Deepening Students' Conceptual Understanding through Meaningful Mathematical Tasks

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Overview

- Eight Effective Mathematics Teaching Practices
- Mathematical Tasks – Definition, Research, Characteristics of Good Tasks
- Design and Implementation of Tasks – What to Consider
- Four Strands of Mathematical Proficiency
- Concept Image and Formation, Interplay between Concept Image and Definition
- Drawing the graph of $y = f'(x)$: Student Difficulties, Proposed Task
- Convergence of Series : Student Difficulties, Proposed Task
- Use of Parametric Equations : Student Difficulties, Proposed Task
- Assessment of Learning Outcomes of Task
- Resources

Eight Effective Mathematics Teaching Practices (Taken from *Principles to Actions*)

- Establish **Mathematics Goals** to focus learning
- Implement **Tasks** that promote reasoning and problem solving
- Use and connect mathematical **Representations**
- Facilitate meaningful mathematical **Discourse**
- Pose **Purposeful Questions**
- Build **Procedural Fluency** from **Conceptual Understanding**
- Support **Productive Struggle** in learning Mathematics
- Elicit and use evidence of **Student Thinking**

MATHEMATICAL TASKS

Definition of Task

- Refer to information that serves as the prompt for student work, presented to them as questions, situations and instructions that are both the starting point and context for their learning **(Watson and Sullivan, 2008)**.

- A set of problems or a single complex problem that focuses students' attention on a particular mathematical idea **(Stein, Grover, and Henningsen, 1996)**.

Why Mathematical Tasks?
(Taken from *Principles to Actions*)

- "The work students do, defined in large measures by the tasks teachers assign, determines how they think about a curricular domain and come to understand its meaning." **(Doyle, 1988)**

- Different kinds of tasks lead to different types of instruction, which subsequently lead to different learning opportunities for students **(Doyle, 1988; Stein, Smith, Henningsen and Silver, 2000)**.

Research on Use of Mathematical Tasks over last 20 years
(Taken from *Principles to Actions*)

- Not all tasks provide the same opportunities for student thinking and learning **(Hiebert et al. 1997; Stein et al. 2009)**.

- Student learning is greatest in classrooms where the tasks consistently encourage high-level student thinking and reasoning and least in classrooms where the tasks are routinely procedural in nature **(Boaler and Staples 2008; Hiebert and Wearne 1993; Stein and Lane 1996)**.

- Tasks with high cognitive demands are most difficult to implement well and are often transformed into less demanding tasks during instruction **(Stein, Grover, and Henningsen 1996; Stigler and Hiebert 2004)**.

Characteristics of **Good** Mathematical Tasks

- Provide **appropriate contexts and complexity**, that stimulate construction of cognitive networks, **thinking**, **creativity** and **reflection**, and that address significant mathematically topics explicitly **(Anthony and Walshaw, 2009)**

- Have a **mathematical focus**, are challenging for the learner and provoke **insights into the structure of mathematics** **(Anthony and Walshaw, 2009)**

- Encourage **use of multiple solution strategies** and **multiple representation** **(Stein and Lane, 1996)**

- Should include opportunities to classify mathematical objects, **interpret multiple representations**, **evaluate** mathematical statements, **create problems**, **analyse reason** and solutions **(Swan, 2005)**

Mathematical Tasks according to Levels of Cognitive Demands (Smith and Stein, 1998)

- **Higher-level Cognitive Demands (Procedural Tasks with Connections, Problem Solving / Doing Mathematics)**
- Tasks that allow students to engage in active inquiry and exploration or encourage students to use procedures in ways that are meaningfully connected with concepts or understanding

- **Lower-level Cognitive Demands (Memorisation Tasks and Procedural Tasks without Connections)**
- Tasks that encourage students to use procedures, formulas, or algorithms in ways that are not actively linked to meaning or that consist of memorisation or reproduction of previously memorised facts

Example of a Task of Lower-level Cognitive Demand

Using your graphic calculator, sketch the graphs of

(a) $y = 2^x$,

(b) $y = \left(\frac{1}{3}\right)^x$,

(c) $y = 1^x$.

Example of the Same Task of Higher-level Cognitive Demand

(a) Using your graphic calculator,

explore the shape of the graph $y = a^x$, where a is a real constant, for different values of a.

(b) Explain the differences between the shape of the graph $y = a^x$ for the cases of $0 < a < 1$ and $a > 1$.

(c) Explain why exponential functions $y = a^x$ are defined for $a > 0$.

What to Consider in the Design of Tasks
(**Peter Sullivan, Libby Knott, Yudong Yang, 2015**)

- **Context of Task** – ranging from Pure Mathematics to semi-reality to reality

  Tasks set in realistic contexts maximise students' engagement, but the context might sometimes detract from the potential of the task to achieve the intended learning.

- **Language of Task** – mathematical precision is part of the desired learning, but clarity is also needed to support the learning.

Examples on Context of Task

Contrast, in terms of context, the 2 tasks on Maximisation and Minimisation.

Task 1:

It is given that $\boldsymbol{\pi r^2 h = 320}$ and that $\boldsymbol{S = 2\pi rh + 2\pi r^2}$ where r and h are measured in cm.

Find the value of h and of r such that S is a minimum.

Examples on Context of Task

Task 2:

Coca Cola company had recently changed the size of the container of its drink to the “thinner and taller” version.

(a) What could be a possible reason for the change?

(b) Using calculus, determine the dimensions of the new container so that the volume of the drink in the container remains the same as before and the total surface area of the new container is a minimum.

(c) Can you suggest a possible reason why the Coca Cola company did not use the dimensions that you found in (b)?

