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How Primary Five Pupils Use the Model Method to Solve Word Problems

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Abstract: One hundred and fifty-one Primary 5 (average age 10.7 years, sd = 0.65) pupils were asked to use the model method to answer five word problems. Pupils used the model method most successfully with an arithmetic-word problem. With the algebraic-word problems, the rate of success decreased from word problems where the nature of relationships was homogenous involving two additive relationships to that which involved non-homogenous relationships. Pupils errors were clustered around relational phrases such as ‘more than’, ‘less than’ and ‘n times as many’. Pupils who used the model method to solve word problems involving fractions showed that they had a superior command of how they visualised the problem. Their solutions showed that they oscillated between the symbolic representations of the problem and its visual analogue.

Introduction

In 1980, the National Council of Teachers proclaimed, in its *Agenda for Action,* that “problem solving must be the focus of school mathematics” (National Council of Teachers of Mathematics, 1980, p. 1). With this statement problem solving became the theme of mathematics education of the 80’s, spawning extensive research into this area.

There were debates looking at what counted as problems and whether working with these problems could be counted as providing learners with appropriate problem-solving experiences (e.g., Burkhardt, Groves, & Stacey, 1988). These debates sparked extensive research investigating how beliefs affected learners’ solving of non-routine problems (e.g., Schoenfeld, 1983; Cobb, 1984; Wheatley, 1984; Frank, 1988), how learners’ metacognitive ability affected their ability to solve problems (e.g., Brown, 1978; Brown & DeLoache; 1978; Silver, Branca, & Adams, 1980; Schoenfeld, 1983; Ng, 1989; Foong, 1990; Yeap, 1997), and how learners’ gender affected their problem solving behaviours (e.g., Buerk, 1982; Burton, 1987).

Within this research activity, the mathematics education community became familiar with the work of Polya through his book *How to Solve It* (1945) in which he introduced the term ‘heuristic’ and the art of problem solving, with suggestions on how to guide learners through difficult problems. Whether or not the teaching of heuristics could improve learners’ problem-solving abilities became an intensive and enduring area of research (e.g., Schoenfeld, 1980a, 1980b, 1983; Taback, 1988; Wong, 2002).

Over this time period, in Singapore, the primary aim of mathematics education became the cultivation of learners who are competent problem solvers (Curriculum Planning and Development Division, 2000). To achieve this aim, primary pupils were taught basic skills and processes as well as problem solving heuristics. One heuristic that received particular attention was 'draw a diagram'. Kho (1987) developed and introduced various types of models to the local teaching community. Because of its specialised nature, this heuristic is known locally as the model method.

Since the model method was introduced in 1983, primary pupils in Singapore have been taught to use this method. In 2003, we conducted a study with 151 Primary Five pupils from five primary schools. The study investigated how pupils' working memory contributed to word problem solving. The study consisted of three mathematics tests. Each pupil completed three parallel versions of the test. In each version, pupils were asked to solve ten word problems of increasing difficulty using (i) the model method, (ii) any method, or (iii) any but the model method. The findings of that study are reported in Lee, Ng, Ng, and Lim (2004). One of the key findings was that working memory, in particular the ability to reason and remember simultaneously plays a more prominent role than previously thought. Not only did it account for a reliable amount of variance in both mathematical problem solving and English literacy, when its total effect was computed, it had greater influence on mathematical performance than did English literacy. From a pedagogical perspective, these findings suggest that improving working memory capacity will improve performance in both mathematics and literacy.

As part of that study, we also analysed how pupils used the model method to solve word problems. Analysis of their responses showed the pupils tended to use the model method to solve five of the easier questions of the ten-question model method mathematics test. In this paper our purpose is to report how Primary Five pupils used the model method to solve these five easier word problems. Based on the pupils' errors we will also discuss some of the difficulties pupils have with such word problems.

We begin with a brief description of the model method, followed by a discussion of what constitutes an arithmetic-word problem versus an algebraic-word problem. A description of the study, the questions used in the study, and the findings are reported next. Implications for teaching are discussed in the concluding section.

The model method – a heuristic for solving arithmetic- and algebraic-word problems

The model method is a visual analogue that captures all the information provided in a word problem – hence providing a global view of the entire problem. Its structure is constructed with a series of rectangles in which the relationships of the rectangles to one another are specified and relationships of all the rectangles are presented globally. Each rectangle is identified as a unit – the standard term used in the Singapore textbooks and teacher's guides. The model method can be used to solve word problems that require arithmetic reasoning as well as those that require algebraic reasoning as the principle underpinning the construction of the model is the same. See Ng (2004) for a discussion of how the model method is developed in the Singapore primary textbooks.

