INVESTIGATING THE PROCESSES OF MATHEMATICAL INVESTIGATION

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Abstract

This paper describes a research study on how and what secondary school students investigate when faced with an open investigative task involving an interesting game that combines magic square and tic-tac-toe. It will examine the strategies that the students use and the mathematical thinking processes that they engage in when doing their investigation. The findings will be used to inform a theoretical model that we have devised to study the cognitive processes of open mathematical investigation, which include understanding the task, posing problems to investigate, specialising, formulating and testing conjectures, generalising, looking back and extending the task.

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1. Introduction

"Investigation [is] just a vehicle for other learning… This other learning might be seen as learning to be mathematical." (Jaworski, 1994, p. 4) Many researchers and educators believe that mathematical investigation can help students develop mathematical thinking processes which are useful in unfamiliar situations. For example, the Cockcroft Report (1982) recognised the importance of investigation in a mathematical problem-solving curriculum: "The idea of investigation is fundamental both to the study of mathematics itself and also to an understanding of the ways in which mathematics can be used to extend knowledge and to solve problems in very many fields." (p. 73)

Others subscribe to the view that mathematics classrooms should reflect the practices of mathematicians (Lampert, 1990; Schoenfeld, 1992) who not only solve problems but pose problems to solve, formulate and test conjectures, construct arguments and generalise (Moschkovich, 2002). Civil (2002) described the distinguishing characteristics of a classroom environment in which students do mathematics as mathematicians: (a) collaboration in small groups on challenging mathematical tasks; (b) the students are encouraged to develop and share their strategies, and to be persistent in the mathematical tasks; (c) mathematical discussions and communication among the students and with the teacher; and (d) the students are responsible for decisions concerning validity and justification. One type of mathematical tasks that will promote this kind of classroom environment is open investigative tasks (Civil, 2002; Lampert, 1986; Schoenfeld, 1991) although the writers may not use this terminology. For example, Lampert (1990) asked fifth graders to investigate the last digit of the square of a number. The purpose, however, is not to find the last digit *per se* but to construct arguments that support or reject conjectures formed. The latter reflects what mathematicians really do. Therefore, the use of open investigative tasks has the potential to create a “microcosm of mathematical culture” (Schoenfeld, 1987, p. 213) in the classroom where students engage in activities that are central to academic mathematicians’ practices.

Many researchers believe that mathematical investigation must be open in its goal (e.g., Orton & Frobisher, 1996), in its process (e.g., Delaney, 1996; Pirie, 1987) and in its answer (e.g., Bailey, 2007). They also agree that investigation must involve both problem posing and problem solving (e.g., Cai & Cifarelli, 2005), thus suggesting that investigation is different from problem solving in that the former is divergent while the latter is closed and convergent (Evans, 1987). But Yeo and Yeap (2009a) observed that some school teachers have often told their students to solve a closed problem by investigation. Yeo and Yeap realised that the conflict is due to the different usage of the same term ‘investigation’ and they resolved the issue by separating investigation into task, process and activity, just as Christiansen and Walther (1986) distinguished between a mathematical task and a mathematical activity.

In this paper, we will first discuss the different usage of the term ‘investigation’ and how it is related to problem solving. Then we will devise a theoretical framework to model the cognitive processes when students attempt open investigative tasks. In particular, we will focus on an open investigative task involving an interesting game that combines magic square and tic-tac-toe, which was given to a group of 20 secondary school students to investigate. The latter were videotaped while thinking aloud their thought processes and the transcripts of their verbal protocol will be used to inform the theoretical model in order to examine how students think when they investigate.

2. Investigation as Task, Process and Activity

Yeo and Yeap (2009a) gave the following example of an open investigative task:

> Powers of 3 are 3^1, 3^2, 3^3, 3^4, 3^5, … Investigate.

