Chapter 7

Solving Mathematical Problems by Investigation

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Most educators would think of heuristics when it comes to solving closed mathematical problems, while many researchers believe that mathematical investigation must be open and is different from problem solving. In this chapter, we discuss the relationship between problem solving and investigation by differentiating investigation as a task, as a process and as an activity, and we show how the process of investigation can occur in problem solving if we view mathematical investigation as a process consisting of specialising, conjecturing, justifying and generalising. By looking at two examples of closed mathematical tasks, we examine how investigation can help teachers and students to solve these problems when they are stuck and how it can aid them to develop a more rigorous proof for their conjectures. We also deliberate whether induction is proof and how heuristics are related to investigation. Finally, we consider the implications of the idea of solving mathematical problems by investigation on teaching.

1 Introduction

The use of problem-solving heuristics or strategies to solve mathematical problems was popularised by Pólya (1957) in his book *How to solve it* (first edition in 1945). Few educators would talk about solving mathematical problems by investigation. In fact, many educators (e.g., HMI, 1985; Lee & Miller, 1997) believe that mathematical investigation must be open and that it must involve problem posing. Thus the idea of solving closed mathematical problems by investigation is a contradictory notion. Although many educators (e.g., Evans, 1987; Orton & Frobisher, 1996) have observed that there are overlaps between problem solving and investigation, they usually ended up separating them as distinct processes: problem solving is convergent while investigation is divergent (HMI, 1985). Some educators (e.g., Pirie, 1987) have even claimed that it is not fruitful to discuss the similarities and differences between them, but we agree with Frobisher (1994) that this is a crucial issue that may affect how and what teachers teach their students. Therefore, the main purposes of this chapter are to clarify the relationship between problem solving and investigation, to illustrate how investigation can help teachers and students to solve two closed mathematical problems when they are stuck, and to discuss how they can make use of investigation to develop a more rigorous proof for their conjectures.

We begin by examining what constitutes a problem to a particular person, whether problems must be closed or whether they can be open, and how investigation is related to problems. Subsequently, we discuss the relationship between investigation and problem solving by first separating investigation into investigative tasks, investigation as a process and investigation as an activity, and then characterising the process of mathematical investigation as involving the four core thinking processes of specialising, conjecturing, justifying and generalising. We argue that investigation as a process can occur when solving closed mathematical problems and we examine how investigation can aid teachers and students to solve these problems when they are stuck by looking at two closed mathematical tasks. In particular, we observe how investigation can help them to develop a more rigorous proof for their conjectures. Then we deliberate whether induction is proof by looking at the different meanings of the terms ‘induction’, ‘inductive observation’ and ‘inductive reasoning’, and we consider how investigation is related to problem-solving heuristics after establishing that investigation is a means to solve closed problems. The chapter ends with some implications for teaching.

2 Relationship between Problem Solving and Investigation

Whether a situation is a problem or not depends on the particular individual (Henderson & Pingry, 1953). If the person is "*unable to proceed directly* to a solution" (Lester, 1980, p. 30), then the situation is a problem to him or her. Reys, Lindquist, Lambdin, Smith, and Suydam (2004) believed that this difficulty must require "some creative effort and higher-level thinking" (p. 115) to resolve. Thus most textbook 'problems' are actually not problems to many students partly because they know how to 'solve' them and partly because the main purpose of these 'problems' is to practise students in the procedural skills that have been taught in class earlier (Moschkovich, 2002). Therefore, it may be a better idea to use the term 'mathematical task' instead of 'mathematical problem' when we are referring to the task itself. For example, the Professional Standards for Teaching Mathematics (NCTM, 1991) used the phrase 'mathematical tasks' instead of 'mathematical problems' (see, e.g., p. 25) and Schoenfeld (1985) wrote, "… being a 'problem' is not a property inherent in a *mathematical task* [emphasis mine]" (p. 74). However, we do use the terms 'mathematical problems' and 'problem solving' in this chapter, but whenever such terms are used, it implies that the task is a problem to the person because if otherwise, then there is no need to solve the task.