How do arithmetic-word problems differ from algebraic-word problems? Although both arithmetic and algebraic-word problems involve comparative relationships represented by phrases such as 'more than', 'less than' and '*n* times as many', the primary difference between arithmetic-word problems and algebraic ones is that arithmetic-word problems are "connected", as a relationship can be easily found between the known state and the unknown state, while algebraic-word problems are "disconnected" (Bednarz & Janvier, 1996). Consider the questions presented in the test (in Table 1) where we examine the relationships among the known and unknown states. Question 1 is an arithmetic-word problem; the remaining four questions are algebraic-word problems.

In Question 1, the unknown is the total enrolment of the three primary schools. The enrolment of Dunearn Primary is the known state (Bednarz & Janvier, 1996) and the enrolments of the remaining two schools are the intermediate unknowns. The enrolment of each individual school can be found, step-by-step, by using the enrolment of Dunearn Primary. This is because the link between the intermediate unknown and the known state is direct. The total, the unknown state, is found by summing the enrolments of each of the three schools. Thus the known state (enrolment of Dunearn Primary) and the unknown state (total enrolment) are "connected" (Bednarz & Janvier, 1996).

Questions 2 – 5 are algebraic-word problems. Take, for example, Question 3. The total mass of the three animals is the known state and it comprises the mass of the three animals which are all unknown and are all related to each other by comparative relationships. Unlike in Question 1, the individual masses of cannot be calculated using step-by-step procedures. Entry into an algebraic-word problem is via a selected unknown and, to solve for the unknown, all three unknown states have to be considered together. Note also that generally any of the 3 unknown states

Table 1
The Five Word Problems

Question Number	Text
1	Dunearn Primary school has 280 pupils. Sunshine Primary school has 89 pupils more than Dunearn Primary. Excellent Primary has 62 pupils more than Dunearn Primary. How many pupils are there altogether?
2	At a sale, Mrs Tan spent $530 on a table, a chair and an iron. The chair cost $60 more than the iron. The table cost $80 more than the chair. How much did the chair cost?
3	A cow weighs 150 kg more than a dog. A goat weighs 130 kg less than the cow. Altogether the three animals weigh 410 kg. What is the mass of the cow?
4	A tank of water with 171 litres of water is divided into three containers, A, B and C. Container B has three times as much water as container A. Container C has 1/4 as much water as container B. How much water is there in container B?
5	A school bought some mathematics books and four times as many science books. The cost of a mathematics book was $12 while a science book cost $8. Altogether the school spent $528. How many science books did the school buy?

can be used as an entry point to the problem. In Question 3, the mass of the cow can be found directly if it is chosen as the "generator" (Bednarz & Janvier, 1996). However the mass of the dog or the goat could also be chosen as the generator and the mass of the cow then found indirectly. Once a generator has been selected, other relationships in the problem can be constructed around the chosen generator.

This does not mean that algebraic-word problems cannot be solved using arithmetic means. In the 'any but model method' mathematics test, pupils used heuristics such as making a systematic list and guess and check to solve some of the parallel problems. The following example in Figure 1a shows a pupil using a guess and check method to solve a question which is parallel to Question 5. A man has some ducks and three times as many chickens. He sold the ducks at $10 each and chickens at $6 each. He received $420 from the sale of his animals. How many chickens did the man have? This pupil chose to enter the problem by using a hypothetical known state, the number of ducks as 30, and then, as in Question 1, following the relationships step-by-step, worked out the number of chickens that eventually satisfied the known state, the total of $420.

However using the model method to solve the chicken and duck problem affords the pupil to ask a more global question – how many groups of $28 are there in $420, which is not possible using a guess and check method (See Figure 1b). How using the model method provides pupils with a global view of the problem is discussed in greater detail later in the results section.