The goal is wide open (Orton & Frobisher, 1996). Students can pose any problem to investigate (Cai & Cifarelli, 2005) or they can just search for any pattern (Height, 1989). Yeo and Yeap (2009a) decided to classify both approaches as goal setting (or problem posing) for convenience: the former approach is setting a specific goal by posing a specific problem; the latter approach is setting a general goal by posing the general problem "Is there any pattern?"

When students attempt such an open investigative task, they are engaged in an open investigative activity (Yeo & Yeap, 2009a). This is consistent with Christiansen's and Walther's (1986) idea of the difference between a mathematical task and a mathematical activity.

However, a further distinction is necessary. Yeo and Yeap (2009a) observed that when students pose a problem to investigate, they have *not* started investigating yet. An analogy is cooking (Berinderjeet Kaur, personal communication). Cooking involves preparing the ingredients, the *actual process* of cooking, and scooping out the cooked food onto a dish and perhaps decorating it. Similarly, an open investigative activity involves what to do before investigation (e.g., understanding the task; and goal setting or problem posing), the *actual process* of investigation, and what to do after investigation (e.g., review and extension).

Separating investigation as a process from the activity itself has a very important implication. If we view investigation as involving both problem posing and problem solving, then problem solving is a subset of investigation. But if we separate the process of investigation from the open investigative activity, then it is possible to solve a closed mathematical problem by investigation. Yeo and Yeap (2009b) discussed two general approaches to solving a closed problem: by using a deductive argument or by investigation, or by using a combination of both methods. To understand these two approaches, we need to characterise the process of investigation in the next section.

3. Theoretical Framework to Model the Cognitive Processes of Open Investigation

We have devised a theoretical framework to model the cognitive processes of an open investigative activity. It involves five phases: entry (which involves understanding the task), goal setting (or problem posing), attack, review and extension. It is similar to the problem-solving model of Mason, Burton and Stacey (1985; first edition in 1982): entry, attack and review (which includes extension), except that our framework involves an addition phase of goal setting (or problem posing) because it is for open investigative activities. The attack phase is actually a combination of Pólya's (1957) second and third stages of his four-stage problem-solving model: devise a plan and carry out the plan; i.e., the attack phase is simply the actual process of problem solving. To understand why the process of problem solving is doing inside a model for an open investigative activity, we will now examine the framework in more details. Figure 1 shows the theoretical model for the cognitive processes of an open investigative activity. The arrows show the logical progression from one process to another although students can skip any of these processes, or they can jump from one process to any other process simply because they change their mind along the way.

