Title and Speaker

Enhancing Mathematical Reasoning at Secondary School Level

Wong Khoon Yoong
Mathematics and Mathematics Education Academic Group
National Institute of Education / NTU
<kywong@nie.edu.sg>

Overview

1. What is "mathematical reasoning" and its importance: 6 reasons.
2. Intuitive-experimental justification; proof.
3. 4 aspects of mathematical reasoning:
   a) Patterns with explanation.
   b) Definitions, accurate.
   c) Cover all cases.
   d) Correct sequence of results.
4. Teaching strategies: Cognitive and Disposition.

What is Mathematics Reasoning?

1. When learning new mathematics:
   a) Why is (….) true? Why is (….) false?
   b) Intuitive-experimental justification, logical inference, deduction, proof.
2. When solving problems:
   State known formulae or principles for intermediate steps.
3. Used by educators, psychologists:
   a) Thinking: e.g., proportional reasoning, algebraic reasoning, visual reasoning.
   b) Problem solving, decision making.

Mathematical Reasoning & Thinking Skills

1. Mathematical reasoning refers to the ability to analyse mathematical situations and construct logical arguments. (MOE, 2007, p. 5).
2. Thinking skills are skills that can be used in a thinking process, such as classifying, comparing, sequencing, analysing parts and wholes, identifying patterns and relationships, induction, deduction and spatial visualisation. (MOE, 2007, p. 5).

Mathematical Reasoning: NCTM

- People who reason and think analytically tend to note patterns, structure, or regularities in both real-world situations and symbolic objects; they ask if those patterns are accidental or if they occur for a reason; and they conjecture and prove.

  (NCTM, 2000, p. 56).
- *Mathematics is a study of patterns; need to justify why the patterns work.*

Why is Reasoning Important #1

1. Mathematics reasoning is part of the 4th R!

Mr Tharman Shanmugaratnam

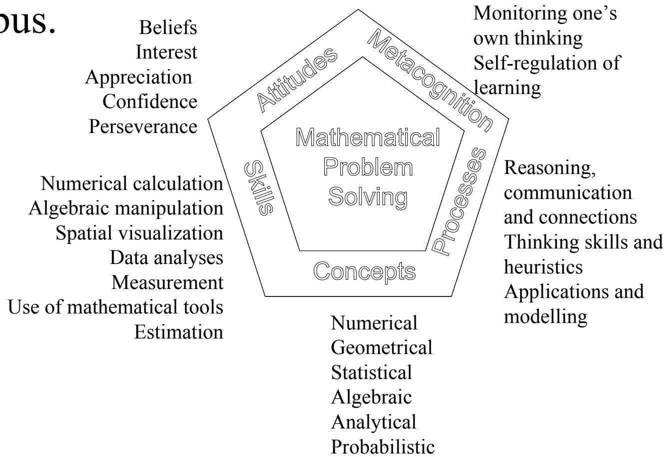
STU *Philosophy in Schools Conference*

(17 April, 2006, Straits Times, 18 April 2006, H7)


Schools need to teach one more 'R' — reasoning

Why is Reasoning Important #2

2. *Process* component of the Mathematics Syllabus.



Why is Reasoning Important #3

3. Rubric of SAIL (Strategies for Active and Independent Learning).

   Secondary Mathematics:
   1. Approach and Reasoning.
   2. Solution.
   3. Connections.
   4. Mathematical language and representation.
   5. Overall presentation.

Why is Reasoning Important #4

4. Essence of mathematics, at all levels, primary to tertiary.
5. If you forget the rule, can work it out from first principles.
6. Disposition: Habit of mathematical mind (MOE, 2007, p. 5); confidence in own thinking/ability; I can make sense of mathematics, I can reason things out, not "just get answer"; an inquiry mindset, etc.

Why is Reasoning Important #5

An encounter:

- A trainee taught Sine Rule, no justification or proof, only worked examples.
- Me: Why didn't you show the proof?
- Trainee: I don't know the proof. My teacher did not teach it.
- Me: But you are a math major. Could you prove it yourself?
- Trainee: Don't know.

Recall!

- Recall the importance of teaching "mathematical reasoning".
- Which reason appeal to you most?