One of the contentious issues among educators concerns the closure or openness of mathematical problems. Henderson and Pingry (1953) believed that a problem must have a clearly defined goal, and Orton and Frobisher (1996) claimed that very few mathematics educators would classify mathematical investigations as problems because they were of the opinion that investigations must have an open and ill-defined goal. But we agree with Evans (1987) that if a student does not know what to do when faced with an investigation, then the investigation is still a problem to the student. Orton and Frobisher (1996) also observed that educators in some countries, e.g., the United States of America, would call investigations 'open problems'. But this phrase is an oxymoron if one holds on to the view that problems must be closed. Nevertheless, this suggests that many educators seem to separate mathematical problems from investigations in that the former must be closed while the latter must be open.

Others (e.g., Cai & Cifarelli, 2005; Frobisher, 1994) have suggested that investigation should involve both problem posing and problem solving. Although many educators have claimed that there are overlaps between problem solving and investigation, they still ended up separating them. For example, HMI (1985) stipulated that there is no clear distinction between problem solving and investigation but it still ended up separating problem solving as a convergent activity from investigation as a divergent activity partly because the writers believed that investigation should involve problem posing as well (Evans, 1987).

However, school teachers are often not so clear about the differences between problem solving and investigation. Some of them even feel that their students are doing some sort of investigation when solving certain types of closed problems (personal communication). For example, consider the following mathematical task which is closed:

> **Task 1: Handshakes**
> At a workshop, each of the 70 participants shakes hand once with each of the other participants. Find the total number of handshakes.

If students do not know how to solve this task, then this task is a problem to them. Some teachers believe that these students can begin by *investigating* what happens if there are fewer numbers of participants, which may help the students to solve the original problem. But there seems to be very little literature on this subject of solving a closed problem by investigation. However, a thorough search has revealed a few writings. For example, in the *synthesis* class in Bloom's taxonomy of educational objectives in the cognitive domain, Bloom, Engelhart, Furst, Hill, and Krathwohl (1956) wrote about the "ability to integrate the results of an *investigation* [emphasis mine] into an effective plan or solution to *solve a problem* [emphasis mine]". The *Curriculum and Evaluation Standards for School Mathematics* stipulated that "our ideas about problem situations and learning are reflected in the verbs we use to describe student actions (e.g., to investigate, to formulate, to find, to verify) throughout the Standards" (NCTM, 1989, p. 10), thus suggesting that the Standards do recognise investigation as a means of dealing with problem situations.

Yeo and Yeap (2009) tried to reconcile the differences between the view that mathematical investigation must be open and the view that investigation can occur when solving closed problems. The conflict appears to arise from the different uses of the same term ‘investigation’. Just as Christiansen and Walther (1986) distinguished between a task and an activity, Yeo and Yeap (2009) differentiated between investigation as a task, as a process and as an activity. They called the following an open investigative task, rather than the ambiguous phrase ‘mathematical investigation’:

Task 2: Polite Numbers

Polite numbers are natural numbers that can be expressed as the sum of two or more consecutive natural numbers. For example,

$$9 = 2 + 3 + 4 = 4 + 5,$$
$$11 = 5 + 6,$$
$$18 = 3 + 4 + 5 + 6.$$

Investigate.

When students attempt this type of open investigative tasks, they are engaged in an activity, which is consistent with Christiansen’s and Walther’s (1986) definitions of a task and an activity. Yeo and Yeap (2009) called this an open investigative activity which involves both problem posing and problem solving: students need to pose their own problems to solve (Cai & Cifarelli, 2005). However, Yeo and Yeap (2009) observed that when students pose a problem to solve, they have not started investigating yet. This led them to separate investigation as a process from investigation as an activity involving an open investigative task.

An analogy is Pólya’s (1957) four stages of problem solving for closed problems. During the first stage, the problem solver should try to understand the problem. But the person has not started solving the problem yet. The actual problem-solving process begins during the second stage when the person tries to devise a plan to solve the problem and it continues into the third stage when the person carries out the plan.

After solving the problem, the person should look back, which is the fourth stage. Therefore, the actual problem-solving process occurs in the second and third stages although problem solving should involve the first and fourth stages also: what the person should do before and after problem solving.

