 12(a): A(4, –4), B(9, 6), C and D are the vertices of a parallelogram. Find the coordinates of D.
Task 12(b): Find the coordinates of a point P that is equidistant from the points (1, 0) and (0, 1).
Task 12(c): Find a general form for the coordinates of a point P that is equidistant from the points (1, 0) and (0, 1).

Task 12(a) does not have sufficient information given in the task statement to solve it: the point D can be any point in the Cartesian plane. Task 12(b) also does not have enough given information to solve it but it is slightly different. If we want, we can still find some kind of an answer for P although it is not unique: P is any point that lies on the line $y = x$ and its coordinates are (x, x) where x is any real number. This means that the goal of the task is poorly stated. If Task 12(b) is rephrased as in Task 12(c), then the goal is clearly stated and there is enough given information to solve it. Does this mean that Tasks 12(a) and 12(b) are tasks that do not have sufficient information to solve them? Yes, but this is the result of poor problem posing. Therefore, you have no choice but to provide all the necessary information to solve a task whose answer is closed if you do not want to end up with a poorly-set task that cannot be solved.

However, when a task has an ill-defined answer, it may not be *possible* to supply all the information necessary to solve it (e.g. Task 5(a)). Then you can choose how much information to give. If you give more information, such as in Task 5(b), then the task will be less open than Task 5(a). In real life, no one will provide you all the necessary information to solve a problem. It may also not be practical to do so because, for example, in Task 5(b), it depends on what the designer prefers and so it is impossible to decide what the boundary of the necessary information to solve the task is (Simon, 1978). The designer will have to do some research to gather all the necessary information, such the size and cost of a slide or a swing etc., according to what he or she wants. So this second type of scaffolding refers more to providing guidance in terms of the amount of information given in the task statement but it depends on the *nature* of the task, unlike the first type of scaffolding. If it is *inherently* impossible to supply all the information necessary to solve it, then we may say that the task is open. Compare this with problem-solving tasks, such as Tasks 2(a), and investigative tasks, such as Task 7. In these tasks, not much information is given but it is enough for a student to solve them without further information or research.

9.5 Extension

Orton and Frobisher (1996) described another type of open-ended tasks. They have used the term ‘process problems’ to refer to mathematical tasks that concentrate “on the mathematics itself and on the mathematical thinking processes for arriving at the solution” (p. 29). These ‘process problems’ are similar to problem-solving tasks described in this paper. Orton and Frobisher called these ‘open-ended problems’ because there is no end to these tasks: we can always extend the task by asking “What if …?” For example, for Task 2(a), we can ask, “What if the index is 2008? 2009? *n*?” or “What if the base is 2? 4? 5? *n*?” Although the original task has a closed goal and answer, the inherent nature of problem-solving tasks allows us to generate more problems to solve, and in this sense, these tasks have ‘no end’ or an ‘open end’. But again, it depends on the student. If the student does not know that the task can be extended, or that he or she is expected to extend the task, or if the student chooses not to, then the task is closed to him or her. But if the student is encouraged to extend the task, will this extension become another task with an open goal and an open answer, just like an investigative task? Moreover, investigative tasks can also be extended and so this type of open-endedness is not unique to problem-solving tasks only.

9.6 Framework Characterising Openness of Mathematical Tasks

To summarise Section 9, when we talk about open-ended or open tasks, we must be clear whether we are referring to the goal being open, or the method being open, or the answer being open, or whether there is no scaffolding so that the task is open, or whether the task can be extended and so the 'end' is open. It is also helpful to distinguish between the two types of open answer, the two types of open method and the two types of scaffolding, and between the terms 'open' and 'ill-defined'. We have also observed that investigative tasks are not the only type of open tasks, but problem-posing tasks and practical tasks, which are not exactly investigative tasks (see Section 7), are also open. Table 1 (see next page) shows a framework characterising the openness of a mathematical task based on the various task variables and a few representative tasks to illustrate how the framework works. Some cells are left empty because it is not important to consider these variables when deciding whether a task is open. For example, an investigative task is already open in the goal and in the answer, and so it is not necessary to discuss whether the method is open or closed because it can be both, depending on which perspective you look from (see Section 9.3). The framework is never meant to be exhaustive: it may be possible for a problem to be open in other aspects which have not been examined in this paper.

Based on the framework in Table 1, it may be helpful to distinguish among three types of openness:

(i) Openness that depends on the student. Problem-solving tasks are open in the sense that they have multiple methods of solution and that they can be extended. But this depends on the student: if he or she does not do it, then the student will not be penalised.