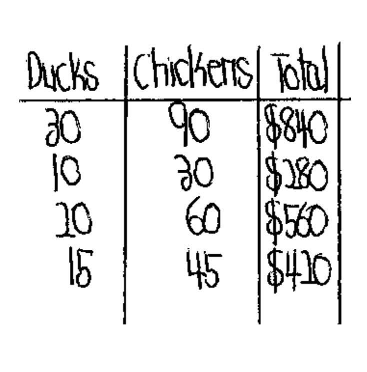

Figure 1a: Using guess and check

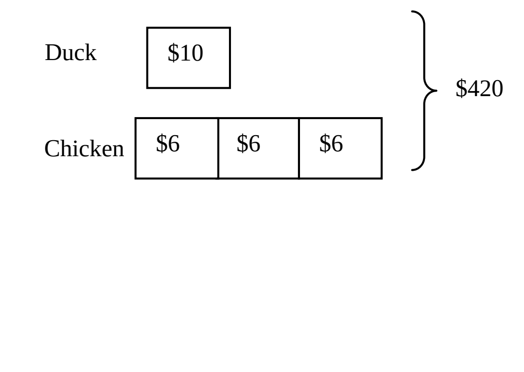

Figure 1b: Using model method to solve provides a more global view

The model method is less abstract than formal algebra and as a visual analogue the "models serve as good pictorial representations of algebraic equations" (Kho, 1987, p. 349). The affordance of a visual and less abstract method enables pupils to solve algebraic-word problems even when they have no formal knowledge of letters representing unknowns, do not know how to construct algebraic equations, nor how to transform them to solve for the unknown. How the value of the unknown unit is determined is also discussed in the results section.

The study

Participants

One hundred and fifty-one Primary 5 pupils from five public schools in Singapore took part in this study. Seventy-seven were boys. The pupils had an average age of 10.7 years. Most pupils in these schools were drawn from middle to lower middle-class areas. In Singapore, Primary 5 classes are streamed according to performances in the languages and mathematics. In this study, one fifth (21%) of the pupils were from the top stream and the remainder was from the middle stream. The lowest stream pupils were not recruited because pupils from this stream followed a different mathematics syllabus. Most pupils in this study have had at least 7 years of English language instruction, including two years of pre-school.

All the pupils who took part in this study were taught how to use the model method to solve word problems. As part of this study the head of mathematics department and one Primary 5 mathematics teacher of each of the five schools were interviewed. All the heads-of-department and the teachers reported that they had in place a programme to introduce the model method to the pupils. All but one school introduced the model method to the pupils when they were in Primary 2. In the remaining school, the model method was introduced in Primary 1. The teachers were asked to solve the five word problems listed in Table 1. Their solutions were similar to the pupils'. The teachers explained that these pupils were not taught how to divide with rational numbers, nor how to construct or transpose equations, as these concepts are taught at the secondary level. The teachers explained that they taught the pupils alternative methods to solve for the unknown unit. These alternative methods will be discussed with reference to pupils' solutions to the word problems.

Instrument

The pupils were given an hour to complete a specially designed mathematics test comprising ten questions. Question selection was guided by the school curriculum as well as research examining how students solve word problems (e.g., Lewis & Mayer, 1987; Mayer, 1989; Hegarty, Carifio, & Nasser, 1994; Mayer & Monk, 1995). Questions utilised concepts of whole numbers, fractions and proportional reasoning. Except for Questions 1 and 5, the remaining three questions involved three unknowns. Pupils were presented with the easiest question first (Question 1) while the rest were presented in a random order. Pupils were advised to use the model method to solve these word problems as we were interested in pupils' facility with the model method.

All questions involved comparative relationships such as 'more than', 'less than' and '*n* times as many'. Except for Questions 1 and 2 where the mathematical relationships between the unknowns are only of an additive nature, that is they have a 'homogenous' or same operation relationship, the relationships in the remaining three questions are non-homogenous. In question 3, for example, the relationship between the unknowns is non-homogenous involving addition and subtraction and therefore is more challenging than Question 2. Similarly for Question 4, although the volumes of the three containers may be linked by the multiplication process, further investigation suggests that because the volumes of the three containers are linked by multiplication involving whole numbers and rational numbers it could therefore be interpreted to mean that they are linked by multiplication and division, that is, linked non-homogenously. Question 5 is distinct from the rest as there are only two unknowns linked by a multiplicative relationship.

Analysis

Pupils' solutions were analysed according to whether or not they responded correctly to the questions. The findings are divided into two sections – section (i) discusses pupils' solution to the arithmetic-word problem while section (ii) discusses pupils' solutions to the algebraic-word problems. Each section ends with a discussion of pupils' difficulties with such word problems.

Results

For ease of comparison, the rate of response to the five questions is presented as percents and is summarised in Table 2.