Similarly, when students attempt an open investigative task, they should first try to understand the task and then pose a problem to solve. However, this is before the actual process of investigation. After the investigation, the students should look back and pose more problems to solve. Therefore, there is a difference between the process of investigation and an open investigative activity: the former does not involve problem posing but the latter includes problem posing. From this point onwards, the term ‘investigation’ will be used in this chapter to refer to the process while the activity will be called an ‘open investigative activity’. This distinction is important because we would like to argue that investigation can occur when solving closed problems. But first, we need to characterise what investigation is.

Yeo and Yeap (2009) observed that when students investigate during an open investigative activity, they usually start by examining specific examples or special cases which Mason, Burton, and Stacey (1985) called specialising. The purpose is to search for any underlying pattern or mathematical structure (Frobisher, 1994). Along the way, the students will formulate conjectures and test them (Bastow, Hughes, Kissane, & Mortlock, 1991). If a conjecture is proven or justified, then generalisation has occurred (Height, 1989). Thus investigation involves the four mathematical thinking processes of specialising, conjecturing, justifying and generalising, which Mason et al. (1985) applied to problem solving involving closed problems. Therefore, mathematical investigation can occur not only in open investigative activities but also in closed problem solving. But if investigation must involve problem posing, then investigation cannot happen when solving closed problems. This is why the separation of problem posing in open investigative activities from the process of investigation is very important.

Hence, if we view investigation as a process involving specialising, conjecturing, justifying and generalising, then we can solve closed mathematical problems by investigation when we are stuck.

3 Solving Mathematical Problems by Investigation

In this section, we will illustrate how investigation can help teachers and students to solve two closed mathematical problems when they are stuck, and how the result of an investigation can be used to develop a more formal or rigorous proof for their conjectures. Furthermore, we deliberate two important issues: whether induction is proof and how heuristics are related to investigation. Let us start by looking at the following task:

Task 3: Series

Find the value of $\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+2008}$.

This task was given to a group of in-service primary school teachers during a workshop at Mathematics Teachers Conference 2008 in Singapore. All of them had not seen this question before and they did not know how to solve it immediately, so this was a problem to them. Most of them were stuck: they did not even know how to begin. After some pondering, some of them tried to evaluate the denominators of all the fractions but it led to nowhere. So the first author guided them to investigate some specific examples by starting with smaller sums, i.e., what is the sum of the first two fractions, the sum of the first three fractions, etc., to see if there is any pattern:

$$S_2=\frac{1}{1}+\frac{1}{1+2}=\frac{4}{1+2}$$

$$S_3=\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}=\frac{9}{1+2+3}$$

$$S_4=\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}=\frac{16}{1+2+3+4}$$

$$S_5=\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\frac{1}{1+2+3+4+5}=\frac{25}{1+2+3+4+5}.$$

Some teachers were able to observe that $S_n = \dfrac{n^2}{1+2+3+...+n}$. The sum of the numbers in the denominator can be found easily as $\dfrac{n(n+1)}{2}$, so $S_n = \dfrac{2n}{n+1}$. Therefore, $S_{2008} = \dfrac{4016}{2009}$.

Unfortunately, most of the teachers thought that this was the answer. Some of them knew that this was only a conjecture because the observed pattern might not be true but they forgot to test the conjecture, while most of them did not even realise that this was only a conjecture. This is probably due to how they were taught number patterns in schools when they were students themselves, and now they are teaching their students the same thing: there is always a unique answer for the missing term in a sequence. For example, in the following sequence, what is the next term?

1, 4, 7, ____

Most of the teachers were taught that the answer must be 10 and so it is unique. However, the missing term is only 10 if the sequence is an arithmetic progression, in which case, the general term is $T_n = 3n - 2$. In theory, the next term can be any number. For example, the fourth term for the above sequence can be 16 if the general term is $T_n = n^3 - 6n^2 + 14n - 8$ (the reader can check that $T_1 = 1$, $T_2 = 4$, $T_3 = 7$ and $T_4 = 16$ using this formula). If you want the missing term in the above sequence to be any number, e.g., 22, all you need to do is to form and solve four simultaneous equations with four unknowns, and a polynomial with four parameters is of degree 3, i.e., the cubic polynomial $T_n = an^3 + bn^2 + cn + d$. So the four equations are:

$$\begin{aligned}
T_1 &= a + b + c + d = 1 \\
T_2 &= 8a + 4b + 2c + d = 4 \\
T_3 &= 27a + 9b + 3c + d = 7 \\
T_4 &= 64a + 16b + 4c + d = 22.
\end{aligned}$$