(ii) Openness that is inherent in the task. Investigative tasks have an open goal and an open answer although the goal can be ill-defined (e.g. Task 2(b)) or well-defined (e.g. Task 2(c)). If a student gives only one answer when there are multiple correct answers, then he or she will be penalised, unlike the student in (i). So an open answer, where there are multiple correct answers, is an inherent property of the task. Tasks that have multiple valid answers are also open because both the method and the answer are ill-defined and thus open. It is also inherently impossible to provide all the necessary information in the task statement to solve many real-life or practical tasks and so these tasks are also open although you can still provide some scaffolding by giving more information in the task statement.

(iii) Openness that depends on the structure of the task statement. If a task is complex enough and there is no scaffolding provided in the task statement with regard to how to solve it, then the task is open. This is not an inherent property of the task because you can always rephrase the task statement to provide more guidance.

The second and third types of openness highlight the difference between the inherent properties of a task (which will be called the nature of the task) and how the task statement is phrased or structured (which will be called the structure of the task statement). In literature, some researchers (e.g. Simon, 1973, 1978) discussed the task structure when they wrote about well-structured and ill-structured tasks, and about structured and unstructured tasks. Do these terms mean the same thing as the structure of the task *statement*? This issue will be discussed next.

Table 1: A Framework Characterising Task Openness (with Some Examples)

Framework		Examples of Mathematical Tasks to Illustrate Framework				
		Task 8 (handshake problem)	**Task 2(b) (investigate powers of 3)**	**Task 2(c) (patterns in powers of 3)**	**Task 5(a) (design playground)**	**Task 10 (make box with biggest volume)**
Task Variables	**Type of Task**	Problem-solving Task	Investigative Task	Investigative Task	Practical Task	Problem-solving Task
Goal	Ill-defined?	Well-defined	Ill-defined	Well-defined	Well-defined	Well-defined
	Open?	Closed	*Open*	*Open*: many goals (affected by open answer)	Closed: only one goal (affected by ill-defined answer)	Closed
Method	Ill-defined?				Ill-defined: no method to guarantee answer; so method is *open*	
	Open?	*Open*: multiple methods of solution				
Scaffold-ing	Guidance to solve it?					No guidance to solve it, and it is also complex enough: so task is *open*
	Possible to provide enough info to solve it?				Not possible to provide enough info to solve it: so task is *open*	
Answer	Ill-defined?	Well-defined			Ill-defined: multiple valid answers; so answer is *open*	Well-defined
	Open?	Closed	*Open*: multiple correct answers	*Open*: multiple correct answers		Closed
Exten-sion		Can extend	Can extend	Can extend		Can extend

10. Ill-Structured or Well-Structured Tasks?

Simon (1978) defined 'ill-structured problems' as those that satisfy all the following three criteria:

> "1. The criterion that determines whether the goal has been attained is both more complex and less definite.
> 2. The information needed to solve the problem is not entirely contained in the problem instructions, and indeed, the boundaries of the relevant information are themselves very vague.
> 3. There is no simple 'legal move generator' for finding all of the alternative possibilities at each step." (p. 286)

Simon gave an example of real-life tasks: design a house. But which aspect of the task is ill-structured? If we look at this from the perspective of the task variables in Section 9, then we will see that Simon's first criterion is the result of an ill-defined or open answer (see Section 9.1). When there is no correct answer, how do we define criteria to determine whether we have finished the task? For example, in Task 5(a), when can you say that you have finished designing the playground? You may have finished the task for now but you may decide to change some features later on. Similarly, for an investigative task where there are multiple correct answers, you will not know when you have found all the answers. So there is no definite criterion to decide whether the goal has been attained because the answer is open or ill-defined. Simon's second criterion is that it is not possible to provide enough information in the task statement to solve it, which is especially true for real-life or practical tasks that have an ill-defined answer, e.g. Task 5(a) again (see Section 9.4). Simon's third criterion is that there is no sure method that will lead to the answer and this may be the result of an ill-defined or an open answer (see Section 9.3). Therefore, if a task has an ill-defined answer, then it is ill-structured because it will satisfy all the three criteria.

But what about investigative tasks, such as Task 2(b), that have an open answer? They satisfy the first and the third criteria but not necessarily the second. If we allow varying degrees of ill-structuredness since Simon recognised that there is no sharp division between well-structured and ill-structured tasks, then investigative tasks can still be considered ill-structured. So it seems that Simon's idea of the structure of a task has more to do with the nature of the task than how the task statement is structured.