Table 2

*Pupils' Rate of Response across the five questions expressed in percents**

	Question 1	Question 2	Question 3	Question 4	Question 5
Correct Solutions	63	44	37	20	15
Incorrect solutions	37	56	63	80	85

**Total number of pupils = 151*

(i) Arithmetic word problem

How pupils solved Question 1: Pupils found this the easiest of the five questions because the unknown enrolment for Sunshine Primary and Excellent Primary could be found directly and the total enrolment could be computed by summing the three enrolments. Figures 2 and 3 show typical solutions presented by pupils. About 46% of the pupils provided the solution in Figure 2 while 17% pupils gave the solution in Figure 3. Although both sets of models have the same structure 17% of the pupils were more efficient than the rest as they were able to abstract the multiplicative structure offered by the model in Figure 3. That fewer pupils found the total enrolment using a combination of multiplication and addition suggests that it might be more demanding for pupils to abstract the multiplicative structure and to use a combination of operations.

Difficulties faced by pupils: About 10% of the pupils constructed correct models but because of erroneous computations gave the wrong answer. The solutions offered by the remaining 27% of the pupils suggest that either these pupils did not read the question carefully and missed the comparative phrase 'more than' or they did not understand the role of such phrases, the latter more likely to be the reason for this error. If these pupils had understood the meaning of the comparative phrase 'more than', they would have drawn two rectangles to represent the enrolment of Sunshine

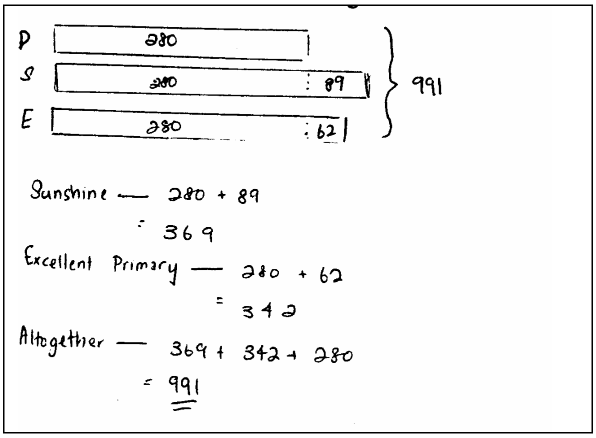

Figure 2. Correct answer with addition only

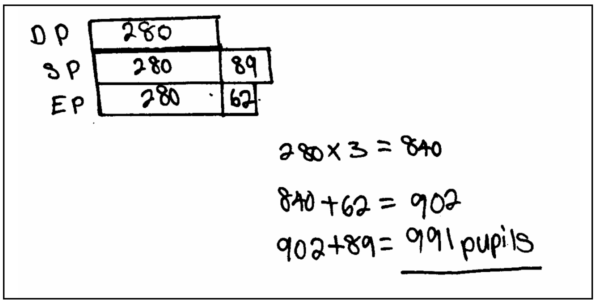

Figure 3. Correct answer with multiplication and addition

Primary, one rectangle identical in length to Dunearn Primary, the generator, and another of proportionate length representing the value 89, the difference in enrolment between Dunearn and Sunshine Primary. A similar reasoning would be used to construct the rectangles to represent the enrolment of Excellent Primary. However because these pupils did not make sense of the comparative phrase ‘more than’ they treated the difference in enrolment between the schools as the actual enrolment. The solution in Figure 4 is an example of a common error where pupils ignored the context of the problem and used the difference in enrolment between the schools as if they represented the actual enrolment of the schools. Similar erroneous solutions were also found for Questions 2 – 5.

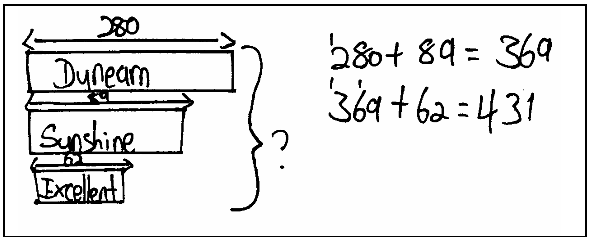

Figure 4. Common error for Question 1

(ii) Algebraic-word problems

In this section, solutions to Questions 2 and 3, Question 4 and then Question 5 are discussed. The focus will be on how pupils who were not taught to construct nor transpose algebraic equations used the model method to solve algebraic-word problems. The difficulties pupils had using the model method to solve such problems are also considered.

How pupils solved Questions 2 and 3: For Question 2, 44% of the pupils found the correct cost of the chair by constructing the model in Figure 5. The cost of the iron was used as the generator and the cost of the chair was represented using a rectangle similar in size as that of the iron plus another rectangle representing the difference in cost of the chair and the iron. A similar principle was used to represent the cost of the table but now a rectangle representing $80, the difference in cost between the table and the chair had to be added to the cost of the chair.