Example on Language of Task

Comment on the language used in this task (taken from Advanced Mathematics Thinking, 1991**) and the possible responses from the students.**

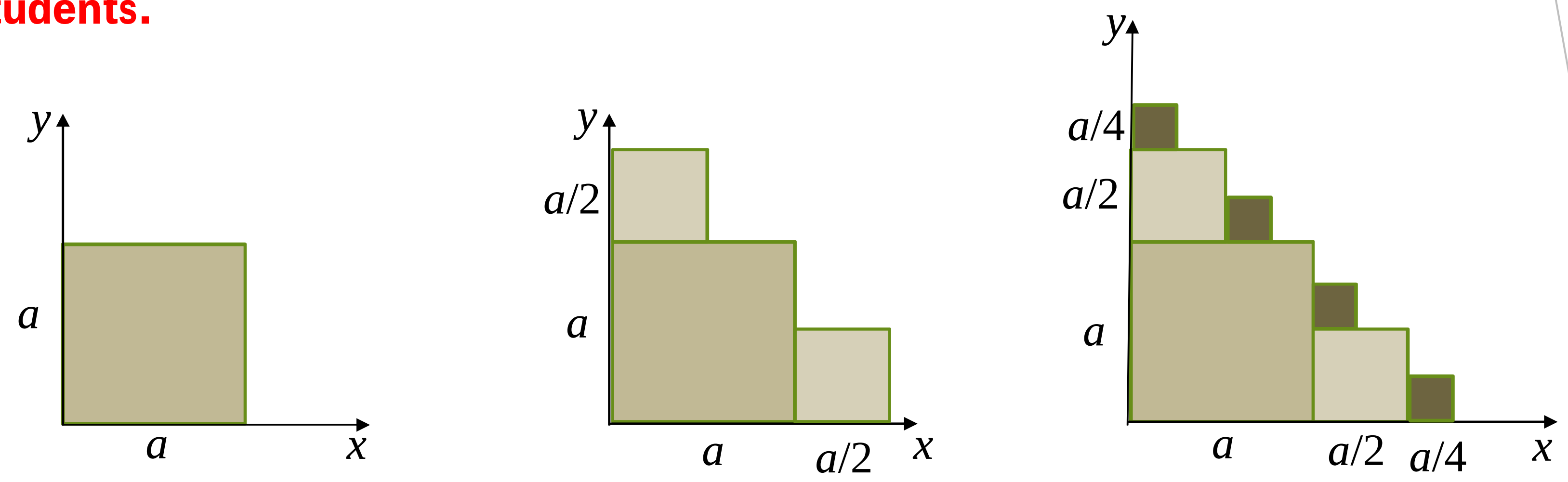


Figure : A Limiting Staircase

(a) If this procedure is repeated indefinitely, what is the “final” result?

(b) What is the area, in terms of a, below the “final” staircase?

What to Consider in the Design of Tasks (Peter Sullivan, Libby Knott, Yudong Yang, 2015)

- **Structure of Task** – degree of openness on task

  Specific questions posed can scaffold student engagement with a task in a more prescribed way while on the other hand, open questions allow students greater opportunity to make strategic decisions on pathways and destinations for themselves.

- **Distribution** – selecting the content to be focused on in the task, which is a function of the cognitive demand of the task

Examples on Structure of Task

Contrast, in terms of structure, the 2 tasks on investigating the shape of the graph of $y = a^x$.

Task 1:

Using your graphic calculator, explore the shape of the graph $y = a^x$, where a is a real constant, for the cases of $a > 1$, $a = 1$, $0 < a < 1$, $a = 0$ and $a < 0$.

Task 2:

Explore the shape of the graph $y = a^x$, where a is a real constant, for different values of a.

What to Consider in the Design of Tasks (Taken from Principles to Actions)

- Need to consider the prior knowledge and experiences of the students who will be engaged in the task.

  Tasks may begin as high-level tasks for students who are initially learning about the underlying mathematics. Eventually, these tasks may become more routine experiences for them.

Most Critical Element of Task Design **(Ernest, 2010)**

- Most critical element - the potential for the task to prompt the learning of the intended mathematical concepts
- Two perspectives of Mathematics to consider – **Practical Perspective** and **Specialised Perspective**

❑ **Practical Perspective** – student learning the mathematics adequate for general employment and functioning in society; the types of calculation one does as part of everyday life such as choosing routes to travel, budgeting, interpreting data in the newspaper and so on.

❑ **Specialised Perspective** – mathematics understanding which forms the basis of university studies in science, technology and engineering; includes an ability to pose and solve problems, nature of reasoning, and intuitive appreciative of mathematical ideas such as pattern, symmetry, structure, etc.

Designing Tasks that Address Specialised Mathematical Goals

- Need to consider the **conceptual ideas** and the **mathematical processes** (make connections, develop mathematical modelling and problem-solving skills, generalise, etc) in which the students might be expected to engage.
- There is an inherent tension between articulating a mathematical goal by teacher to students and having students discover or investigate a mathematical concept or idea in a lesson.
- The articulation of the goal by teacher needs to be rather general, so as to not reveal the concept to be discovered or investigated.

Designing Tasks that Address a Practical Perspective (Goos, Geiger and Dole, 2010)

- Focus on **real-life contexts**, **application of mathematical knowledge**, use of **representational, physical and digital tools** and emphasizes **cultivation of positive dispositions** towards Mathematics

- **Application of Mathematical Knowledge** – fluency with accessing concepts and skills, problem-solving strategies and the ability to make sensible estimation

- **Use of Tools**
  - Representational (symbol systems, graphs, maps, diagrams, drawings, tables)
  - Physical (models, measuring instruments)
  - Digital (computers, software, calculators, internet)

- **Cultivation of Positive Dispositions** – a willingness and confidence to engage with tasks and apply their mathematical knowledge flexibly and adaptably

Task Design Process

- Three-stage, backward-design process (**Ron, Zaslavsky, and Zodik, 2013**)
- State goal(s) and connect the task to the goal(s).
- Design a generic task that addresses the goal(s).
- Carefully choose the specific examples to “plug in” the generic task.

Implementation of Task

- Selecting tasks that promote reasoning and problem solving does not guarantee that students will engage in the task at a high cognitive level.

- Teachers should avoid providing students with a pathway for solving the task when students struggle.

- Teachers must decide **what aspects of a task to highlight**, how to **organise and orchestra the work of the students**, what **questions to ask** to challenge those with varied levels of expertise, and how to **support students without taking over the process of thinking** for them and thus eliminating the challenge (**NCTM, 2000**).
- Depending on profile of students and cognitive demand of tasks, students could work individually, in pair or in groups.

CONCEPTUAL UNDERSTANDING

- Comprehension of mathematical concepts, operations and relations

Four Strands of Mathematical Proficiency (Kilpatrick, Swafford, Findell, 2001)

- **Conceptual Understanding** – comprehension of mathematical concepts, operations and relations
- **Procedural Fluency** – carrying out the procedures flexibly, accurately, efficiently and appropriately, and in addition to these procedures, having factual knowledge and concepts that come to mind readily
- **Strategic Competence** – the ability to formulate, represent and solve mathematical problems
- **Adaptive Reasoning** – the capacity for logical thought, reflection, explanation and justification

Concept Image (Tall and Vinner, 1981)

- According to Tall and Vinner (1981), concepts are more like nodes with connections to **mental images**, **past experiences**, **techniques** and **typical problems.**

- Notion of a 'concept image' captures what it means to have a sense of a concept.