Frederiksen (1984) interpreted Simon's (1978) ill-structured problems as those that "lack a clear formulation, a procedure that guarantees a correct solution, and criteria for evaluating solutions" (p. 367). The first criterion, lacking a clear formulation, seems to deal with how the task statement is phrased. This may include an ill-defined or open goal in the task statement (e.g. investigative tasks) or insufficient information given in the task statement to solve the task (which is the same as Simon's second criterion). It may also refer to little or no scaffolding provided in the task statement when the task is complex enough (e.g. Task 10 in Section 9.4). Evidence from what Frederiksen discussed in his paper suggests that he will not regard Task 11(a), where there is no scaffolding, as an ill-structured task. Frederiksen's second criterion is similar to Simon's third criterion which have been discussed earlier. Frederiksen's third criterion seems to be the same as Simon's first criterion. For example, in practical tasks such as Task 5(a), there are no criteria for "testing the correctness of a solution" (Frederiksen, 1984, p. 367) and so you do not know when the goal has been attained. But there is a subtle difference between both interpretations. This becomes evident when we look at investigative tasks: you will not know when you have found all the answers (i.e. it satisfies Simon's first criterion) but you can still test the correctness of a solution (i.e. it fails Frederiksen's third criterion). It seems that Frederiksen's idea of the structure of a task includes the structure of the task statement (first criterion) but these two terms, i.e. 'structure of the task' and 'structure of the task statement', are not exactly the same because his second and third criteria have nothing to do with how the task statement is structured.

To add to the confusion, some educators use the phrase 'structured questions' to describe mathematical tasks that have scaffolding or guiding part-questions in the task statement (Ministry of Education of Singapore, 2004). For example, Task 11(b) is a structured task because it is guided. Interestingly, the Ministry of Education of Singapore did not refer to Task 11(a) as an unstructured task but as a 'long-answer question'. However, according to both Simon's (1978) and Frederiksen's (1984) criteria, these two tasks are well-structured. So there is a difference between the terms 'well-structured' and 'structured'. The Ministry of Education of Singapore (2004) used the word 'structured' to refer to the first type of scaffolding (see Section 9.4) and so, in this sense, the task structure refers to how the task statement is phrased. But it is not that simple because Frederiksen (1984) used the word 'structured' in a different sense. He created a category called "structured problems requiring productive thinking" (p. 367) which are similar to his 'well-structured problems' except that the method of solution must be generated by the problem solver. In other words, procedural tasks are an example of Frederiksen's well-structured problems, and problem-solving tasks are an example of Frederiksen's structured problems requiring productive thinking.

To summarise this section, it seems that there are at least two different meanings when we talk about the structure of a task. The first one refers to the nature of a task: whether the answer is ill-defined or open, which may affect whether the method is ill-defined and whether it is possible to provide enough information to solve it. The second one refers to how the task statement is structured or phrased. However, both the nature of a task and how the task statement is structured are closely linked together. For example, Task 2(b) is an investigative task and this affects how the goal in the task statement is phrased: investigate. If we were to rephrase the goal as in Task 2(a), then this not only restricts the scope of the task but it also changes the nature of the task from an investigative to a problem-solving task. But there may be some leeway as to how to rephrase the goal without changing its nature. For example, Task 2(b) can be rephrased as Task 2(c) without changing its nature as an investigative task although sometimes it may change its scope because not all investigative tasks are about finding patterns. Therefore, the nature of a task determines, to a certain extent, how the task statement is phrased. On the other hand, how the task statement is structured may affect the nature of the task.

11. Mathematically-Rich Tasks

After the attempt to clarify various terminologies in the previous sections, we realise that there are still grey areas and different interpretations. But now we have a framework to clarify which aspect or variable the person is referring to when he or she talks about open or ill-structured tasks. Although it may be useful to be clear about whether a task is open or ill-structured, these characteristics are not as important to a teacher as the teaching purpose of the task because the teacher will need some useful criteria to choose appropriate tasks for different types of learning and assessment. (There is a difference between the teaching purpose and the goal of a task. For example, the goal of Task 1 is to find the midpoint but the teaching purpose of Task 1 is to let students practise on procedural skills.) For example, procedural tasks are useful for students to practise some procedures after they have been taught in class (Lester, 1980). These tasks are important to a certain extent because there is a need for students to be proficient in some basic skills (Cockcroft, 1982; NCTM, 1980). But these skills are not helpful in unfamiliar situations (Schoenfeld, 1988). So there is a growing support for problem-solving and investigative tasks that help students develop problem-solving strategies which, hopefully, can be applied to new situations (Arcavi, 2002).