Pupils offered two types of correct models to Question 3. About 29% of the pupils provided the correct response as shown in Figure 6 where the mass of the dog was taken as the generator. Here a rectangle was drawn to represent the mass of the dog. Because the cow was 150 kg heavier than the dog, a rectangle longer than that of the dog was used to represent the mass of the cow with the difference in weight between these two animals indicated on the rectangle representing the mass of the cow. As the goat was 130 kg less than the cow, the difference in weight between the mass of the dog and the goat had to be calculated first before the rectangle representing the mass of the goat could be drawn.

Nine percent of the pupils gave the correct solution where the mass of the cow was used as the generator as in Figure 7. Here a rectangle was drawn to represent the mass of the cow. As the mass of both the dog and goat were less than the cow, rectangles shorter than the cow were drawn to represent their respective masses and the differences in mass indicated on the respective rectangles. By using the cow as the generator, a homogenous operation – addition of the differences in mass to the total mass - was used to solve for the mass of the cow. This solution is more

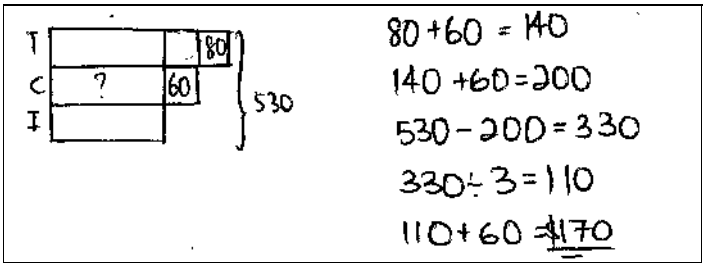

Figure 5. A common model for Question 2

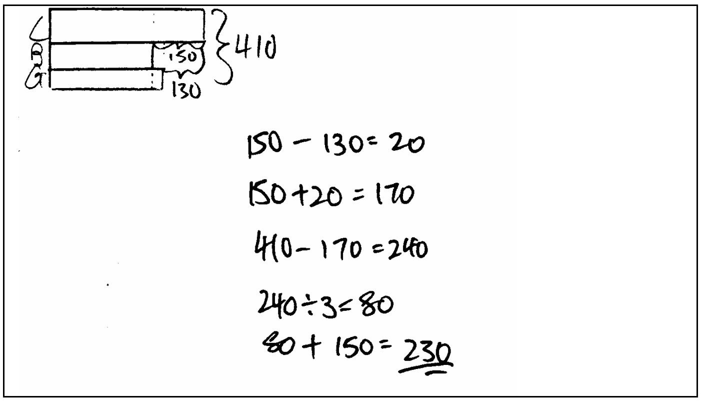

Figure 6. Solution to Question 3 – dog as generator

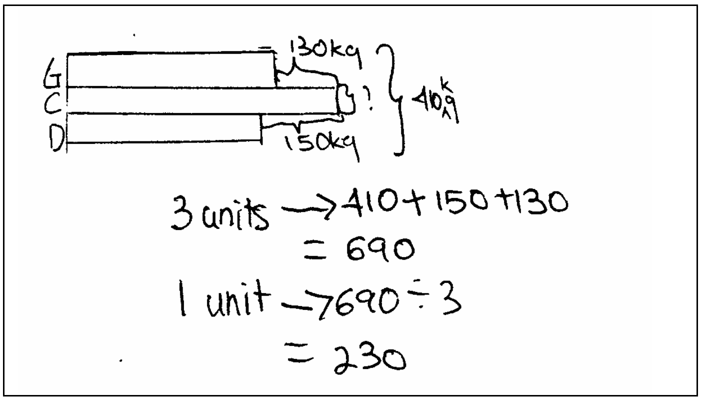

Figure 7. Solution to Question 3 – cow as generator

efficient and elegant than the former as the mass of the cow can be found directly without having to refer to the mass of the dog first. It is noteworthy that while nine percent of the pupils chose the cow as a generator which gave the solution of the mass of the cow directly, none of these pupils chose to use the cost of the chair as a generator for Question 2. If they had done so, they would found the cost of the chair directly. However this choice would necessitate use of non-homogenous operations – addition and subtraction of the difference in cost between the table and the chair and the difference in cost of the iron and the chair respectively to and from the total cost. These pupils were able to ‘see’ which generator to use and for which type of questions.