- Three interwoven dimensions of a concept image corresponding to **emotion**, **behaviour** and **awareness**:

- **Emotion** – Source problems and contexts in which the topics reappear
- **Behaviour** – familiar and new language as well as techniques associated with the concepts
- **Awareness** – images, connections and sense of the concepts, together with confusions, errors or misconceptions students have or make

Concept Images Framework (continued)

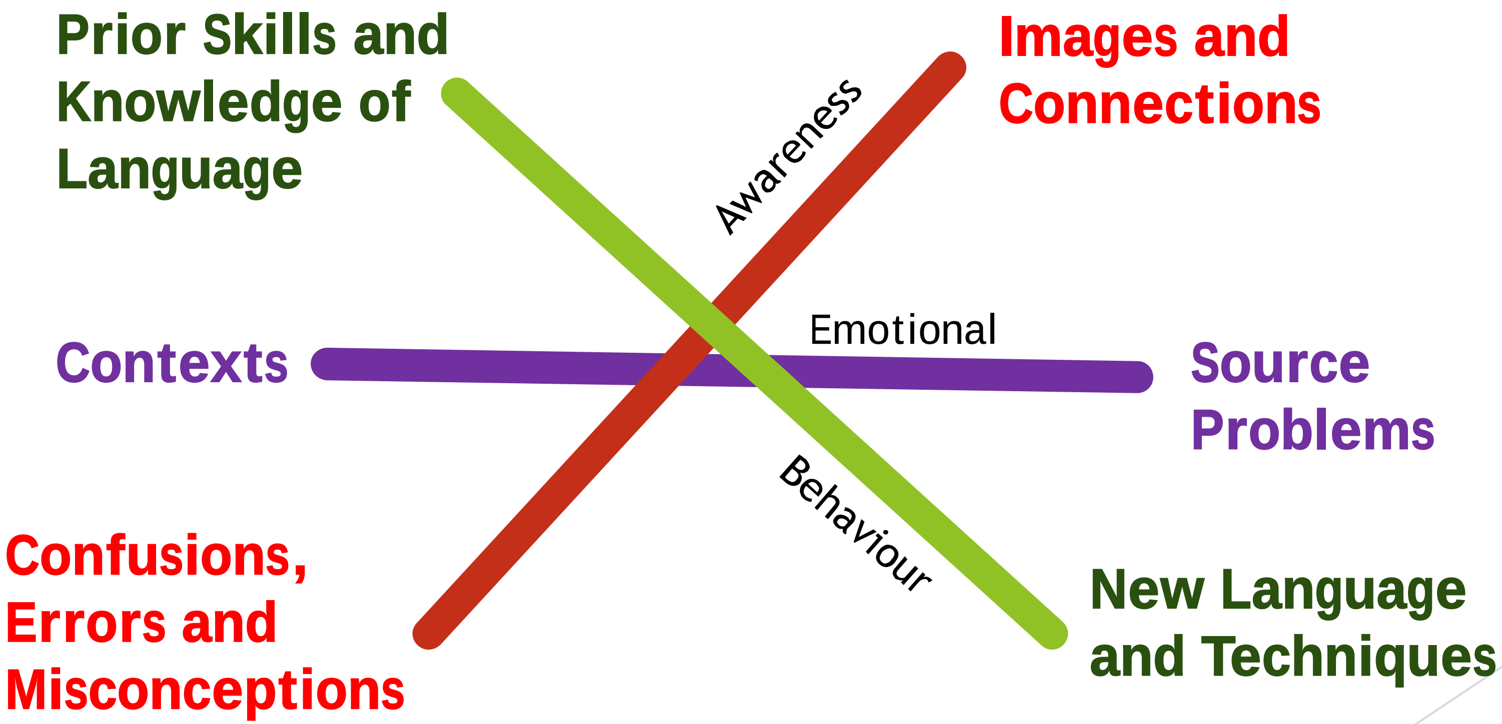


Diagram taken from page 191, Mathematics Teaching Practice, 2002

Implications for Teaching

- The **Emotional Strand** of the Concept-Image Framework suggests that it is useful for the teacher to remind himself/herself of (and so be in a position to indicate to the students) the original problems whose solutions gave rise to this topic. It is also useful to consider the range of contexts and problems in which the same idea has already turned up, as this provides motivation for the students.

- The **Awareness Strand** suggests that it is important for the teacher to refresh his/her memory of the sorts of things that students struggle over or mis-construe, and see if he/she can find what it is about what he/she usually says or does that causes the problems.

- The **Behaviour Strand** suggests that the teacher should remind himself/herself of the techniques that the students will have to carry out, and to be explicit about the inner thoughts, that he/she has when he/she uses them, to the students.