Solving the equations simultaneously, we obtain $a = 2$, $b = -12$, $c = 25$ and $d = -14$. So $T_n = 2n^3 - 12n^2 + 25n - 14$ (the reader can check that $T_1 = 1$, $T_2 = 4$, $T_3 = 7$ and $T_4 = 22$ using this formula). However, the coefficients may not always be 'nice' integral values or the simultaneous equations may have no solutions. For the latter, you can always try another polynomial that has more parameters, e.g., a polynomial of degree 4, and sooner or later, you will find a suitable polynomial. You can even try non-polynomials like a sine function.

Therefore, there is no unique answer for the missing term of a sequence. The answer that we want when we set this type of question is 'the most likely number' and what this means is that we prefer the formula for the general term to be less complicated. Thus the more terms we give for a sequence, the pattern should become more obvious and most of us may agree on one 'most likely number'. For example, 'the most likely number' for the missing term in the above sequence is 10 but some people may disagree. So, to avoid ambiguity, if we increase the number of given terms as shown below, then fewer people would disagree that 'the most likely number' for the missing term in the following sequence is 10, although it can still be any other number if we settle for a complicated formula for the general term, such as a polynomial of degree 6.

1, 4, 7, ____, 13, 16, 19

However, we cannot go for 'the most likely number' if the sequence has a context and is linked to some underlying pattern. For example, if we just consider the following sequence, then 'the most likely number' is 32 because the general term $T_n = 2^{n-1}$ is less complicated than a formula such as $T_n = {}^nC_4 + {}^{n-1}C_2 + {}^nC_1$.

1, 2, 4, 8, 16, ____

But if this sequence has a context and is linked to some underlying pattern, then we cannot just assume that the missing term is 32. For example, consider the following circle:

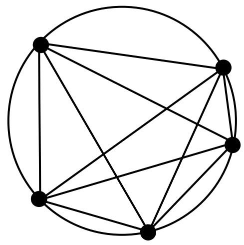

Figure 1. Circle with five points

There are five arbitrary points on the circumference of the circle, and each point is connected to every other point by a chord such that no three chords interest at the same point inside the circle. The chords divide the circle into regions. In this case, when n = 5 (where n is the number of points on the circumference of the circle), there are 16 regions inside the circle. If we consider the case when n = 1, 2, 3, 4, 5, … , then the total number of regions inside the circle, T_n, will form the following sequence:

1, 2, 4, 8, 16, …

If n = 6, what will be the total number of regions? The teachers in the workshop predicted that there would be 32 regions although a few of them suspected that this might not be the answer, or else the first author would not be giving them this counter example. Then the teachers counted the total number of regions for the following circle manually:

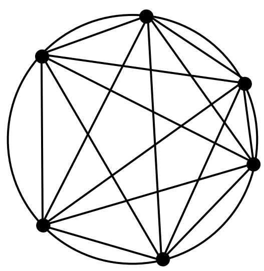

Figure 2. Circle with six points

When they found out that there were only 31 regions in the circle in Figure 2, some of them thought that they had counted wrongly and so they recounted the number of regions, while others realised that it was possible to have a sequence as follows:

$$1, 2, 4, 8, 16, 31, \ldots$$

However, some of them concluded that the above sequence has no pattern. The first author reiterated that there is still a pattern in the above sequence, but the underlying pattern is not $T_n = 2^{n-1}$ which is the 'more obvious' observed pattern in the sequence 1, 2, 4, 8, 16, … In fact, there is even a formula for the total number of regions: $T_n = {}^{n}C_4 + {}^{n-1}C_2 + {}^{n}C_1$ (the reader can check that $T_1 = 1$, $T_2 = 2$, $T_3 = 4$, $T_4 = 8$, $T_5 = 16$ and $T_6 = 31$ using this formula).