Pupils’ models showed that they kept all the unknown units and the known values which they represented as specified rectangles to the left of the bracket. The total value was kept to the right of the bracket. This visual analogue then represented a ‘pictorial equation’ (Ng, 2004) which gave a global representation of the problem. Pupils could then solve for the unknown unit in one of two ways. The procedures in Figure 7 are close to an algebraic equation while the arithmetic procedures in Figure 6 showed how pupils worked part-by-part from the model. In the former, the pupil constructed an equation where all the known numerical values to the left of the bracket were mentally transposed to the right of the arrow sign. This mental transposition reflects a good understanding of the properties of the operations, that is, the operation to the right of the bracket is the complement of the operation on the left of the bracket – subtraction is the complement of addition and division is the complement of multiplication. Kieran (2004) emphasised the importance of the knowledge of inverses of operations and the related idea of doing and undoing. This particular method was found in only one pupil’s solutions.

The most common form of solution is of the type shown in Figures 5 and 6. Here because no algebraic equations were constructed to represent the information captured by the models, the unknown to be solved for was not made explicit. However these pupils knew they had to calculate the value of the unit, knowledge they held mentally. The pupils then proceeded to solve for the unknown value by undoing, step-by-step, the operations suggested by the relationships of the rectangles to the left of the bracket – a rectangle with a specific rectangle added on would suggest subtraction of that specified value from the given total while three identical rectangles would suggest division of the total on the right. The work of these pupils suggests that they exhibited good metacognition skills as they knew what to do next, after each operation and they knew when they had the answer.

How pupils solved Question 4: Question 4 required pupils to work with rational numbers. Most pupils did not have problems drawing three rectangles to represent the volume of water in container B. Also they did not have problems drawing the model for container C as they worked backwards from the given information that the volume of container C was 1/4 of that of container B; that is, the volume of container B was 4 times the container C. However the majority of the pupils had problems using the model to represent the relationships among all three containers. The solutions in Figures 8 and 9 are examples of correct solutions by pupils.

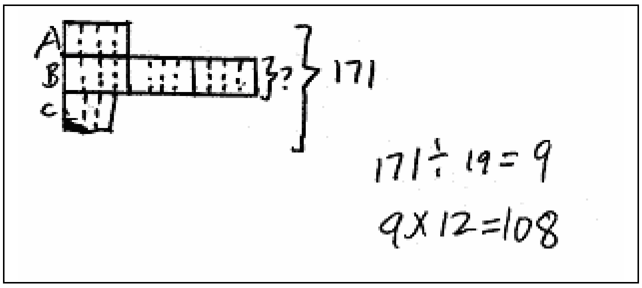

Figure 8. Solution to Question 4

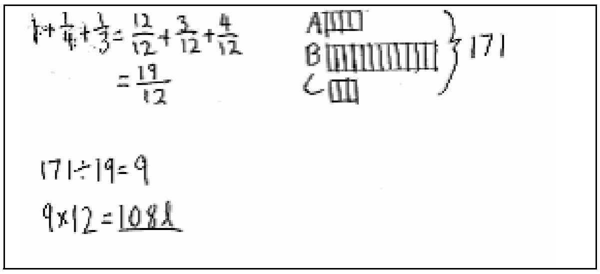

Figure 9. Solution to Question 4

The solution in Figure 8 shows that the pupils conducted the operations mentally before they drew the model for the question. The two different types of lines – full and dotted – that were used to draw the rectangles suggest that the pupils first drew the rectangles for containers A and B. After they had evaluated the relationship between B and C, pupils then drew in the dotted lines for all the three containers to show how the volumes of the three containers were related. This solution suggests the following train of thought: If the volume of the container B is three times that of container A and if volume of C is 1/4 that of B, the common multiple of 3 and 4 is 12. Therefore the volume of container B must have 12 smaller units while A has four and C, three. Because there was a total of 19 small units, then the value of one small unit was 1/19 of the total. Pupils then found the volume of container B.

Unlike the solution presented in Figure 8 where pupils drew the models after mental computation, the solution in Figure 9 shows how pupils used fractions to help them draw the model. Here pupils used the volume of container B as one whole and they demonstrated the relationships between A and C with B symbolically. Once they noted that B had 12 units, A four and C three, pupils then drew the rectangles to represent the calculated relationship. By translating the numerical information into a model, pupils could solve for the unknown volume of container B. In both cases, without the model drawing, pupils would not be able to solve for B as they have not been taught how to solve the equation 19 units/12 = 171 as they have yet to learn how to transform equations and to divide whole numbers by a rational number. The model method allowed pupils to work with whole numbers rather than fractions and this simplified the solution process as it now involved dividing 171 by 19 and then multiplying the result by 12. The model method allowed pupils to circumvent the need to evaluate the expression 171 x 12/19, a skill taught only in secondary school mathematics.

