A Personal View on Teaching Probability

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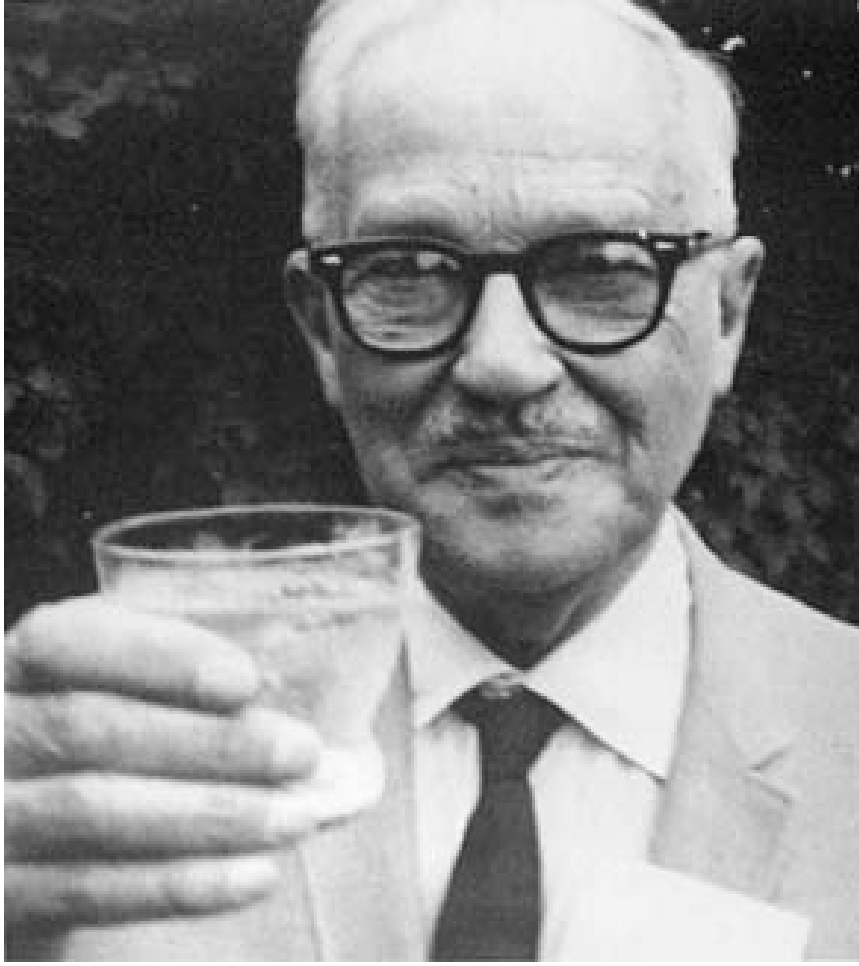
Outline

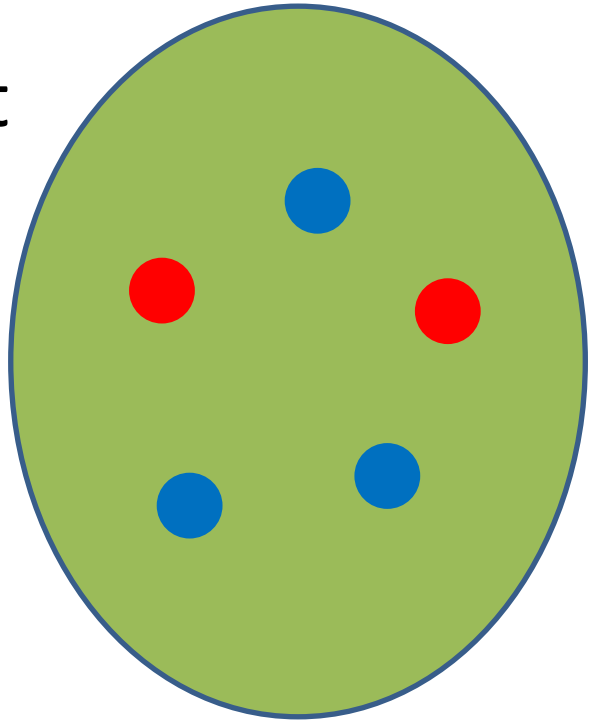
- Repeated experiments
- Conditional probability, dependence
- Population, sampling, hypothesis testing
- How to communicate

Each morning before breakfast every single one of us approaches an urn filled with white and black balls. We draw a ball. If it is white, we survive the day. If it is black, we die. The proportion of black balls in the urn is not the same for each day, but grows as we become older… Still there are always some white balls present, and some of us continue to draw them day after day for many years.