Let us return to the observed pattern in Task 3. The teachers finally realised that this was only a conjecture and they needed to test it. At first, no one was able to prove or refute it. After some time, a teacher managed to develop a rigorous proof. In fact, this teacher did not even solve the problem by investigation: she did not follow the hint of the first author above but she did the following on her own:

$$\begin{aligned}
&\frac{1}{1}+\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\ldots+\frac{1}{1+2+3+\ldots+2008} \\
&=1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\ldots+\frac{1}{1+2+3+\ldots+2008} \\
&=1+\left(1\div\frac{2\times 3}{2}\right)+\left(1\div\frac{3\times 4}{2}\right)+\left(1\div\frac{4\times 5}{2}\right)+\ldots+\left(1\div\frac{2008\times 2009}{2}\right) \\
&=1+\frac{2}{2\times 3}+\frac{2}{3\times 4}+\frac{2}{4\times 5}+\ldots+\frac{2}{2008\times 2009} \qquad \nwarrow \text{Line \#3} \\
&=1+\left(\frac{2}{2}-\cancel{\frac{2}{3}}\right)+\left(\cancel{\frac{2}{3}}-\cancel{\frac{2}{4}}\right)+\left(\cancel{\frac{2}{4}}-\cancel{\frac{2}{5}}\right)+\ldots+\left(\cancel{\frac{2}{2008}}-\frac{2}{2009}\right) \longleftarrow \text{Line \#5} \\
&=1+\frac{2}{2}-\frac{2}{2009} \\
&=\frac{4016}{2009}
\end{aligned}$$

All the other teachers were very impressed that this teacher was able to devise such a proof[1]. The first author asked the teacher how she managed to think of Line #3 and Line #5 which were the key steps in her proof, but she herself could not explain how and why she did it this way. All the other teachers agreed that they themselves would never have thought of this type of rigorous proofs that seem to come out of nowhere, which agrees with what Lakatos (1976) wrote when he observed that "it seems impossible that anyone should ever have guessed them" (p. 142).

There is a more elegant but similar proof:

$$\text{Let } S = \frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \frac{1}{1+2+3+4} + \ldots + \frac{1}{1+2+3+\ldots+2008}$$

$$= 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \ldots + \frac{1}{1+2+3+\ldots+2008}.$$

$$\text{Then } \frac{1}{2}S = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots + \frac{1}{2(1+2+3+\ldots+2008)}$$

$$= \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \ldots + \frac{1}{2008\times 2009}$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \ldots + \left(\cancel{\frac{1}{2008}} - \frac{1}{2009}\right)$$

Line #5

$$= 1 - \frac{1}{20009}$$

$$= \frac{2008}{2009}$$

$$\therefore S = \frac{4016}{2009}$$

Similarly, most people would never have thought of finding half the sum in this second proof. But how did the originator of this proof know what to do? The person most likely had to do some investigation first.

[1] Actually, there is more to the (first) proof than is shown here. There must be good reasons to believe that the patterns in Lines #3 and #5 will continue. We will leave it to the reader to find the reasons.

However, what might have helped in the investigation were some prior mathematical knowledge and skills which the person might have relied upon, which Schoenfeld (1985) called resources which were necessary for effective problem solving. For example, the person might have known the method of differences (which is the key step of the proof: see Line #5 of the second proof), and he or she might also be familiar with expressing $\frac{1}{n(n+1)}$ as $\frac{1}{n}-\frac{1}{n+1}$. The person might also have recalled that the numbers 1, 3, 6, 10, … , which appear in the denominators of the original series, are triangular numbers, and that the general term for triangular numbers is $T_n = \frac{1}{2}n(n+1)$, which is one step away from getting $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. These might have helped the person to think of starting with half the sum after some investigation. But if anyone does not have all these resources at his or her disposal, then the person may have to do more investigation to discover these first, or perhaps the person can conjure the first proof provided by the teacher above (this teacher has admitted that she knows the method of differences) and then refine it later to become a more elegant proof like the second one.

To summarise, this example (Task 3) illustrates the two main approaches to solve a closed mathematical problem: by investigation or by ‘other means’ (which is rigorous proof in this case), and that very few teachers were able to solve it using a rigorous proof directly.