Concept Formation

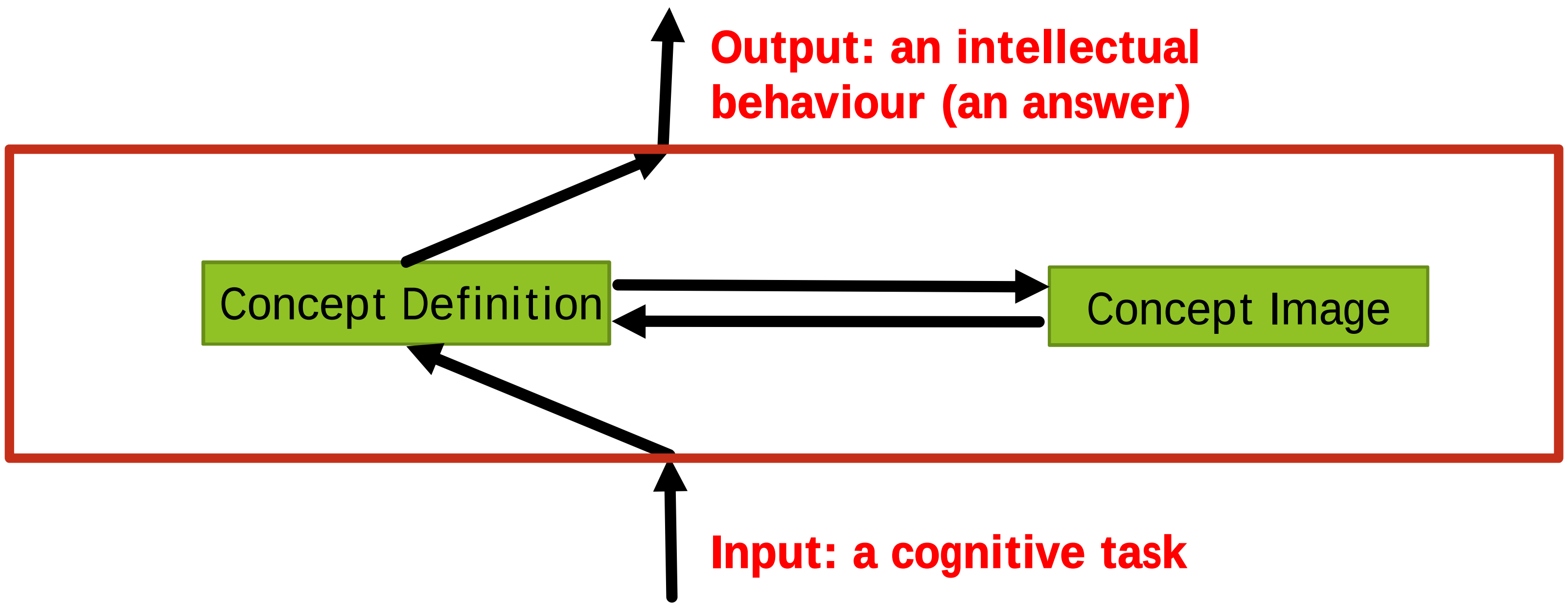
- Most concepts in Mathematics are introduced by definitions.
- Definition of concept helps to form a concept image.
- Once the concept image is formed, the concept definition may become dispensable.
- The definition of the concept may remain inactive or even be forgotten when statements about the concept are considered.

Interplay between Concept Definition and Concept Image (Shlomo Vinner, 1991)

- When a concept is first introduced by means of a definition, the concept image cell is initially empty (as some meaning is not associated with the concept name).
- After several examples and explanations, the concept image cell is gradually filled.
- But the concept image does not necessarily reflect all the aspects of the concept definition.

Diagram taken from page 70, Advanced Mathematics Thinking, 1991

Scenario 1 When a Cognitive Task is Posed (Shlomo Vinner, 1991)



Interplay between concept definition and concept image

Diagram taken from page 71, Advanced Mathematics Thinking, 1991

Scenario 2 When a Cognitive Task is Posed (Shlomo Vinner, 1991)

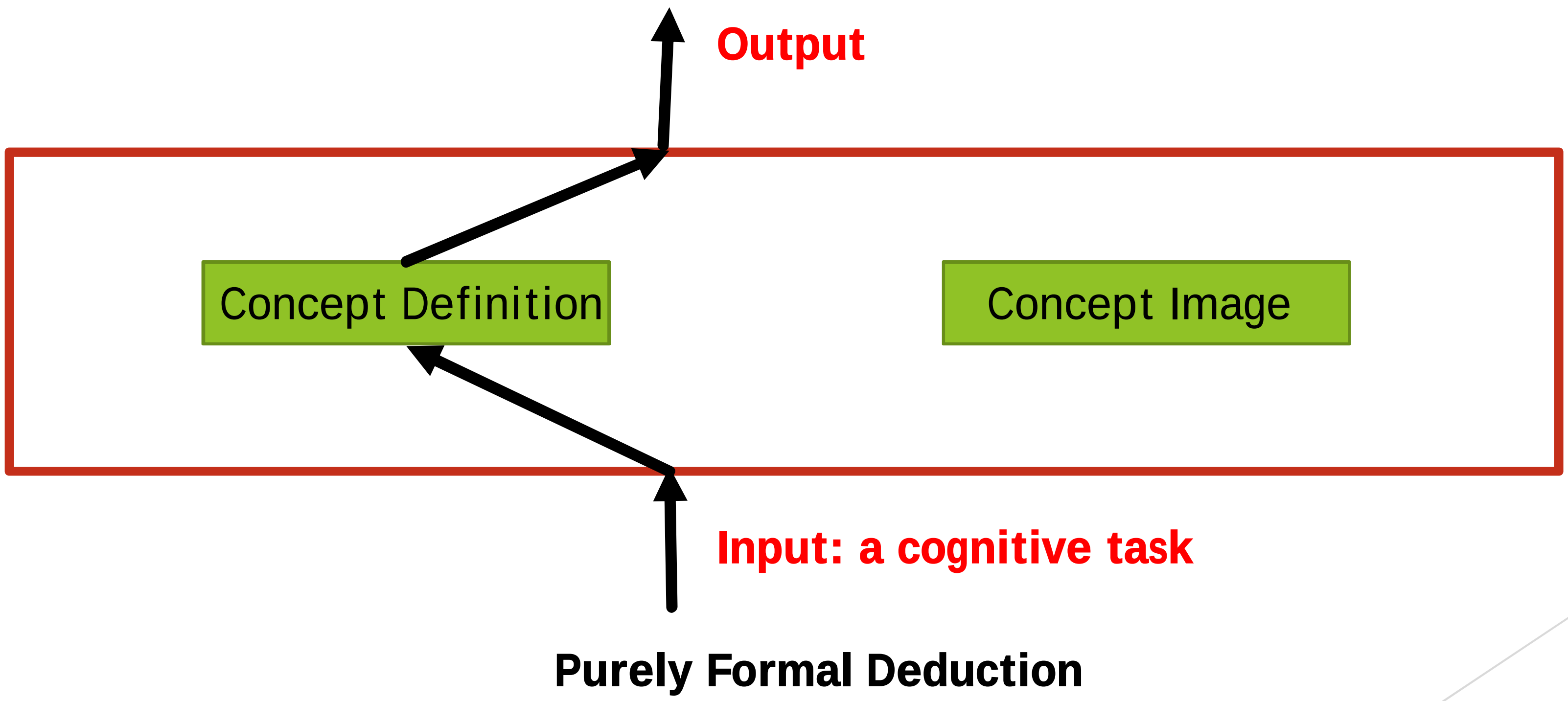
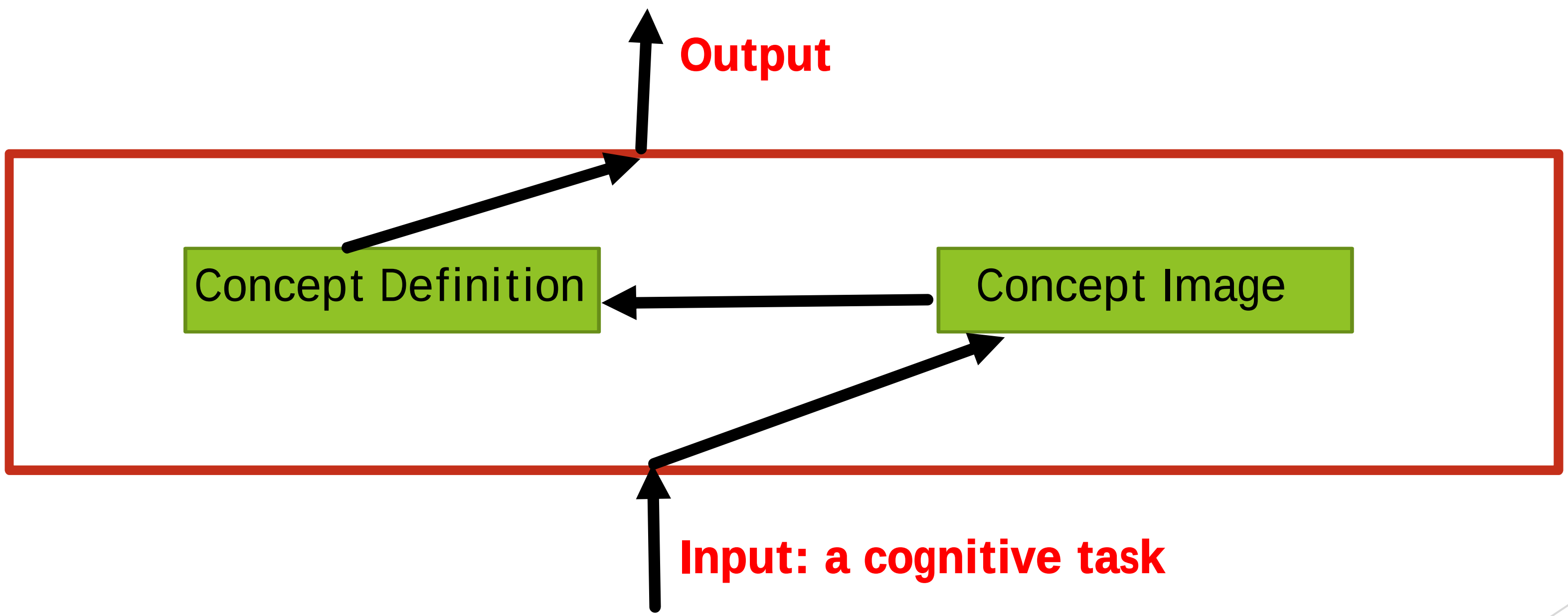