These tasks help students develop mathematical processes that are generally more analytical, although the thinking involved in producing a plan to solve a problem may be more creative than analytical, and this is classified in the 'synthesis' category of the Bloom Taxonomy (Bloom et al., 1956) which includes the "ability to integrate results of an investigation into an effective plan or solution to solve a problem" (p. 170). But the main purpose of these tasks is to develop students' analytical thinking and so the term 'analytical tasks' will be used to describe them in this paper. However, there are certain tasks that specifically develop students' creativity or ability to synthesise, for example, problem-posing tasks and composing of mathematical poems (see Section 5). So the term 'synthesis tasks' will be used to describe these tasks where the main purpose is to develop students' creativity.

This type of tasks are rich in mathematics in the sense that students can develop rich mathematical processes or learn new mathematics such as concepts, formulae and procedures. Although some researchers (e.g. Orton & Frobisher, 1996) claimed that the answer is not as important as the method of solution, others believed that students should also learn the content that they have discovered during problem solving or investigation because this content will become part of the resources needed to solve other similar or new problems. For example, Schoenfeld (1985) believed resources, which include mathematical knowledge or content, are necessary to help students solve problems: without them, the students have nothing to think. Bell (1983) also advocated picking out key mathematical concepts and skills from mathematical investigation for further development and practice.

Therefore, mathematically-rich tasks are defined as mathematical tasks that provide a student with the opportunity to learn *new* mathematical content such as concepts and procedures, or develop mathematical processes such as analytical skills, creativity and metacognition. These include problem-solving, investigative and synthesis tasks. Non-mathematically-rich tasks include procedural tasks and 'word problems' which a student practises what he or she has been taught earlier by the teacher, rather than learning new mathematics from doing the tasks. "Practice of [procedural tasks and 'word problems'] is not an appropriate method for the development of *new* [emphasis mine] knowledge and its contribution to mathematics *learning* [emphasis mine] is minimal." (Orton & Frobisher, 1996, p. 27) The phrase 'mathematically-rich' is borrowed from educators who called this type of mathematically-rich tasks "rich mathematical activities" (e.g. Neyland, 1994, p. 106) and researchers who described using "mathematically rich games" (e.g. Civil, 2002, p. 49) to provide students opportunities to learn mathematics. Others refer to this simply as "mathematical activity" (e.g. Bell, 1983, p. 587; Love, 1988, p. 249). Figure 1 on the next page shows a classification of mathematical tasks according to their teaching purpose.

The classification in Figure 1 is never meant to be comprehensive. As in many classifications, there are always grey areas. For example, some practical tasks, such as Task 5(a), develop both students' analytical and creative skills and so it is not appropriate to classify them under 'analytical tasks' or 'synthesis tasks'. Moreover, Figure 1 does not contain all the mathematical tasks. For example, some assessment tasks are missing from the classification because the tasks in Figure 1 are classified according to their learning or practice purpose and not according to their assessment purpose although there are certainly overlaps between the tasks in the above classification and assessment tasks which will be discussed next.