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Intuitive-Experimental Justification

- Main purpose: Enough evidence to convince oneself or others than something is true/false.
- Intuitive-experimental Justification: O-level Elementary Math; not rigorous, but adequate for most students; should be in most topics.
  - Variety of specific examples.
  - Enactive: Activities with manipulative, applets, GSP, spreadsheet.
  - Patterns with explanation.
  - Visual illustrations ("visual proofs").

Proof

- Main purpose: Deductive thinking.
  - Chains of statements, based on accepted/proved previous statements.
  - Axiomatic proof (not required).
- More rigorous, difficult for many students.
- Additional Math (p. 19): Proofs in plane geometry.
- van Hieles' Theory: Recognition → Analysis → Ordering → Deduction → Rigour.

Key Ideas in Reasoning

1. True statement: Apply to *every* relevant case; citing a few cases is NOT proof.
2. False statement: Only one case (counter-example) is required; more cases may be given for teaching purpose.
3. Conjectures with intuitive-experimental justification.

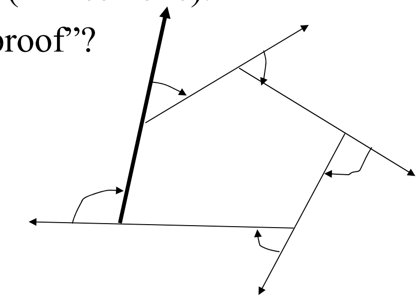
? Not included in Syllabus:
  - Truth values and logical statements *(implication, contrapositive, converse, inverse*).
  - Different methods of proof.
  - Axiomatic proof.

Authority

- ❖ In secondary schools, some results have to be accepted; justifications/proofs too difficult.
- ❖ Teachers should know the proofs.
- π is irrational.
- $\sqrt{2}$ is irrational. (maybe?).
- Irrational numbers have non-terminating or non-recurring decimals.
- Converse of Pythagoras Theorem.

Sum of Exterior Angles of a Convex Polygon is 360º: Justification

1. Construct a triangle, a pentagon, etc. and measure in each case. Close to 360º.
2. Cut out exterior angles and assemble.
3. GSP.
4. Walk around a polygon (kinaesthetic).
5. The diagram a "visual proof"?



Sum of Exterior Angles of a Convex Polygon is 360º : Proof

- Apply to specific, simple polygons, say, triangle, quadrilateral, pentagon.
- Exterior angle = 180^0 – interior angle; apply sum of interior angles.
- Note the sequence: interior angle → exterior angle.
- Students go over the proof for different polygons.
- Generalise to *n*-gon.

If *ab* = 0, then *a* = 0 or *b* = 0

- Textbooks do not give justification/proof.
- Intuitive-experimental Justification:
  1. Test a variety of values.
  2. Area of rectangle; GSP. ab = 0
- Proof:
  - If $a \neq 0$, then divide by *a*. $b = {}^0/_a = 0$.
  - Same if $b \neq 0$.
- Discuss: If $ab = c$, where $c \neq 0$, then $a = c$ or $b = c$. GSP activity.

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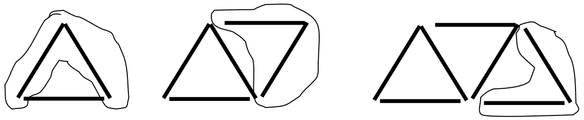
Patterns Require Explanation

- Pattern → Formula → Explanation (structural) → Predict and validate.
- Algebraic pattern:

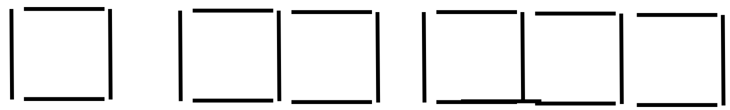


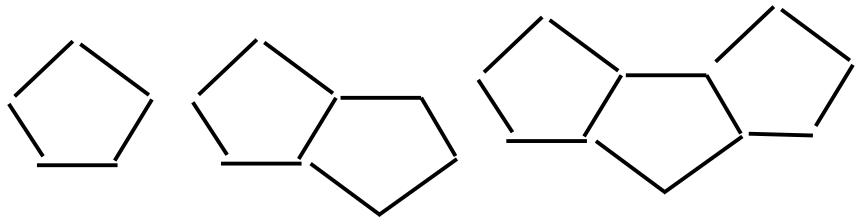
  - Tabulate, pattern, and write a formula.
  - How many sticks are needed for the 1000th case?
  - Explain the formula; should be included. $1 + 2n$

Look back: Extend the logic to squares, pentagons, etc. $1 + kn$

Patterns: Extension

- $1 + kn$.