Diagram taken from page 72, Advanced Mathematics Thinking, 1991

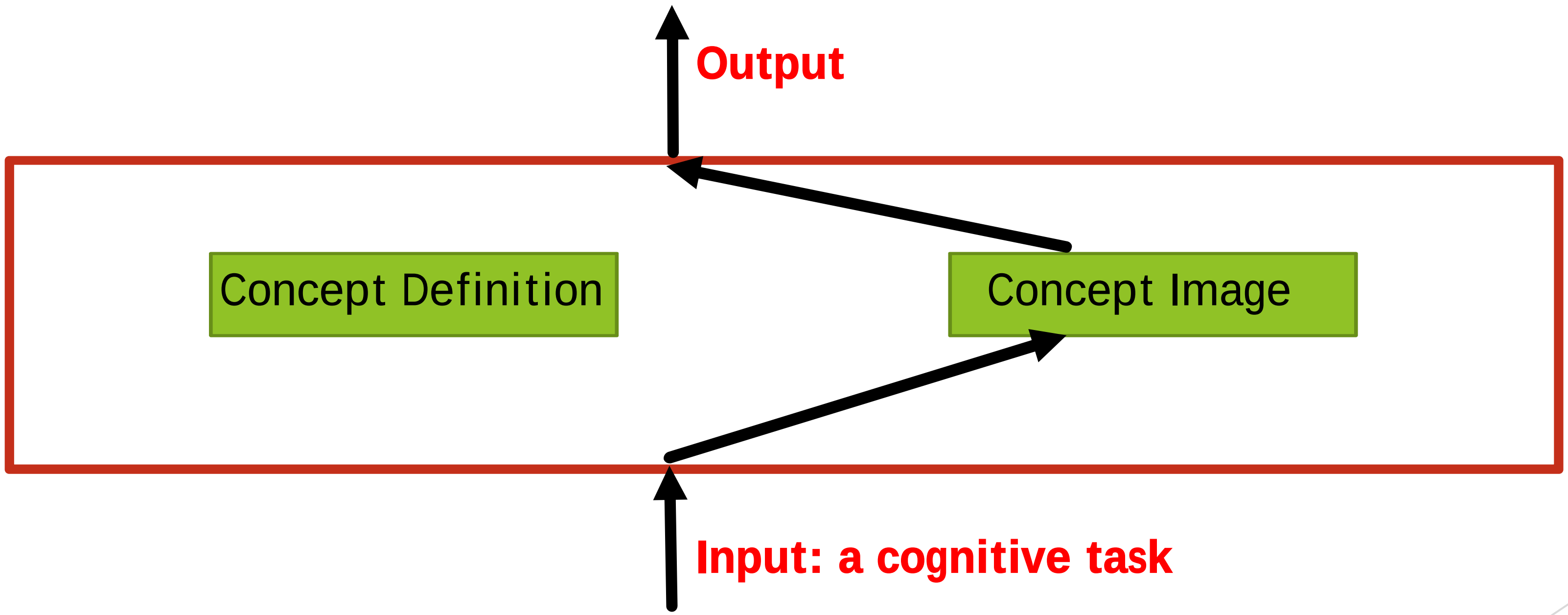
Scenario 3 When a Cognitive Task is Posed (Shlomo Vinner, 1991)



Deduction Following Intuitive Thought

Diagram taken from page 72, Advanced Mathematics Thinking, 1991

Scenario 4 When a Cognitive Task is Posed (Shlomo Vinner, 1991)



Intuitive Response

Diagram taken from page 73, Advanced Mathematics Thinking, 1991

Implications

- Scenario 4 seems to be the common model for the processes to occur in practice.
- Usually, the concept definition cell, even if non-void, is not consulted during the problem solving process.
- Everyday life habits take over and the student is unaware of the need to consult the formal concept definition.
- In most of the cases, the reference to the concept image cell will be quite successful.
- Only non-routine problems, in which incomplete concept images might be misleading, can encourage students to refer to the concept definition.

