DISCOVERING MATHEMATICS

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MTC 2016 @ NIE

2 June 2016

What does a Mathematician do?

“If you asked a mathematician, you might be surprised to hear him/her say that what he/she does for a living feels closer to writing poetry than doing rote calculations. Indeed, mathematics is more about planning, exploring, creating, and experimenting than it is about memorizing and labeling. It is elegant, logical, beautiful and inspiring. Mathematicians solve problems; they search for structure and truth; they seek to understand why shapes, ideas, numbers and patterns interact and behave the way they do.”

Shai Simonson, 2011

Why then do our students have a different view of Mathematics?

- “Mathematics is about formulas and calculations, how to get the correct answer …”
- Is it *what* we teach or *how* we teach?

All Telling – All Discovery Continuum

- Marrongelle & Rasmussen, 2008
- All lectures on one end

Laurentius de Voltolina,
14th Century

All Telling – All Discovery Continuum

- Marrongelle & Rasmussen, 2008
- All lectures on one end, Inquiry Based Learning (Moore Method) at the other end.
- RL Moore (1882-1974)
  - No textbooks, students are given a list of definitions and theorems to prove.
  - Expected to work alone without discussion nor references.
  - Students *present* all the material.
  - “That student is taught the best who is told the least.”
  - http://legacyrlmoore.org/

Evidence for Active Learning

PNAS

Active learning increases student performance in science, engineering, and mathematics

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To test the hypothesis that lecturing maximizes learning and course performance, we metaanalyzed 225 studies that reported data on examination scores or failure rates when comparing student performance in undergraduate science, technology, engineering, and mathematics (STEM) courses under traditional lecturing versus active learning. The effect sizes indicate that on average,

225 studies in the published and unpublished literature. The active learning interventions varied widely in intensity and implementation, and included approaches as diverse as occasional group problem-solving, worksheets or tutorials completed during class, use of personal response systems with or without peer instruction, and studio or workshop course designs. We followed guidelines for

- Freeman et. al. 2014, Proceedings of the National Academy of Sciences (USA)
- Meta-analysis of 225 studies of Active Learning vs Lecture in undergraduate STEM courses
- Average examination score (+6%), failure rates (22% vs 34%)

GPS Nav sends American across Iceland

- Noel Santillan (Feb 2016) directed his GPS to guide him from Keflavik Airport to a hotel in Reykiavik but ended up 5.5 hours later at a small village.
- “GPS eliminated much of the need to pay attention”

- Greg Milner

A teacher is supposed to *teach*

"I had stopped all telling and eliminated any type of evaluation of students' answers. I tried to do nothing but ask questions and remain neutral.… I accepted all individual answers but was at a loss for how to move forward. I was beginning to feel that math needed to be more than just a time to share ideas…I had begun to feel as if I ought to be doing something more with responses. I was a *teacher*. I was supposed to *teach*."

- Heaton, 4th grade teacher, university teacher educator, educational researcher, 2000.

Considerations for Active Classrooms

- “First, changing teaching practice does not mean wholesale abandoning of past practice. This means that teachers must go beyond simply replacing old strategies with new ones; rather teachers must determine how to integrate new strategies into their existing repertoire.”
- “Second, the role of the teacher needs to include more than bringing tasks to the classroom and standing back as students solve problems.”

Marrongelle & Rasmussen (2008). Meeting New Teaching Challenges: Teaching Stratgies that Mediate Between All Lecture and All Student Discovery.

Available at http://www.fisme.science.uu.nl/en/fius/rmeconference/2009/handouts/rasmussen/marrongelle_rasmussenPCTs.pdf

Example 1

- Prove by induction that

$$\sum_{k=2}^{n} \frac{1}{k(k-1)} = 1 - \frac{1}{n}.$$

- Alternative: Define

$$s(n) = \sum_{k=2}^{n} \frac{1}{k(k-1)}.$$

- Find s(2), s(3), s(4).
- Conjecture a value for s(n).
- Prove your conjecture.

Example 2: Integration By Parts

5.3 Integration by Parts

Integration by parts is an application of the product rule – essentially the product rule in reverse! It is typically used when the integrand is the product of two functions, one which you can easily anti-differentiate and one which becomes simpler (or at least no more complicated) when you differentiate.