J Neyman and EL Scot, The distribution of galaxies



Draws Without Replacement



2 draws are made at random without replacement. The balls are of the same size and weight. Let

B_1 = {1st draw is blue}

B_2 = {2nd draw is blue}

True or false?

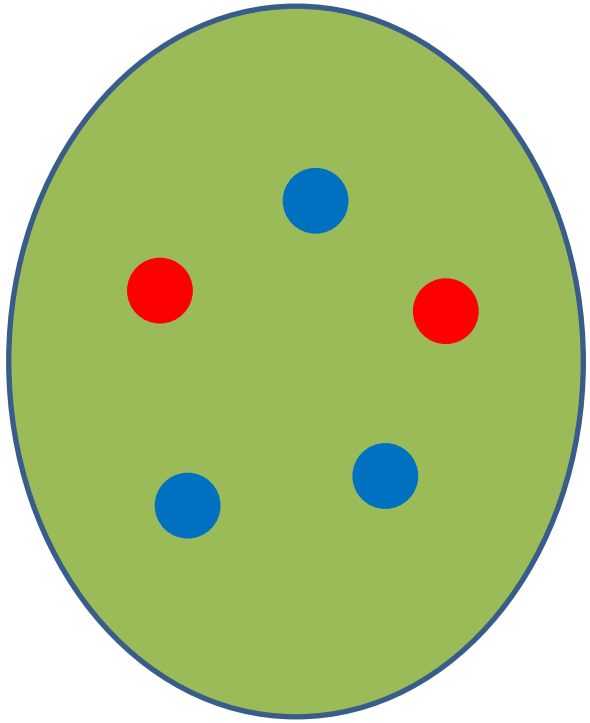
(i) B_1 and B_2 are not independent.

(ii) $Pr(B_1)$ is equal to $Pr(B_2)$.


WHAT?
CAN YOU REPEAT?
memegenerator.net

Data from Draws w/o Replacement

Repeat drawing twice many thousands of times. Record results in a table like below. Each row shows the outcome of a trial.



Trial	1st draw	2nd draw
1	R	B
2	R	R
3	R	R
4	B	R

100 trials

- Let X be the number of times BB occurs. What is its value, roughly? What is its distribution?
- Let Y be the number of times BR occurs. What is its value, roughly? What is its distribution?

- How would you describe X+Y? What is its distribution? Are X and Y independent?

Repeating 100 trials

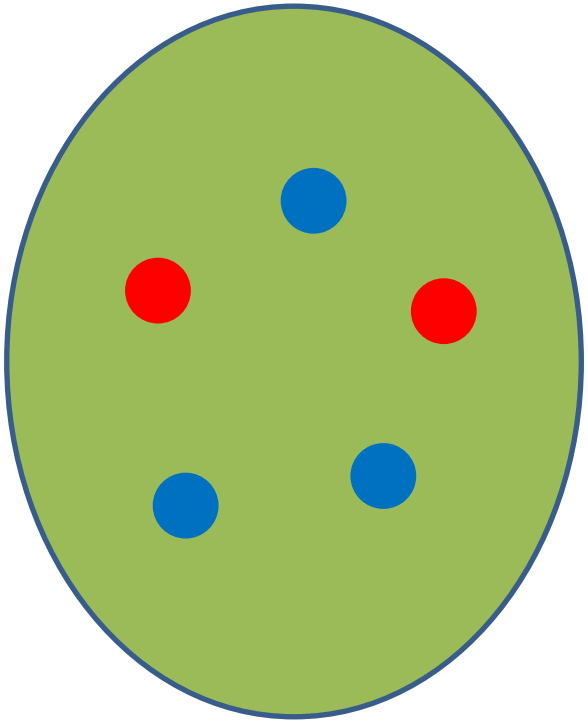
- By repeating the 100 trials many times, we get observed values from X: x_1, x_2, x_3, …

- Roughly what are the mean and SD of the observed values?