Cognitive Obstacles

- The notion of "**cognitive obstacle**" helps to identify difficulties encountered by students in the learning process, and to determine more appropriate strategies for teaching.
- 3 main types of cognitive obstacles

- **Genetic and Psychological Obstacles** – due to personal development of the student
- **Didactical Obstacles** – nature of the teaching (e.g. oversimplifying concepts, use of imprecise language, did not take in extreme cases)
- **Epistemological Obstacles** – nature of the mathematical concepts themselves (e.g. limits, continuity, infinity)

Mathematical Tasks for
Some Concepts in H2 Mathematics

Drawing the Graph of $y = f'(x)$

Student Conceptions of $f'(x)$
(Multiple Representations of $f'(x)$)

- The function obtained by applying the usual rules of differentiation on $y = f(x)$
- The gradient of the tangent to the curve $y = f(x)$
- The limit of the ratio $\frac{f(x+h)-f(x)}{h}$ as h tends towards 0

Why Students are Able to Sketch the Graph of $y = f'(x)$ from Graph of $y = f(x)$ when f is a Quadratic Polynomial (equation of f not given)

- They are able to deduce the equation of f (from secondary school knowledge) from the graph if f is a quadratic polynomial.

- In fact, from differentiation, they could deduce that f′ is a linear polynomial if f is a quadratic polynomial. So they just need to decide whether the gradient of the straight line is positive or negative, and determine the x-intercept.

- They can also memorise that if the shape of graph of f is "U" shape, then positive gradient for the graph of f′ with x-intercept of f′ at the x-coordinate of the minimum point of f (dually for "∩" shape).

Why Students are Able to Sketch the Graph of $y = f'(x)$ from Graph of $y = f(x)$ when f is a Cubic Polynomial (equation of f not given)

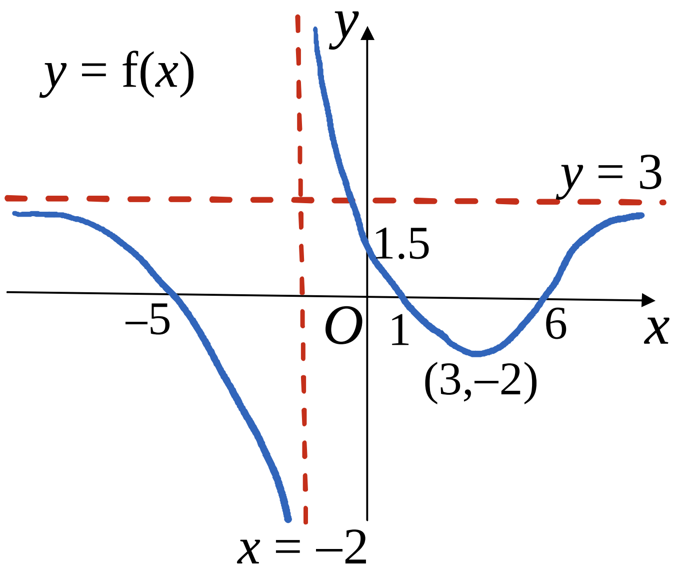
- In fact, from differentiation, they could tell f′ is a quadratic polynomial if f is a cubic polynomial.

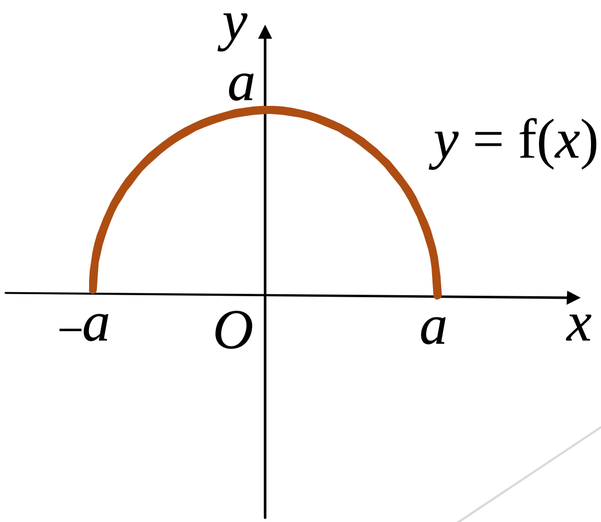
- They need to know that the x-coordinates of the stationary points on the graph of f give the x-intercepts of the graph of f′.

- Once they get the x-intercepts of the graph of f′, they just need to decide whether "U" shape or "$\cap$" shape. To do this, they need to know the concept that as x increases, $f(x)$ increases, $f'(x) \geq 0$ (and dually).

Activity : Group Discussion (10 min)

In your group, discuss the student difficulties in sketching the graph of $y = f'(x)$ from the graph of $y = f(x)$ when the equation for f is not given and f is not a quadratic or cubic polynomial. You might wish to make use of the following 2 examples in your discussion.





Student Difficulties

- In secondary schools, when students draw graphs, they either draw them from memory (e.g. quadratic polynomials) or they “plot” them using a table of values.

- So drawing the graph of $y = f'(x)$ from the graph of $y = f(x)$ when the equation of f is not given requires the students to know how to trace the gradient of the tangent to the graph of f as x varies. They also need to take note of the rate of change of gradient as x varies. Difficulties arise for those parts of the graph of f that change curvature.

Student Difficulties (Continued)

- Teacher focusing on first getting the asymptotes, the x-intercepts and the stationary points of the graph of $y = f'(x)$ might detract students' attention on the main concepts to learn.

- Tabulating the characteristics of the graph of $y = f(x)$ and the corresponding characteristics of the graph of $y = f'(x)$ might lead to students memorising; they may be able to identify the main features of the graph of $y = f'(x)$, but unable to draw the full graph of $y = f'(x)$.

Main Concepts Taught Prior the Task

- When the tangent to the graph of $y = f(x)$ at a given point makes an **acute** angle with the positive x-axis, then the gradient of the tangent (which is $f'(x)$) is **positive**. When the tangent to the graph of $y = f(x)$ at a given point makes an **obtuse** angle with the positive x-axis, then the gradient of the tangent (which is $f'(x)$) is **negative.**

- When the tangent to the graph of $y = f(x)$ at a given point is **horizontal**, then the gradient of the tangent (which is $f'(x)$) is **zero**. When the tangent to the graph of $y = f(x)$ at a given point is **vertical**, then the gradient of the tangent (which is $f'(x)$) is **undefined.**

Main Concepts Taught Prior the Task

- If as x increases, the gradient of the tangent to the graph of y = f(x) changes from 2 to 9, then to 30, we say **f ′(x) increases** as x increases.