Problem 226. *Find two functions f and g so that* $\int f(x)\cdot g(x)\,dx \neq \int f(x)\,dx \cdot \int g(x)\,dx$.

Example 23. *Evaluate the integral* $\int x\ln(x)\,dx$.

We begin by guessing a function which has $x\ln(x)$ as a part of its derivative when we use the product rule. From the product rule for derivatives we have,

$$(x^2\ln x)' = 2x\ln x + x^2(\frac{1}{x}).$$

Integrati

Now we

Where k

Problem 227. *Use the idea from the example to compute the following anti-derivatives.*

1. $\int x^{45}\ln x\,dx$
2. $\int xe^x\,dx$
3. $\int x\cos(x)\,dx$

Source:
http://www.jiblm.org/downloads/dlitem.php?id=100&category=jiblmjournal

Example 2: Integration by parts

- Instead of teaching ILATE or LIATE
- In the following integral (by parts) which would you choose to be your u.

$$\int e^x x^5 dx$$

- Without attempting the problem, explain your choice.
- Now attempt the problem, trying the alternative if you get stuck
- Explain to your friend which is the better choice. (POE)

Example 3: Simple integral

- Evaluate

$$\int_0^2 (x-1)dx$$

- Explain why the integral is 0.
- Find different limits which keep the integral value at 0.
- For arbitrary limits, find an integrand which keep the integral value at 0.

$$\int_b^a (mx+c)dx$$

Discovering Mathematics

“What you have been obliged to discover by yourself leaves a path in your mind which you can use again when the need arises.”

- G. C. Lichtenberg, German Scientist, 1742-1799.

Problem 1: Folds and Creases

- Imagine a long thin strip of paper stretched out in front of you. (Better still, tear out a strip of paper and try it!)
- Place the right hand end on top of the left and press the strip flat so that it is folded in half.
- It now has one fold, one crease and two sections.
- Repeat the whole operation on the new strip two more times.
- How many folds, creases and sections do you have now?
- What if you repeated 10 times in total?

Problem 1: Folds and Creases

- Understanding the problem; Heuristic: make a list/table

Folds, n	Creases, c(n)	Sections, s(n)
1	1	2
2	3	4
3	7	8

- Conjecture: $s(n) = 2s(n-1)$; $c(n) = s(n)-1$
- Can you prove them? Have you solved the problem?
- What other patterns do you observe?

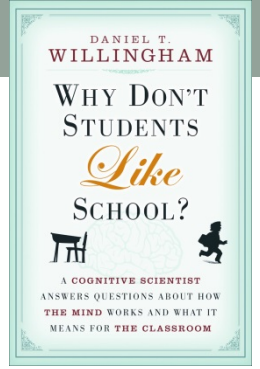
Problem 1: Folds and Creases

Folds, n	Creases, c(n)	Sections, s(n)
1	1	2
2	3	4
3	7	8
4	15	16

- Relations: $s(n) = 2s(n-1)$; $c(n) = s(n)-1$; $c(n)= c(n-1)+s(n-1)$
- Deductions: $s(n) = 2^n$; $c(n) = s(n-1)+ s(n-2) + \ldots +s(1)+c(1)$
- $c(n) = 2^{n-1} +2^{n-2} + \ldots +2^2 +2^1 +1 = 2^n -1$
- Polya's stage 4: Can I generalize? Is it true that

$$x^{n-1} +x^{n-2} + \ldots +x^2 +x^1 +1 = x^n -1 \text{ ?}$$

What about drilling?