Let us look at another example: the *Handshakes* task in the previous section (see Task 1). The first author has given this task to primary and secondary school students, and pre-service and in-service teachers. Some of the teachers and students have seen this question before, and they were able to give the answer almost immediately, so this task was not a problem to them. For those who saw this for the first time and were unable to solve it immediately, this was a problem to them. After a while, the teachers and the better students were able to solve it by ‘other means’, which in this case is simple deductive reasoning: since the first participant must shake hand with the other 69 participants, the second participant must shake hand with the remaining 68 participants and so forth, then the total number of handshakes is 69 + 68 + 67 + … + 1. Some high-ability students can even use a combinatorics argument that the total number of handshakes is equal to the total number of different pairs of participants, i.e., ${}^{70}C_2$, because every different pair of participants will give rise to one distinct handshake. This type of deductive proofs, unlike the formal proofs for Task 3, is within the grasp of many teachers and students.

But for the weaker ones who were unable to reason it in this way, many of them tried to solve the problem by drawing a diagram for smaller numbers of participants (see Figure 3 where n is the number of participants and T_n is the total number of handshakes) in order to observe some patterns so as to generalise to 70 participants. This is specialising in order to form a conjecture towards a generalisation, which are essentially the core processes in a mathematical investigation.

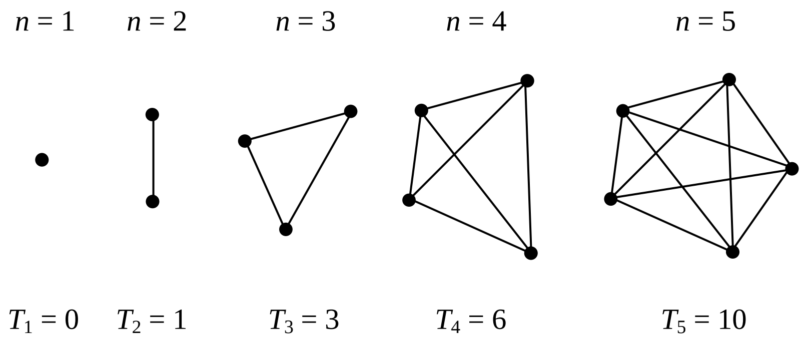

Figure 3. Handshakes task

Many of them were able to observe from their diagrams that the total number of handshakes for n participants is 0, 1, 3, 6, 10, … for n = 1, 2, 3, 4, 5, … respectively. However, most of them were unable to find a formula for the general term of this sequence. But they were able to observe this pattern:

$$0, \qquad 1, \qquad 3, \qquad 6, \qquad 10, \qquad \ldots$$

$$+1 \qquad +2 \qquad +3 \qquad +4$$

Using this pattern as a scaffold, the first author guided the teachers and students by asking them how to obtain T_4 from T_2. This enabled most of them to observe that $T_4 = 1 + 2 + 3$. Similarly, to obtain T_5 from T_2, most of the teachers and students were able to see that $T_4 = 1 + 2 + 3 + 4$. Therefore, they were able to observe that $T_{70} = 1 + 2 + 3 + \ldots + 69$, which is the total number of handshakes for 70 participants. Unfortunately, most of them, including the teachers and the better students, thought that this was the answer, without realising that this was only a conjecture to be proven or refuted. If the conjecture is wrong, you can refute it by using a counter example. But if the conjecture is correct, then do you really need a formal or rigorous proof to prove it? Some educators (e.g., Holding, 1991; Tall, 1991) believe in using rigorous proofs while others (e.g., Mason et al., 1985) support justification using the underlying mathematical structure. We shall illustrate these two approaches of justification using the *Handshakes* task.

The first author began by asking the teachers and students whether there was any reason to believe that the observed pattern would continue. Not a single person was able to find a reason. So the first author guided them with this question: if you go from $T_4 = 1 + 2 + 3$ to T_5, what happens? Some of them were able to observe that if you add the fifth participant to T_4, then the fifth participant must shake hand with each of the four participants, so there are four additional handshakes and thus $T_5 = 1 + 2 + 3 + 4$. Using the same argument, if you add the sixth participant to T_5, then the sixth participant must shake hand with each of the five participants, so there are five additional handshakes and thus $T_6 = 1 + 2 + 3 + 4 + 5$. Therefore, this is a good reason to believe that the observed pattern will continue in this manner because this argument can always be applied from T_n to T_{n+1}. But this is not a proof. However, Mason et al. (1985) believed that this type of argument using the underlying mathematical structure is good enough for school students.