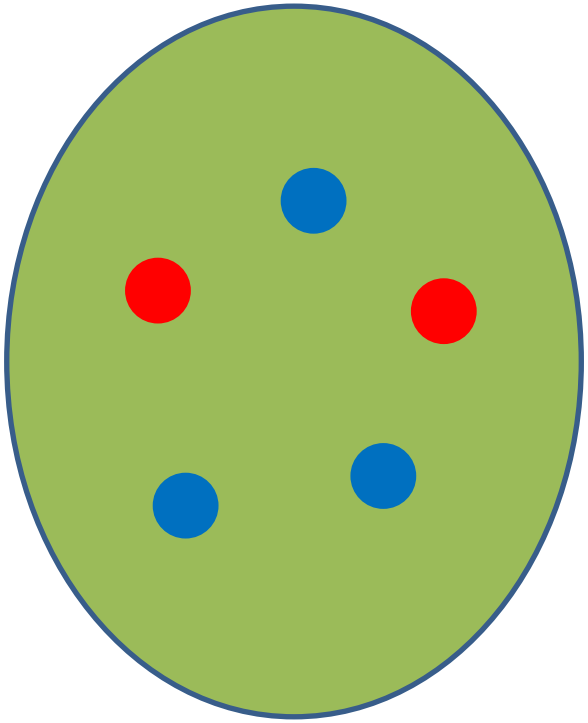
Another look at data

Referring to the table of many rows and 2 columns:

- What is the fraction of B in column 1, roughly?
- What is the fraction of B in column 2, roughly?
- Consider only rows with column 1 showing B. What is the fraction of B in column 2, roughly?



More draws



Make 3 draws without replacement.

- What is $Pr(R_3)$?
- What is $Pr(R_1R_2R_3)$?
- What is $Pr(R_3|R_1R_2)$?