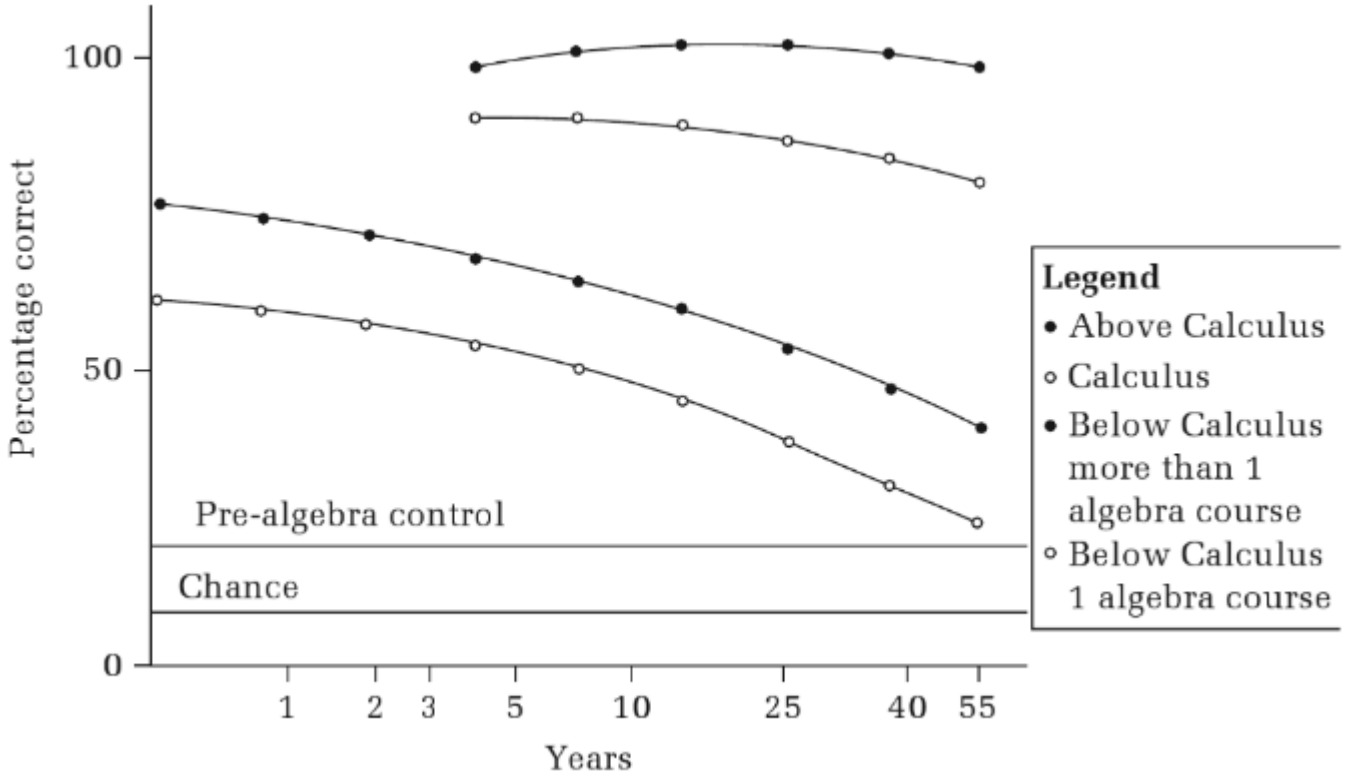


- "It is virtually impossible to become proficient at a mental task without extended practice"

  -Willingham, cognitive scientist, 2009
- Evidently, practice allows one to gain confidence and improve.
- Practicing even when one has mastered a skill is beneficial for: 1) protects against forgetting, 2) reinforcing basic skills for learning, 3) improves transfer

Deliberate Practice

- 1) Makes memory more long lasting



Extracted from Pg 89 of Willingham's Book

Deliberate Practice

- 1) Makes memory more long lasting
- 2) Practice helps mental processes achieve automaticity, thus freeing up working memory.
- 3) Deep understanding improves transfer
- Recommendations
- 1) Space out the practice
- 2) Fold practice into more advanced skill

Polya's Three Principles of Learning

1. For efficient learning, the learner should discover by himself as large a fraction of the material to be learnt as feasible under the given circumstances.
2. For efficient learning, the learner should be interested in the material to be learnt and find pleasure in the activity of learning.
3. Learning begins with action and perception, proceeds from thence to words and concepts, and should end in desirable mental habits.

Polya (1963), On Learning, Teaching and Learning Teaching, *The American Mathematical Monthly*, v70,605-619