Patterns: Different Ways

- Other ways to justify the formula? $1 + 2n$

3 | $3 + 3 - 1$, $3 \times 2 - 1$ | $3 + 3 + 3 - 2$, $3 \times 3 - 2$

$$3n - (n - 1) = 2n + 1$$

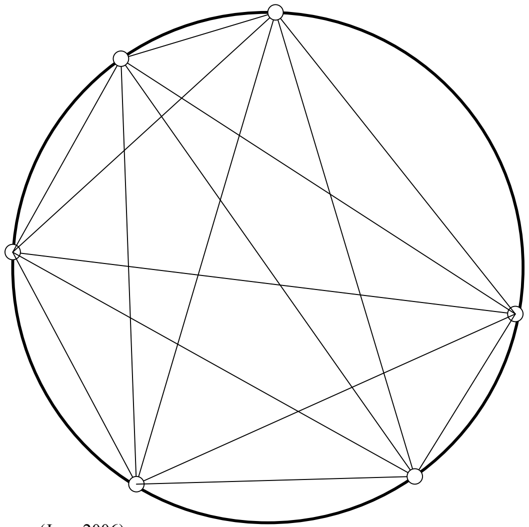
- Promote creativity with alternatives.

How Many Regions? #1

- Without reason, pattern may break down.

  1, 2, 4, 8, 16, …
- What is the next number?
- Take any two points on a circle. Join them with a chord. This divides the circle into two regions.
- Repeat this process with 3, 4, 5, 6, and 7 points. In each case, find the *greatest* number of regions formed inside the circle.

How Many Regions? #2

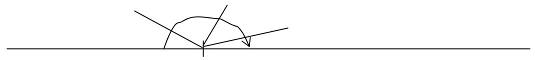


For 6 points,
31 regions!

$$1+\binom{n}{2}+\binom{n}{4}$$

Definitions or Properties?

1. Adjacent angles on a straight line = 180^0.



2. $\pi = \dfrac{\text{circumference}}{\text{diameter}}$
3. $180^0 = \pi$ radians.
4. 1 is not a prime number.
5. p(H) = ½ for a fair coin.

- Cannot prove definitions; discuss rationale.

Definition Crucial for Proof

- Some terms do not have universally accepted definitions; no “world authority”!
- Inconsistent definitions.
  - Is an equilateral triangle isosceles? No?
  - Euclid, Definition 20: An equilateral triangle is that which has its three sides equal, and isosceles triangle which has two of its sides alone equal, … [Artmann, p. 18].
  - Primary textbooks: An isosceles triangle has two equal sides. [meaning *at least two?*]
  - Everyday language: I have two brothers. [only two]

Squares and Rectangles

- Euclid: Definition 22: Used “oblong” for “rectangle”; so a square is NOT a rectangle.
- Modern usage: Be *inclusive:*
  - A square is a rectangle.
  - An equilateral triangle is isosceles.
- How do you show that a square is a rectangle? By properties and by GSP. square

Use More Precise Definitions

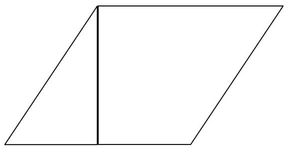
1. A quadratic expression in x is one in which the highest power of x is 2. *Is this clear enough?*
2. Teh & Looi: The general form of a quadratic polynomial is $ax^2 + bx + c$, where a, b and c are real numbers and $a \neq 0$.
   - *Is x^2 a quadratic polynomial?*
3. Teh & Looi: A polynomial is an algebraic expression that contains more than two terms, especially the sum of terms containing different integral powers of the same variable(s).
   - *Is x^2 a quadratic polynomial?*

Definitions: A Personal Proposal

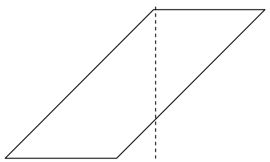
1. Primary level: Use an inclusive and more precise definition; "one" versus "at least one".
2. Secondary level: Continue with primary school definition; explore alternative definitions (reasoning). Discuss rationale why certain definitions are more prevalent than others.