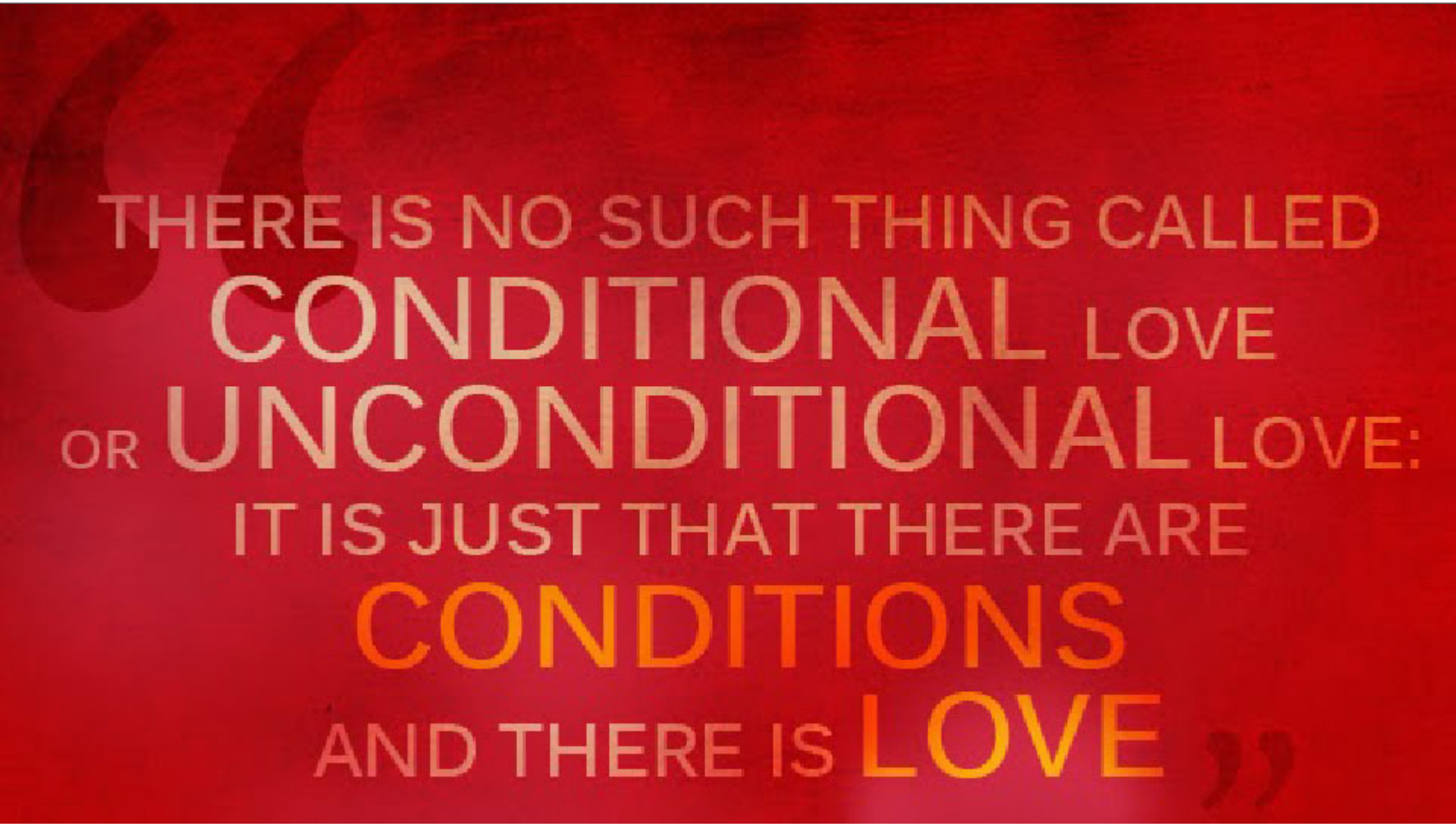
Working Backwards

Two draws are made at random from a box of red and blue balls. The experiment is repeated many thousands of times, resulting in a table with two columns and many rows. Which comparison can help decide whether the draws are made with or without replacement?

(A) fraction of R's in col 1 vs fraction of R's in col 2.

(B) fraction of R's in col 2 among rows with col 1 showing R, vs fraction of R's in col 2 among other rows.

(C) None of the above.



Probability is not like **this** love.

http://www.krazyinlove.com/unconditional-love/there-is-no-such-thing-called-conditional-love/

Events A and B are dependent if…

- …if A influences B. https://revisionmaths.com/advanced-level-maths-revision/statistics/probability
- …the outcome of A depends on the outcome of B. http://www.gcestudybuddy.com/emath-classroom#TOC-Probability
- …occurrence of B affects the probability of A. http://www.cut-the-knot.org/Probability/IndependentEvents.shtml

- A and B are dependent if P(A|B) is different from P(A| not B).

http://www.sampling.com/TeleScoop.html

Sampling

From a city of a million adults, a sample of 1,000 are interviewed to find out if the average monthly income of the city's adults exceeds $2,500. For which sampling method is the z-test appropriate?

(i) Simple random sampling

(ii) Stratified sampling

(iii) Systematic sampling

(iv) Quota sampling

A Population

A number of items, each having an attribute, numerical or categorical, which can be determined accurately.

- Items: residents in a country. Attribute: height, to nearest cm.
- Items: students in a school. Attribute: favourite rainbow colour.

Summarising a Population

- Numerical: mean, SD, quantiles.

- Categorical: frequencies of categories.

- Summaries are known exactly if every item is studied, i.e., a census. Often populations are too large for complete study to be practical.