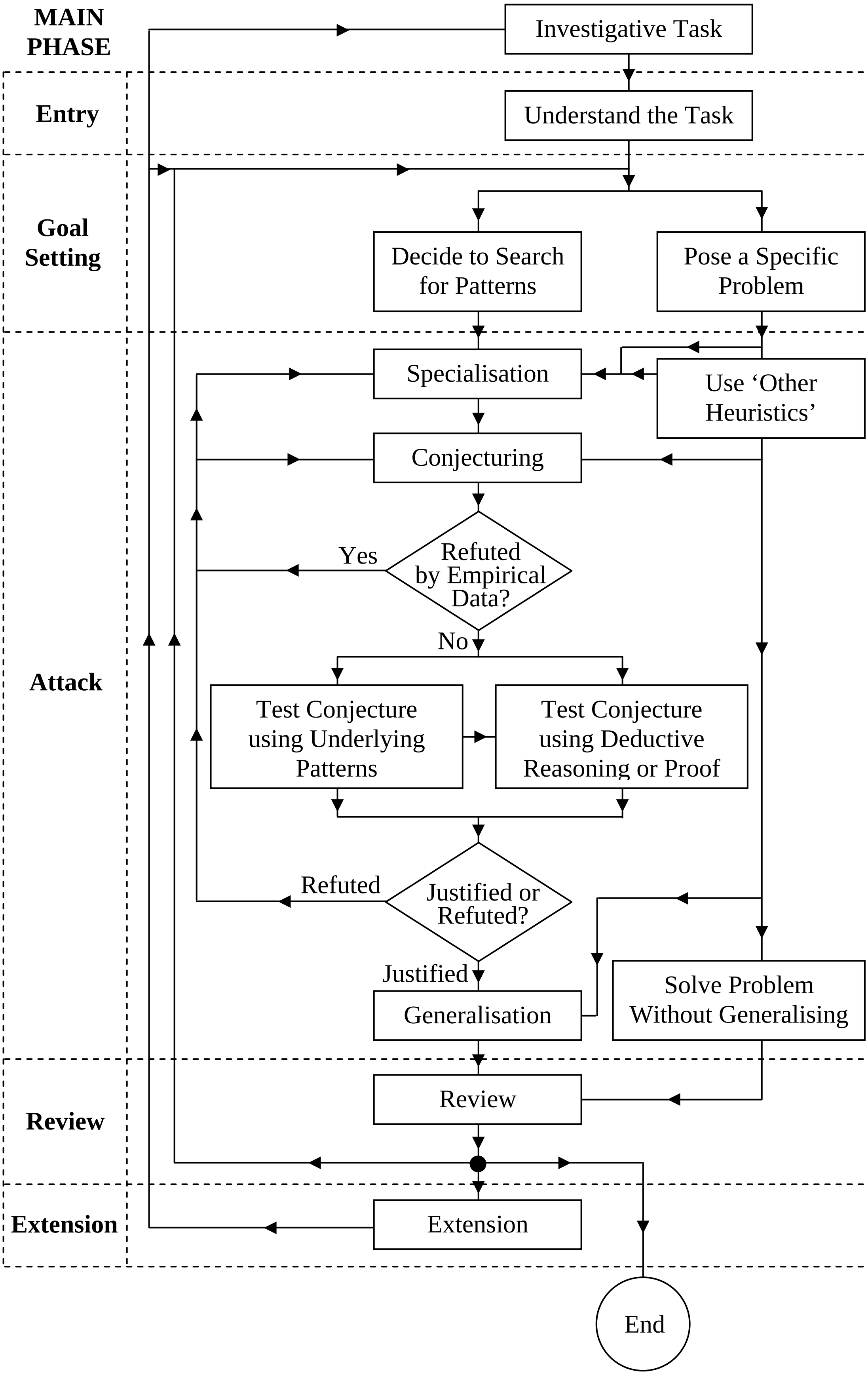

Figure 1. Theoretical Framework: A Mathematical Investigation Model

At the goal setting phase, as explained in the previous section, there are two approaches. Usually, students may not even know what specific problems to pose; so they may set out to just search for any pattern (Frobisher, 1994). To do that, they will have to examine specific cases or empirical data, which Mason et al. (1985) called specialisation. If they succeed in finding a pattern, that is just a conjecture which has to be proven or refuted. To refute a conjecture, students can look at more empirical data. If it is refuted, the students may be able to modify the conjecture, or if it is not possible, then they may have to go back to specialisation again. If a conjecture cannot be refuted by further empirical data, students can test it using two methods, or a combination of both. Some educators (e.g., Mason et al., 1985) believe that it is good enough to test a conjecture using the underlying pattern while others (e.g., Holding, 1991; Tall, 1991) believe in the use of a more rigorous deductive reasoning or a formal proof.

But Lakatos (1976) argued that it is almost impossible for anyone to conjure a formal proof out of nowhere, and so there is a need to use what he called ‘heuristic reasoning’, which is often based on induction and analogy, as a scaffold to construct a formal proof. Yeo and Yeap (2009b) have demonstrated how this could be done with a specific problem: how the use of specialisation can help students to discover the underlying pattern and how this in turn can help students to construct a more formal proof (for more details, please refer to Yeo & Yeap). However, Yeo and Yeap have observed how a teacher can conjure a formal proof out of nowhere, and that it is still possible to think of a deductive proof if it is simple enough: this explains why it is possible to have an arrow in the framework going from conjecturing to testing a conjecture using deductive reasoning or formal proof. If a conjecture is proven or justified, then generalisation has occurred (Height, 1989).