- If as x increases, the gradient of the tangent to the graph of y = f(x) changes from 1000 to 50, then to 5, we say **f ′(x) decreases** as x increases.

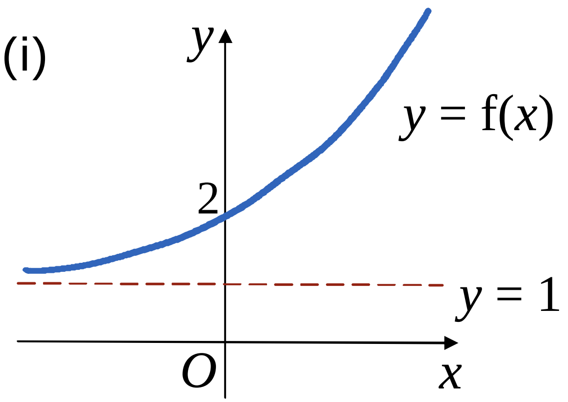
- If as x increases, the gradient of the tangent to the graph of y = f(x) changes from 0 to –2, then to –8, and then to –30, we say **f ′(x) decreases** as x increases.

- If as x increases, the gradient of the tangent to the graph of y = f(x) changes from –130 to –52, then to –18, and then to 0, we say **f ′(x) increases** as x increases.

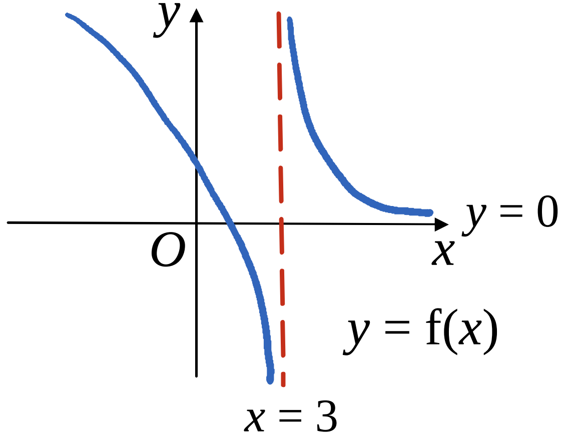
One Proposed Task

(a) For the graph $y = f(x)$ in (i) and (ii), what can you say about $f'(x)$ as x increases? (Trace the gradient of the tangent to the graph as x varies. You might wish to use a table of values of x and $f'(x)$.)

(i)



(ii)



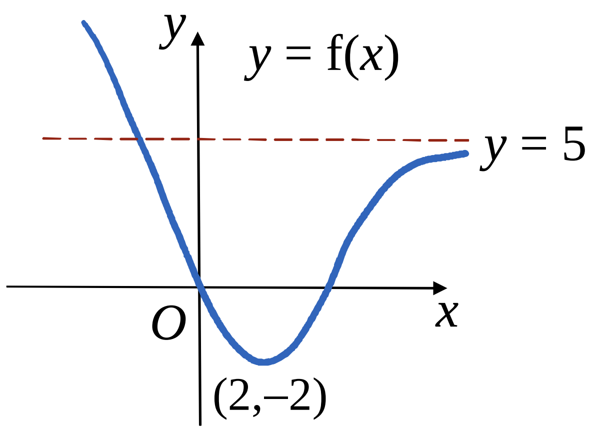
(b) Sketch the graph of $y = f'(x)$ for (i) and (ii).

What happen to the asymptote(s) on the graph of $y = f(x)$ when you draw the corresponding graph of $y = f'(x)$? Explain your answer.

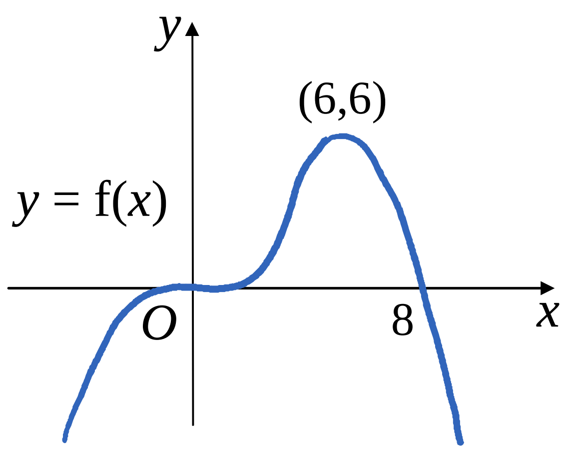
One Proposed Task (continued)

(c) Sketch the graph of $y = f'(x)$ for (iii) and (iv).
(Trace the gradient of the tangent to the graph as x varies. You might wish to use a table of values of x and $f'(x)$.)

(iii)



(iv)



(d) What happen to the stationary points on the graph of $y = f(x)$ when you draw the corresponding graph of $y = f'(x)$? Explain your answer.

A Note on the Graph of $y = 1/f(x)$

- Main concepts are :
  - $f(x)$ **positive**, $1/f(x)$ **positive**
  - $f(x)$ **negative**, $1/f(x)$ **negative**
  - $f(x)$ **increases** as x increases, $1/f(x)$ **decreases**
  - $f(x)$ **decreases** as x increases, $1/f(x)$ **increases**
  - $f(x) = 0$, $1/f(x)$ is undefined
  - $f(x) \to 0^+$, $1/f(x) \to \infty$; $f(x) \to 0^-$, $1/f(x) \to -\infty$
  - $f(x) \to \infty$, $1/f(x) \to 0^+$; $f(x) \to -\infty$, $1/f(x) \to 0^-$
  - $f(x) \to \left(\frac{1}{3}\right)^-$, $1/f(x) \to 3^+$; $f(x) \to \left(\frac{1}{3}\right)^+$, $1/f(x) \to 3^-$
- Encourage students to use a table of values of x and $1/f(x)$ if they encounter difficulties.
- Avoid tabulating the characteristics of the graph of $y = f(x)$ and the corresponding characteristics of the graph of $y = 1/f(x)$.