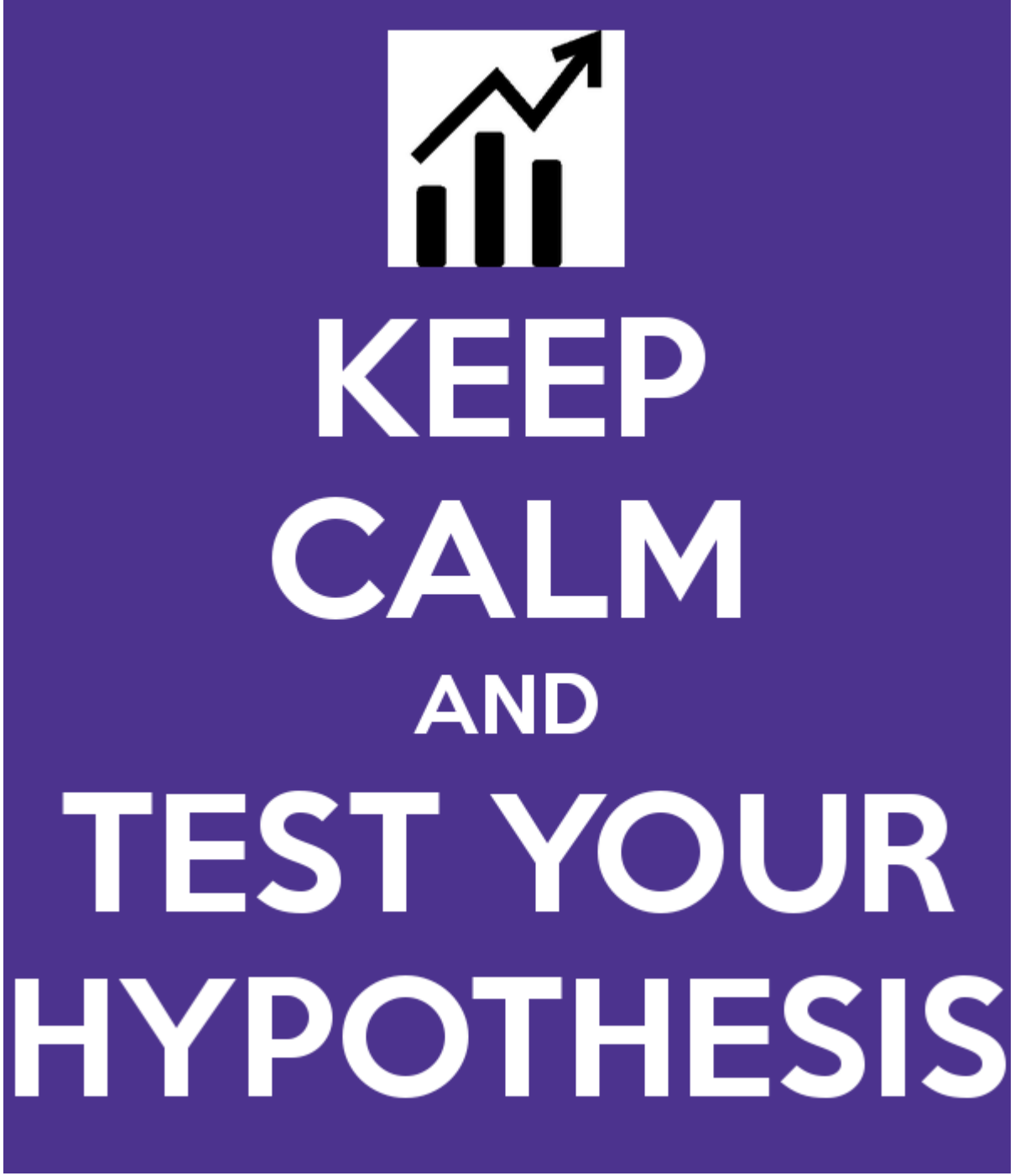
Simple Random Sampling

- Sample mean estimates population mean.
- If sample size is large, but small relative to population size, the sample mean has an approximate normal distribution.
- Estimation and test procedures for population mean are based on simple random sampling.

Other Sampling Methods

- Stratified, systematic or other probability sampling methods can give good estimates of population mean, but the usual procedures may not work.

- Quota, or other convenience sampling methods: usual procedures more questionable.



https://www.keepcalm-o-matic.co.uk/p/keep-calm-and-test-your-hypothesis-19/

Yes, if at least 80% of the households have at least one durian lover!

Should I set up a durian stall in my neighbourhood?

Answering the question

- Take a simple random sample of households. Population: all households in neighbourhood.

- Ask each chosen household if anyone likes durian.

Null hypothesis: Population percent is 80%.

Alternative: Population percent is more than 80%.

Disclaimers

- Issue is about the value of a population number: parameter.
- Random sampling needed to get good sample. Otherwise, test result is hard to interpret.
- Simple random sampling usually needed to justify taught procedure.

Violation of conditions

- Ask people at market who agree to talk. Two problems.

1. Tend to ask certain people.
2. People may refuse to talk.

- Can’t calculate the chance of the observed percentage, or more extreme: the P value.

H2 Paper 2

“random sample” appears in

- 2007#7, 2008#6, 2009#10, 2010#6, 2011#10, 2012#6, 2014#9, 2015#8, 2016#6 (stratified)

- 2013#9: A motoring magazine editor believes that the figures quoted by car manufacturers for distances travelled per litre of fuel are too high. He carries out a survey into this by asking for information from readers. For a certain model of car, 8 readers reply…


I believe in gravity
That kind of language could invite questions. Just say I "accept" gravity.
©2009 B.A.Miller

Some summary

- Repeated experiments – key concept for understanding probability. *Confidence interval.*
- Conditional probability, dependence – *stick to the symbols*.
- Population, sampling, hypothesis testing

– Is there a population parameter?

– Is there (simple) random sampling?

Legal Trial

S is charged with committing a crime, and the court has to decide whether S is innocent or guilty. What about this approach?

- Null hypothesis: S is innocent.
- Alternative hypothesis: S is guilty.

English vs Mathematics

- “I want to study mathematics because I dislike writing essays.”

- Thesis: Basic language skills help mathematics learning.

Example 1

Binomial B(n,p) $\mathrm{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$

- What does the formula **not** tell you?