Mason et al. (1985) have used the four mathematical thinking processes of specialising, conjecturing, justifying and generalising for solving closed problems, but Yeo and Yeap (2009a) have observed that these processes are also the core processes of the *process* of investigation. Therefore, mathematical investigation, as a process, involves specialising in order to formulate and test conjectures so as to generalise or to find the underlying pattern or mathematical structure.

Let us now look at the other approach for goal setting. Sometimes, some students may be able to pose a specific problem. If they do not know how to solve it, then they can examine specific examples and so go through the process of investigation. But some students can use 'other heuristics' (which in this paper means heuristics other than specialisation) like deductive reasoning. Sometimes, this may result in only a conjecture which has to be tested; at other times, using a deductive argument may lead to a generalisation directly, or it can lead to a solution of the specific problem without any generalisation.

Therefore, in the attack phase, it is essentially the two approaches of problem solving as advocated by Yeo and Yeap (2009b): solving by investigation or by 'other means'. These two approaches are not necessarily mutually exclusive: it is possible to begin with 'other heuristics' and end up using the *process* of investigation; or the students can start investigating and then decide to use 'other heuristics' later. What the model does not and cannot possibly show is what happens when students change their minds because they can jump from any process to any other process. For example, a student may be trying to test his conjecture and then he or she decides to form another conjecture, or to pose another specific problem or to try to understand the task again.

In the review phase, the students should review what they have done. After that, they have three choices. They can go back to pose more problems to solve or to search for more patterns (all within the scope of the original investigative task); or they can extend the original task with or without changing the task; or they can end the investigation. Very often, most models for open investigation either show a linear pathway (e.g., Frobisher, 1994), or a cycle (e.g., Lakatos, 1976), but a model for investigation should include both (e.g., Height, 1989): although investigation is cyclic, it has to stop somewhere.

4. Research Method

The sample consists of 20 Secondary Two students from a school in the top end. They were taught by the first author for six lessons of two hours each. The first lesson was to expose the students to this type of open investigation which they are not familiar with. They then took a pre-test which consists of two open investigative tasks. The purpose of the remaining five lessons was to develop in the students various cognitive processes, after which they took a post-test which consists of three open investigative tasks. Each student was videotaped thinking aloud while attempting the tasks in both the pre-test and post-test. The duration for each task was 30 minutes. The verbal protocol was then transcribed and coded according to the cognitive processes in the model in the previous section.

The first two tasks of the post-test were parallel in design to the two tasks of the pre-test, but the third task of the post-test was an unfamiliar task about playing a game. The students were never taught how to deal with this kind of tasks in the lessons, such as to find if there is a winning strategy. The purpose of this third task was to find out if the students knew how and what to investigate when given a totally different type of open investigative task. The focus of this paper will be on this third task, which was adapted from Mason et al. (1985).

Investigative Task: Fifteen

Nine discs marked with the digits 1 to 9 are placed on the table. Two players take turn to pick one disc from the table. The winner is the first player to obtain the sum of exactly 15 among any three of his or her discs. Investigate.

This is an interesting game which combines tic-tac-toe and magic square. In a 3-by-3 magic square, the three numbers in each row, each column and each diagonal, all add up to 15. Anecdotal evidence suggests that few people know about any winning strategy for tic-tac-toe, and that most people do not even try to look for any winning strategy, probably because there is no winning strategy for most of the games that they play. Although there is no sure-win strategy for tic-tac-toe, there is a strategy that will *maximise* your chances of winning if your opponent is unaware of this strategy. Hence, there is a way to win this game of fifteen. In this paper, a winning strategy is defined to be a strategy that will maximise your chances of winning, and so it is different from a sure-win strategy.