The next question is how to guide these teachers and students to construct a more rigorous proof for their conjecture from their investigation. From the underlying mathematical structure discovered in the above investigation (i.e., if you add one participant to *n* participants, then the new participant must shake hand with the *n* participants, thus resulting in *n* additional handshakes and so the total number of handshakes for T_{n+1} is $1 + 2 + 3 + \ldots + n$), a few teachers and students were able to realise that they could use the same argument in the reverse manner: start from the first participant, and he or she has to shake hands with all the other 69 participants; then the second participant has only 68 participants to shake hand with, and so forth; thus the total number of handshakes for T_{70} is $69 + 68 + 67 + \ldots + 1$. In this way, the teachers and students have managed to use their investigation to develop a more rigorous proof for their conjecture. This agrees with what Pólya (1957) believed when he wrote that "we need heuristic reasoning when we construct a strict proof as we need scaffolding when we erect a building" (p. 113). According to Pólya, heuristic reasoning is based on induction or analogy, but both induction and analogy involve specialising in order to discover the underlying mathematical structure. Therefore, Pólya's idea of heuristic reasoning is very similar to the concept of the process of investigation outlined in the previous section.

One major issue to deliberate in this section is whether induction is proof. Yeo and Yeap (2009) believed that the problem lies in the different meanings of the terms 'induction', 'inductive observation' and 'inductive reasoning'. If students observe a pattern when specialising, the pattern is only a conjecture and Lampert (1990) called this 'inductive observation'. But if students use the underlying mathematical structure (Mason et al., 1985) to argue that the observed pattern will always continue, then it involves rather rigorous reasoning and so this can be called 'inductive reasoning' (Yeo & Yeap, 2009). Thus there is a big difference between inductive observation and inductive reasoning: inductive observation is definitely not a proof but inductive reasoning is considered a proof by some educators (e.g., Mason et al., 1985). Unfortunately, some educators (e.g., Holding, 1991) have used the phrase 'inductive reasoning' to mean 'inductive observation'. The same goes for the word 'induction': it can mean either 'inductive observation' or 'inductive reasoning' or both. For example, Pólya's (1957) idea of induction is inductive observation only. Therefore, whether induction is proof or not depends on which meaning you attach to the term 'induction'. In this chapter, the term 'induction' is used to include both inductive observation and inductive reasoning.

Another main issue to discuss in this section is the relationship between heuristics and investigation as a means to solve closed mathematical problems. Literature abounds with problem-solving heuristics (see, e.g., Pólya, 1957; Schoenfeld, 1985) but very few of them mention the use of investigation to solve closed problems, probably because few educators have ever characterised the process of investigation. Now that we have observed that investigation involves the four core processes of specialising, conjecturing, justifying and generalising, we can compare investigation with heuristics. Any heuristic that makes use of specialising can be considered an investigation (Yeo & Yeap, 2009). For example, if students use the heuristic of systematic listing or the heuristic of drawing a diagram for some specific cases, then it involves specialising and so this can be viewed as an investigation from another perspective. But if students use a deductive argument *directly*, then this is not an investigation. It does not mean that students cannot use deductive reasoning during an investigation. For example, students can use a deductive argument when proving a conjecture that is formulated during their investigation.

4 Conclusion and Implications

Differentiating between investigation as a task, as a process and as an activity has helped to separate problem posing from the process of investigation. This is important because if investigation entails both problem posing and problem solving, then investigation cannot happen during problem solving. Characterising the process of investigation as involving specialising, conjecturing, justifying and generalising, it becomes clear that investigation can also occur when solving closed mathematical problems. This agrees with what some teachers believe when they ask their students to investigate to solve a closed problem but most of them have no idea what investigation actually involves. If teachers have a vague idea of what investigation entails, then they may not be able to teach their students how to investigate properly (Frobisher, 1994). Therefore, the implication of defining the process of investigation more clearly in this chapter is to help teachers understand more fully what investigation means and how to help their students to investigate more effectively by focusing on each of the core thinking processes of specialising, conjecturing, justifying and generalising.

Another implication for teaching is how to make use of the results of an investigation as a scaffold to construct a more rigorous proof for a conjecture (Pólya, 1957) instead of conjuring a formal proof out of nowhere (Lakatos, 1976).

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