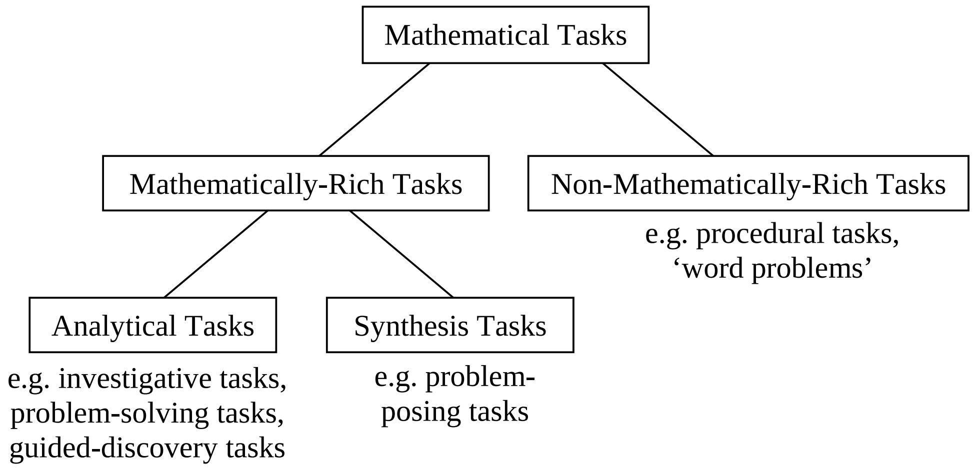

Fig. 1. A Classification of Mathematical Tasks According to Teaching Purpose

12. Rich Assessment Tasks

Some assessment tasks are very similar to the mathematical tasks in Figure 1 that students use in their learning or practice. For example, tasks that assess students’ procedural skills are usually the same as the procedural tasks that these students use in practice. Tasks that assess students’ problem-solving skills are also very similar to the problem-solving tasks used to develop their problem-solving strategies. A teacher can also use the same type of investigative tasks to assess a student’s ability to investigate although some researchers cautioned that this may lead to stereotyping certain mathematical processes as a set of procedures to be learnt by students “as a device for tackling the investigations rather than seen as part of being more generally mathematical” (Jaworski, 1994, p. 7). Although synthesis tasks can be used to develop students’ creativity, these tasks need to be assessed using a scoring rubric because there is no such thing as the correct answer. Becker and Shimada (1997) proposed a rubric based on three criteria: mathematical knowledge, strategic knowledge and mathematical communication. Teh et al. (2007) used different rubrics for different types of tasks. These include assessing both the processes and the product, or assessing the three elements of creativity: fluency (production of many ideas), flexibility (production of different categories of ideas) and originality (Pope 2005; Torrance, 1984). However, there are certain tasks that are not included in Figure 1. For example, consider this task:

Task 13: Explain Geometrical Meaning
Solve the simultaneous equations $x + 2y = 7$ and $x^2 - 4x + y^2 = 1$. Explain clearly the geometrical meaning of your answer.

The second part of this task attempts to assess students' conceptual understanding of what it means by solving simultaneous equations graphically. This is not a procedural task and it is also not a mathematically-rich task because the purpose is not for learning new knowledge but to assess their existing understanding.

Another example of an assessment task not included in Figure 1 is journal writing. Waywood (1992) described three types of journal writing: recount, summary and dialogue. In the recount mode, students report what they have observed in their lessons. In the summary mode, students summarise what they have learnt during their lessons. In the dialogue mode, students discuss what they have learnt. Waywood also described how to assess students' journal writing using a scoring rubric. However, teachers can also use journal writing as a formative assessment where they learn more about their students' learning difficulties from their journals and then take steps to remedy the situation (Miller, 1992). This type of journal writing should not be graded, or else the students may pretend that they know everything. Journal writing can also help students learn how to communicate mathematically when they try to explain what they have learnt. This may also help them to clarify their own understanding (Stempien & Borasi, 1985).

Therefore, some assessment tasks are very different from the mathematical tasks in Figure 1 that students used in their learning or practice. We may also want to distinguish between assessment tasks that are rich and those that are not so rich. In this paper, the term 'rich assessment tasks' will be used to refer to tasks that attempt to assess students' conceptual understanding, mathematical communication and thinking processes, rather than direct application of procedural skills or the mere regurgitation of facts.

13. Implications for Teaching and Learning

The classification according to purpose helps teachers to see that practising procedural tasks and solving 'word problems' are just two of the many different types of mathematical tasks. Some educators believe that there is a place for practising basic skills (Cockcroft, 1982; NCTM 1980) and these procedural skills form part of the resources that Schoenfeld (1985) believed are necessary to solve problems. However, school mathematics should not concentrate on just practising these basic skills (Burton, 1984; Lampert, 1990). There are many kinds of mathematically-rich tasks that help students to learn new mathematics or to develop mathematical processes which some educators (e.g. Moschkovich, 2002a; Schoenfeld, 1988) believed are useful and essential for students to solve unfamiliar mathematical and real-life problems. The classification above suggests to teachers what are the appropriate tasks to choose if they want to focus on developing students' analytical skills, or if they want their students to be creative, or both. The discussions on the various characteristics of the tasks help teachers to further refine their needs, for example, whether to make mathematics more realistic by using more real-life tasks, or whether to narrow the scope of investigation for weaker students by using the less open problem-solving and guided- discovery tasks than the more open investigative tasks. Teachers can also choose various types of rich assessment tasks if they want to assess their students' understanding, mathematical communication and thinking processes.

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