However, there is still a need to prove that there are only eight combinations of three numbers from 1 to 9 that add up to 15, and these are the three rows, the three columns and the two diagonals of the magic square; because if there are more than eight combinations, then you cannot just apply the winning strategy for tic-tac-toe to this game of fifteen. For example, in a possible extension of the task where the players are allowed to combine *any* number of discs (instead of exactly three discs) to obtain a sum of 15, there are more than eight combinations of numbers that add up to 15, and so you cannot just apply the winning strategy for tic-tac-toe because there is another sure-win strategy.

5. Analyses

In this section, we will analyse the transcript of a particular student (pseudonym: Stella) to study her cognitive processes as she attempted the task so as to inform our theoretical model.

Excerpt 1: Entry Phase and Goal Setting Phase

Line	Time	Transcript	Processes
01	00:00	"Nine discs marked with the digits 1 to 9 are placed on the table. Two players take turn to pick one disc from the table ... Ok, so, the winner is the first player to obtain the sum … of [exactly] 15 among any three … of his or her discs. Investigate." …	Understanding the Task: Read task
02	00:25	So, nine discs. So let's, let's imagine and draw. So nine discs. [Draw nine circles] Eight, nine. So [number discs as she counts] 1, 2, 3, 4, 5, 6, 7, 8, 9.	Understanding the Task: Visualise given info
03	00:40	"Two players take turn to pick one disc from the table."	Understanding the Task: Re-read part of task
04	00:44	So, right, um, let's investigate. [Start writing] Investigate, if they do not, um … players do not [cancel 'do not'] … Ok, let's investigate: players, um, follow sequence, sequence of discs … If, let's investigate: if players follow sequence of discs, um, who will win first?	Pose a specific problem [Misinterpret task: has misconception that players must pick discs in order]
05	01:21	Ok. "Two players *each* take turn to pick."	Understanding the Task: Re-read part of task

Explanatory Note:

When the student read from the given task, it will be put in inverted commas. If the student missed out a phrase when reading from the given task, the phrase will be put in square brackets. If the student added a phrase into the given task while reading it, the phrase will be in italics. The three dots … indicate a short pause of less than one second. For longer pause, the duration will be specified.

Stella tried to understand the task by reading it aloud (Line 01) and drawing nine discs to visualise the given information (Line 02). Then she re-read part of the task to further her understanding of the task (Line 03). But when she posed a specific problem to investigate, she misinterpreted the task that the players must pick the discs in order (Line 04). Although you can investigate what happens if the players were to pick the discs in order, it is still a strange way to play a game like this. Stella then re-read part of the task (Line 05). This shows that it is possible to jump from any part of the mathematical investigation model to any other part (in this case, from the Goal Setting Phase back to the Entry Phase) which the model cannot possibly indicate all these.

Excerpt 2: Attack Phase

	Time	Transcript	Processes
06	01:24	[Continue writing] So players, two players, player 1 and player 2.	Specialisation
07	01:36	So, um, "the sum of exactly 15".	Understanding the Task: Re-read part of task
08	01:39	So player 1 will take 1 [continue writing], player 2 will take 2, this one will get 3, this one will get 4, this one will get 5, this one will get 6.	Specialisation
09	01:47	So now, is there a pattern?	Specialisation [in order to look for patterns]
10	01:50	[Take calculator and press] 5 plus 3, plus 3 plus 1 is 9. Ok, then, um, so, 2 + 4, 6 + 4 + 2 = 12.	Specialisation
11	02:04	So, if they continue, right, so you have 9, so you must [continue writing] plus 7, and this one plus 8. So, right, [press calculator] 9 plus 7 equal 16, and [use mental calculation] this will become 20.	Specialisation
12	02:16	So, right, um, we found out that [continue writing] if	Solve Problem

	Time	Transcript	Processes
		players follow sequence, he or her [*sic*] will not get the sum of 15. Yes, yeah.	Without Generalisation

Stella began the investigation by examining a specific case (specialisation) in order to look for a pattern. But she ended up realising that there would not be any winner, which solved her first specific problem. Therefore, it is possible to solve a problem by specialisation without forming any conjectures and without any generalisation. However, there is a missing arrow going from 'specialisation' to 'solve problem without generalisation' in our mathematical investigation model. This is an example of how research data can inform a theoretical model.