Cover all cases: Area of Parallelogram #1

- Standard justification:

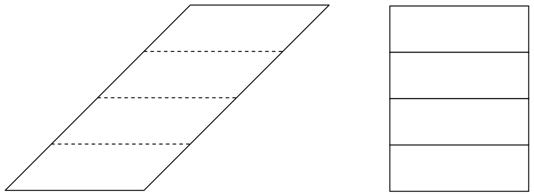


- Tip "falls" outside base?

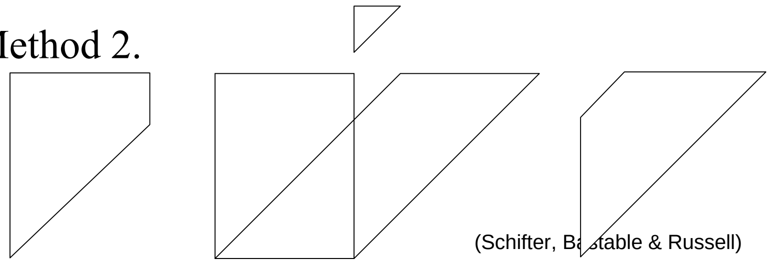


Cover all cases: Area of Parallelogram #2

- Method 1.



- Method 2.



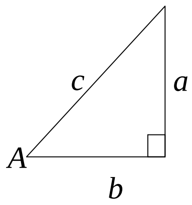
(Schifter, Bastable & Russell)

Prove Pythagoras Theorem

$\sin^2 A + \cos^2 A = 1$ (*)

$(a/c)^2 + (b/c)^2 = 1$

$a^2 + b^2 = c^2$



Your response:

1. Simple and easy to understand.
2. Need to know trigonometry; difficult for students.
3. Need Pythagoras Theorem to prove (*).
4. I will teach this proof.

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Teaching: Uncertainties

- Unresolved:
  - *skill* → *reasoning*
  - *reasoning* → *skill*
- Each step makes sense, but the whole proof is puzzling.
- Not easy to see the needed constructions in geometry proofs, insight or mystery!?
- Too difficult for my class! Just learn the rule!!
- No time to justify/prove; just teach the rule.

No Time: Rush to Cover Syllabus?

- Can they reason things out?
- Enough time with content reduction, "white space", and teaching assistants?

Some Teaching Ideas

1. Include intuitive-experimental justification for most topics to enable *all* pupils to practise reasoning.
2. Start with pupils' reasoning; explain, discuss, train students to listen to one other's reasoning, challenge, convince others, conclude.
3. Students write about their thinking process.
4. Disposition: Students learn from your disposition; model habit to reason things out.
5. Students ask questions and look for alternatives.

Students Ask Questions

1. Why Questions:
   - Does this rule always work? Sometimes? Why?
   - Why does this method work here?
   - I agree/disagree with … because …
   - If … then … because …
2. Alternatives:
   - Do you (classmates) have a different answer? Explain your reasoning. Why do you think so?
   - What happens if ….? [conjecture]
3. Make into laminated cards.
4. Internalise these questions as "private talk".

Conjecture: Example

1. Find conjecture and justify; individual or group.
2. Example:
   1. Take a natural number and find its square.
   2. Take its preceding number and find its square.
   3. Conjecture?

$$8 \rightarrow 64$$
$$7 \rightarrow 49$$

Conclusion: Only Teachers Can Deliver

✓ Only you, the teachers, can deliver a successful program on Reasoning.

✓ MME inservice modules to help you. Enrol now!

- *Creativity in Teaching Mathematics*: Mr Lee Ngan Hoe (Jun 2006).
- *Mathematical Thinking for Upper Secondary Teachers*: A/P Lim-Teo Suat Khoh (Jul 2006).
- *Pathways to Reasoning*: A/P Berinderjeet Kaur (Sep 2006).

Thank You!

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