Example 2

Poisson Po(λ)

$$\mathrm{P}(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

Recommendations

- Define symbols, as far as possible
- Specify relevant sets

Example 3

Unbiased variance estimate from a single sample

$$s^2 = \frac{1}{n-1}\left(\sum x^2 - \frac{(\sum x)^2}{n}\right) = \frac{1}{n-1}\sum (x - \bar{x})^2$$

Use words!

- Where does the variance come from?
- Where do the x’s come from?
- How do the x’s come about?
- What does “a single sample” mean?

Example 4

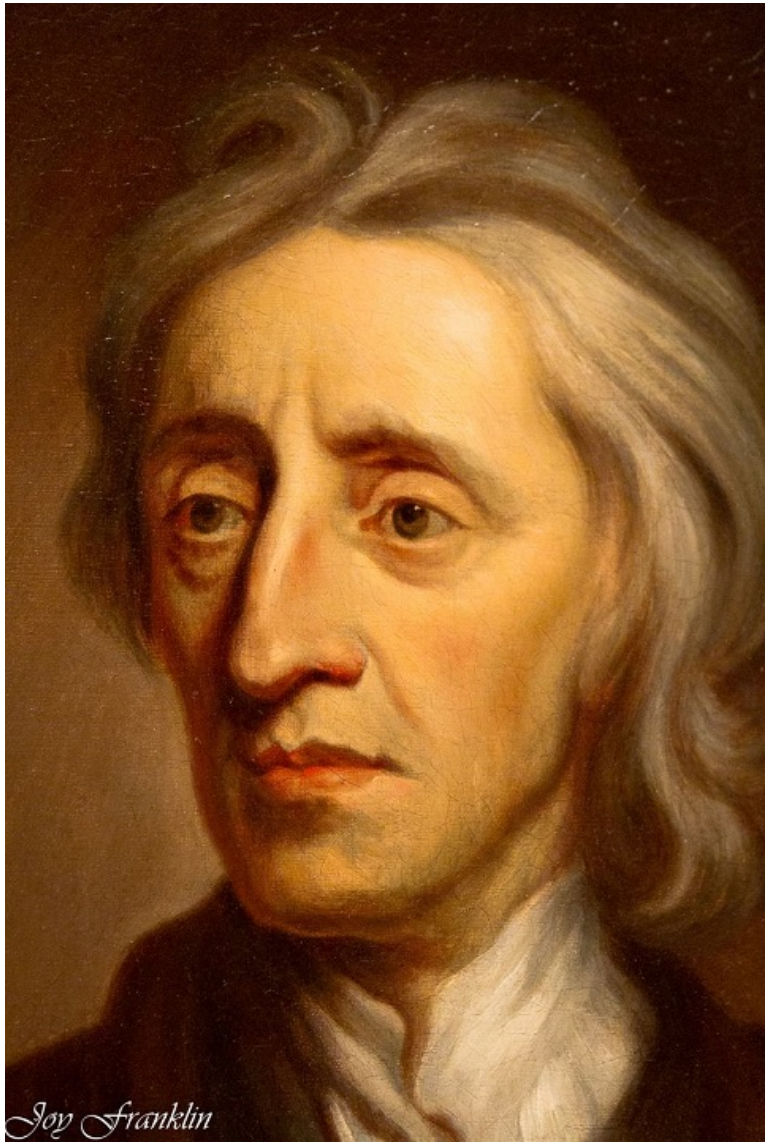
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

John Locke

Since we cannot know everything, we would be well-advised to observe and respect the extent and limitations of human knowledge.

~A Guide to Locke's Essay

http://www.philosophypages.com/locke/g05.htm



Affirmation

- Why does this equation have no solution in R?

$$x^2 + 1 = 0$$

- Some students will answer correctly, some may not. Good to make reason explicit:

For every $x \in R, x^2 \geq 0$.

Exception

- True or false?

$$\tan\frac{\pi}{2} = \frac{\tan\frac{\pi}{3} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{3}\tan\frac{\pi}{6}}$$

- Reason from

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

for any $A + B \neq \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \ldots$

Extension

- Knowing

$$\text{For every } x \in R, x^2 \geq 0.$$

might make complex numbers better received.

- To

$$\text{For every } x \in \mathrm{C}, x^2 \geq 0.$$

respond with something like

$$\text{There is an } x \in \mathrm{C} \text{ such that } x^2 \leq 0.$$

“The more we use poor language, the poorer our thoughts become.”