Excerpt 3: Review Phase

Line	Time	Transcript	Processes
36	11:13	So after, um, 5 trials … [continue writing] works after 5 times.	Solve Problem Without Generalisation
37	11:20	So Case 3 is correct [**pause** to think for 2 s] so Case 3 is correct ... [draw a line, write the word 'Correct' and box it] so Case 3 is correct …	Review: Check solution partially?
38	11:38	So, is there other method … to explain this?	Review: Look for alternative method; Pose a specific problem
39	11:42	[Pause to think for 1 s] Ok, so [pause to think for 4 s] so, right, um [pause to drink and think for 9 s] other method … to … explain this [pause to think for 3 s].	Pause for longest time to think
40	12:04	Ok, so, right … um, there is er … some [start drawing discs] there is a pattern like this, right? [Continue drawing discs in 3 × 3 array but not row by row. After drawing 8 discs, pause for less than 1 s and then start counting] 1, 2, 3, 4, 5, 6 [and then draw last disc at (2,3)-position.]	Specialisation: make use of previous knowledge of magic square

From Line 12 to Line 36, Stella realised that the players need to pick the discs randomly, but her idea was that the discs were arranged randomly and the players still had to pick the discs in order! She did find a winner eventually (Line 36). But she hesitated for a while (Line 37). It may be possible that she was checking her solution, perhaps only partially, before she decided that it was correct. So this could come under the Review Phase. After that, she asked whether there were other ways to solve the problem (Line 38), which is one of the things that, according to Pólya (1957), you could do at this stage. However, this will also coincide with goal setting: posing another specific problem.

Then Stella paused for a long time to think (Line 39): at least 17 seconds. This was the longest time that she ever paused so far to think when attempting this task and this was going to be the biggest breakthrough that she had achieved in this task. We did not know what went inside her mind as she did not think aloud her thoughts. We could only guess that she might be trying to recall what she had learnt before which she could use to solve this problem, a strategy that the first author taught the class, which was based on Schoenfeld's (1985) idea of 'resources' that students can call upon to solve problems. In the end, she succeeded in linking the sum of 15 to magic squares which she had learnt before, which was evident when she started drawing one (Line 40).

Excerpt 4: Review Phase and Extension Phase

Line	Time	Transcript	Processes
85	28:48	[Look at what she has written and **pause** for 7 s before turning to p. 1, then **pause** for 2 s]	Review; or try to pose new problem
86	28:58	Ok. "Nine discs marked with the digits 1 to 9 are placed on the table. Two players take turn to pick one disc from the table. The winner is the first player to obtain the sum of exactly 15 among the three … of …"	Review: by reading the task again; or try to pose new problem

Line	Time	Transcript	Processes
87	29:11	Oh, among any three … [turn to p. 6 and p. 7] oh, so [*unintelligible word*] two is the extension.	Review: realise her mistake, that she has already extended the task
88	29:18	[Turn to p. 1] Ok [**pause** to think for 2 s] investigate, investigate [turn to p. 4 and p. 5] we investigate 3 tries, right? … Ok, yeah [turn to p. 6 and p. 7]	Try to pose new problem
89	29:27	[Turn to p. 8] So, right, is there er, is anything other [*sic*] to investigate?	Try to pose new problem
90	29:30	[**Pause** to drink and think for 4 s, then turn to p. 1, and flip through some pages and back to p. 1] Oh, is there anything other at … um … we can investigate? [I interrupted to tell her to continue talking and to speak louder] Ok.	Try to pose new problem
91	29:50	[**Pause** to think for 3 s] Among 15, so, right … er, so two tries we try already, three tries we try … so players try to increase the number.	Try to pose new problem
92	30:02	[Turn to p. 8] So, let's try [continue writing] try to increase the number, the number of [*sic*] the discs. Ok, so, so, right [turn to p. 1 to look at task very briefly and then turn back to p. 8] add 10 [continue writing] 10 to each number.	Pose new problem which is an extension of the task
93	30:32	So the number will be … 11 [turn to p. 1 to look at task very briefly and then turn back to p. 8] 12, 13, 14 and so on. [Time is up at 30:44]	Pose new problem which is an extension of the task