How pupils solved Question 5: Pupils provided solutions similar to that presented in Figure 10. Here one unit of 'mathematics books' is \$12 and four units of 'science books', \$32. How many \$44 units were there in \$528? Since there were 12 such units, therefore there should be 48 science books in all. Those who were successful in answering Question 5 marked the costs of the books in the individual rectangles and identified the total with a "?" even though they knew the total was \$44. Perhaps they had a global view of the problem and the "?" was to remind them that they needed to find out how many groups of \$44 there were in \$528. Hence for Question 5, pupils had to extend their understanding of the model further to include possibly the following "grouping method" (Ng, 2003) (see Figure 11) which perhaps was carried out mentally and not made visible to the reader.

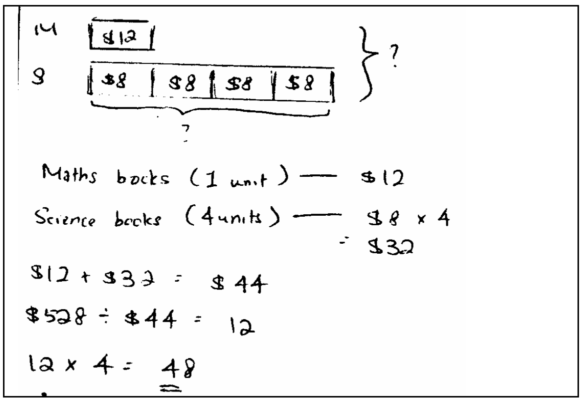

Figure 10. Solution to Question 5

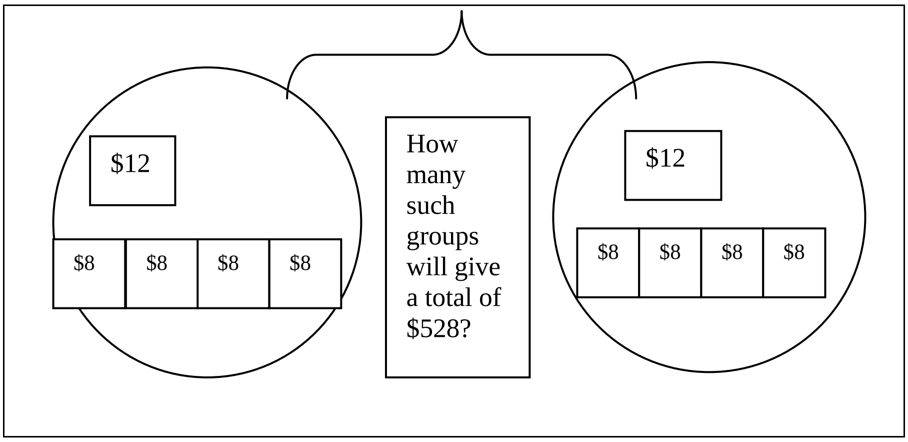

Figure 11. Grouping method held mentally

How did pupils who were not taught to construct nor transpose algebraic equations use the model method to solve algebraic-word problems?

Although Questions 2 – 5 were algebraic-word problems, pupils' responses suggest that writing equations to represent the information captured by the models and then working with the equations instead of the model was not part of their repertoire. Rather they preferred to write arithmetical calculations to solve the problems, part-by-part. Teacher's guides for textbooks (e.g., *Primary Mathematics 5A Teacher's Guide*, Curriculum Planning & Development Division, 1999) and examples from textbooks (e.g., *Primary Mathematics 5A,* Curriculum Planning & Development Division, 1999) offer possible solutions based on models constructed. The following shows a possible solution for Question 3 where the mass of the cow was used as the generator and equations were an inherent part of the process.

3 units = 410 + 150 + 130
1 unit = 230
Mass of the cow = 230 kg

However teachers from the participating schools explained that the pupils were not taught to construct the above series of equations because pupils found it easier to work from the model and use arithmetic methods instead to solve for the value of the one unit rather than to construct and work from the equation. Also as specified by the *Mathematics Syllabus Primary* (Curriculum Planning and Development Division, 2000), pupils were not taught to construct equations.

Difficulties faced by pupils: Although Questions 2 – 4 have a similar structure and the same number of unknowns, pupils found Question 4 which involved concepts of fractions most difficult and Question 2, which involved whole numbers, the least.