Convergence of Series

Student Difficulties in the Concept of Convergence of Series

- Difficulty in understanding the limit of a sequence
- Do not understand the condition for the sum to infinity of a geometric series to exist
- Do not see the link between the sum to infinity of a geometric series and the convergence of any series
- Do not see the link between the sum to infinity of a convergent series and the limit of the corresponding sequence of partial sums
- Memorise the procedure of obtaining the sum to infinity of a convergent series from the expression for the sum of first n terms of the series

Student Interpretation of “Limit”
(From the study by Bernard Cornu, 1983)

- An impassable limit which is reachable
- An impassable limit which is impossible to reach
- A point which one approaches, without reaching it
- A point which one approaches and reaches

Student Interpretation of "Tends Towards" (From the study by Bernard Cornu, 1983)

- Students already had a certain idea of the term "tends towards" based on their daily experiences and continued to rely on these meaning that they had.

- Contrary to what may be imagined by most teachers, these ideas did not disappear, but mixed with newly acquired knowledge, modified and adapted to form the students' personal conceptions.

- The different meanings for "tends towards" held by students
- To approach (eventually staying away from it)
- To approach … without reaching it
- To approach .. just reaching it

Findings (Aline Robert, 1982)

- Initial teaching tends to emphasise the process of approaching a limit, rather than the concept of the limit itself.

- The concept images, associated with this process, contain many factors which conflict with the formal definition (e.g. “approaches but cannot reach”, “cannot pass”, “tends to”, etc).

- Students develop images of limits and infinity which relate to misconceptions concerning the process of “getting close” or “growing large” or “going on forever”.

Definition of Limit of a Sequence (James Stewart, 2006)

A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, we say the sequence $\{a_n\}$ **converges** or is **convergent**.

Definition of Convergence of Series (James Stewart, 2006)

Given a series $a_1 + a_2 + a_3 + \ldots + a_r + \ldots$,

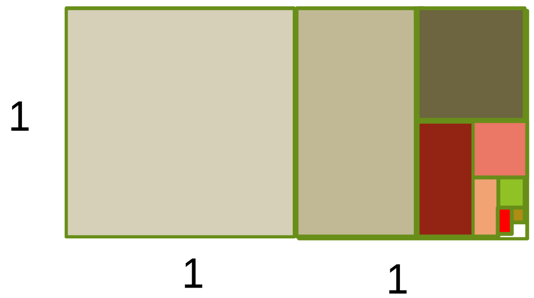
let $S_n = a_1 + a_2 + a_3 + \ldots + a_n$.

If the sequence $\{S_n\}$ is **convergent** and $\lim_{n\to\infty} S_n = S$ exists as a real number,

then the series $a_1 + a_2 + a_3 + \ldots + a_r + \ldots$ is **convergent** and S is called the **sum (to infinity)** of the series.

One Proposed Task

(a) What series does the total area of the shaded region (if the process goes on) represent? State the limiting value of this area.



(b) Determine the type of series (arithmetic, geometric, or others) that you have stated in (a).

One Proposed Task (Continued)

(c) Consider the following sequence:

$$\boldsymbol{S_1, S_2, S_3, S_4, \dots, S_n, \dots}$$

where S_n is the sum of the first n terms of the series that you have stated in (a).

(i) Write down an expression for S_n.

(ii) State the relation between the area of the shaded region (if the process goes on) and S_n.

(iii) Verify that the limiting value that you have obtained in (a) tallies with the limit of the sequence in (c).

One Proposed Task (Continued)

Let S_r be the sum of the first r terms of a series.

We call the **limit** (if it exists) of the sequence

$$S_1, S_2, S_3, S_4, \ldots, S_n, \ldots$$

as the **sum to infinity** of the series.

A series is **convergent** if the sum to infinity of the series exists (that is, a distinct real value).

(d) Does the sum to infinity of an arithmetic series exist?

Explain your answer.

Use of Parametric Equations

Student Difficulties

- Do not understand the applications of parametric equations in real life
- Do not understand the need for a third independent parameter
- Do not understand what it means by "a point P on the curve, defined by parametric equations, with parameter 2"
- Think it is always possible to convert parametric equations to its corresponding cartesian equation

One Proposed Task

Two persons A and B are running and competing against each other on a circular track, with equation given by $x^2 + y^2 = 16$, relative to an origin O. Both persons start at the same point $(4,0)$ on the track. The organiser of the race attaches a tracking device on each of the two persons and wants to view their positions on the running track, at time t after the start of the race, on his monitor screen in an air-conditioned room.

(a) Do you think the equation $x^2 + y^2 = 16$ is sufficient for the organiser to track their positions on the running track as t varies?
Explain your answer.

One Proposed Task (Continued)

It is given that A runs at a constant angular speed of 2 rads^{-1} while B runs at a constant angular speed of 3 rads^{-1}.

(b) Write down the x and y coordinates of A and B in terms of t.

(c) Work out $\boldsymbol{x^2 + y^2}$ for your answer in (b) for the case of A and of B. What do you notice?

The pair of equations in (b) are called the **parametric equations** (as the equations for x and y are written in terms of a third independent **parameter** t). The equation in (c) is called the corresponding **cartesian equation**.

One Proposed Task (Continued)

(d) Determine the positions of A and B when

$t = 0$, $t = \pi/6$, $t = \pi/2$, $t = 2\pi/3$, $t = \pi$, $t = 5\pi/4$.

For each of the values of t, indicate the relative positions of A and B in the x-y plane.

Assessment of the Learning Outcomes of the Task

- Reflection component (**concepts learnt from the task, further questions to ask the teacher, takeaways from doing the task, etc**) for the students to complete at the end of the task

- Short exit pass question related to the concept at the end of the task

  E.g. for the task on sketching the graph of $y = f'(x)$, we can have an exit pass question on the path, given by $y = f(x)$, that a roller-coaster travels in an amusement park and have the students to sketch the corresponding graph of $y = f'(x)$.

Resources

- One useful website to source for ideas - http://nrich.maths.org

  E.g. **http://nrich.maths.org/4750** (Maclaurin's Series)

  http://nrich.maths.org/7084 (Calculus)

  http://nrich.maths.org/6960 (Rational Functions)

- Mathematics Competition questions could be used to set task (**Toh, T.L., 2013**)

- Use and modify questions in textbooks. Good to refer to university texts on calculus, etc, especially those with contexts and applications

  E.g. **Calculus, Concepts and Contexts** by James Stewart

- From observation of our surroundings, reading from newspapers, etc

- Sharing of ideas and resources with colleagues

THANK YOU !

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