After Stella had finished investigating her problem, she was either reviewing what she had done or she was trying to think of a new problem to investigate (Line 85). Then she read the task again (Line 86) and discovered a mistake that she had made (Line 87): the task requires the players to sum only three numbers to get 15, but she allowed the players to use the sum of two numbers as well. Instead of dismissing her solution, she realised that what she had done could be an extension of the original task: the players could use any two discs to obtain a sum of 15. Then she tried to think what else was there for her to investigate and she thought for about 45 seconds (Line 88-91) before deciding to increase the number on the discs by adding 10 to each number (Line 92), which was an extension of the original task. It did not occur to Stella that a more interesting extension would be to allow the players to use *any* number of discs to obtain a sum of 15, since she had just realised that allowing the players to use two discs (instead of three discs in the original task) was an extension (Line 87).

6. General Discussion

Stella had learnt the processes of mathematical investigation rather well. She knew she had to understand the task first. But still, she misunderstood the task to mean that the players had to pick the discs in order or according to some prearranged sequence. Even after she realised that she could make use of magic squares, she still made the players pick in order according to the sequence of the numbers in the row or column. She did not pause to question her assumption and so she lacked metacognitive monitoring, which had also been taught during the teaching experiment.

She also knew that she needed to pose problems to investigate. But the quality of the problems posed left much to be desired, esp. when she misunderstood the task. The idea of a winning strategy did not cross her mind. In fact, very few students in the sample actually investigated if there was a winning strategy.

She knew how to specialise to look for patterns. But she did not try to generalise any of her findings. For example, when she found that it was possible for a player to win this game by choosing certain discs (which is just one specific case), she did not go further to examine other cases to see if she could generalise her findings. But it was precisely her failure to generalise that led to the discovery that you could specialise and solve a specific problem without generalising, and this possibility is missing in the theoretical model of cognitive processes in open mathematical investigation. This is how research data can be used to inform a theoretical framework in order to improve it.

When she reached the review phase, she did not check through her solutions or question her assumption that the players must pick the discs in order or according to some prearranged sequence. But she did pose a good question (according to Pólya, 1957) in the review phase: is there other method to explain this?

Her best moment occurred when she spent one of the longest durations (about 17 seconds) thinking during the investigative activity but she kept silent during this period. The outcome was that she recalled that the magic sum in a 3-by-3 magic square was also 15. In fact, very few students in the sample realised the connection between this game and magic squares. Stella also did not see the connection between this game and tic-tac-toe.

Another important insight happened when she discovered during another review phase that she had made a mistake: she allowed the players to use two discs to sum to 15 instead of three discs as in the original task. But instead of rejecting her previous solution, she realised that this could be an extension of the original task. However, she failed to extend the task further by letting the players the freedom to use any number of discs to sum to 15. Instead, she just changed the task by increasing each number on the discs by 10.

Overall, Stella knew how to investigate to a certain extent although she still did not do it well enough. But she did not really know what to investigate.

7. Conclusion and Implications

The theoretical model serves well to describe the cognitive processes in open mathematical investigation and how the processes are related to one another. The research data obtained from the student's work has helped to refine the model to a certain extent. The model has served as a window into the mind of the student as she attempted the open investigative task. But knowing how to investigate rather well will not guarantee the quality of the investigation if the student does not know what to investigate. Posing good problems is a skill that needs to be developed further in the student. Metacognitive regulation of problem solving is also important to keep the student from misunderstanding the task and going down false trails.

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