Between Questions 2 and 3, Question 3 was the harder of the two. A possible reason could be that in Question 3, pupils had to consider non-homogenous relationships, addition and subtraction, between the unknowns, while in Question 2 pupils need only work with homogenous relationship involving addition between the unknowns. Although Questions 2 and 3 have the same structure, constructing a model for Question 3 is a more complicated process than for Question 2. If the dog is taken as a generator, the mass of the cow is represented by two rectangles - the mass of the dog and another rectangle representing the difference in mass between these two animals. Representing the mass of the goat required more thought as pupils had to work backwards so as to refer the mass of the goat to that of the dog. Constructing the model for Question 2 is more direct as the cost of the chair and the cost of the table can be found by adding rectangles representing the difference in cost between the iron and chair and iron and table respectively to a rectangle representing the cost of the iron.

Analysis of pupils' incorrect solutions showed that pupils had difficulties understanding the questions. Pupils erroneous solutions showed that when they could not make sense of the questions, they 'converted' the algebraic-word problems into arithmetic-word problems by using the numbers in the relational phrases as the values for the unknowns (see Figures 12 and 13).

Pupils had difficulties with phrases involving multiplicative reasoning. Wrong solutions to Question 4 showed that pupils had misinterpreted the relationship between container B and C. About 11% of the pupils misinterpreted the relationship between B and C as additive rather than as multiplicative, that is the volume of container C as 1/4 more than B. Because of the misinterpretation, these pupils presented models as in Figure 14a and 14b. One pupil drew seven rectangles for C, seven being the total of four and three. Two pupils saw the relationship between B and C as multiplicative, with C four times as much as B. Therefore there were 12 rectangles for C and three for B (see Figure 15).

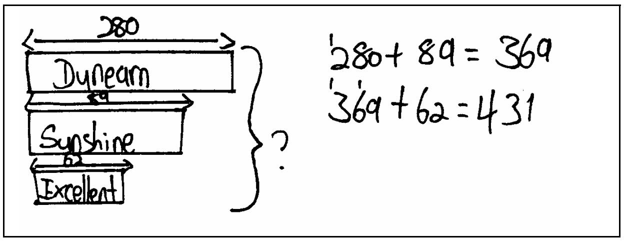

Figure 12. Wrong solution to Question 2

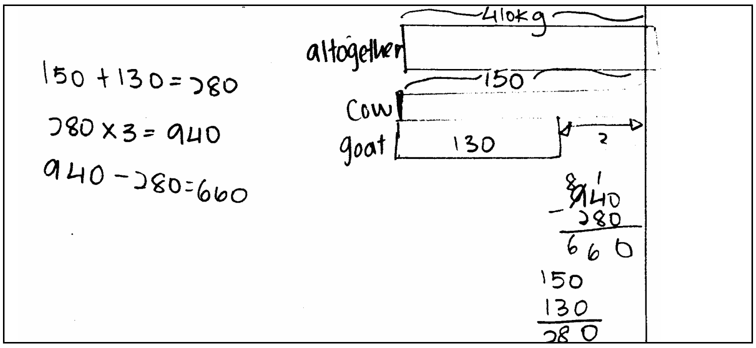

Figure 13. Wrong solution to Question 3

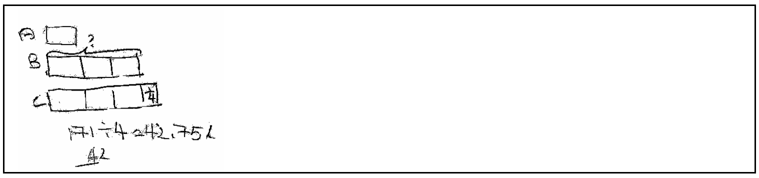

Figure 14a. Relationship between B and C as additive

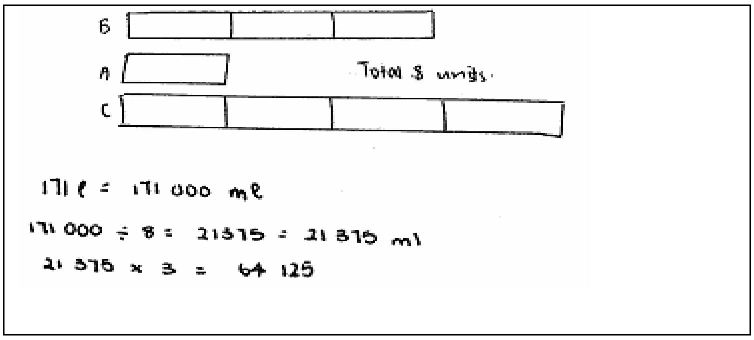

Figure 14b. Relationship between B and C as additive

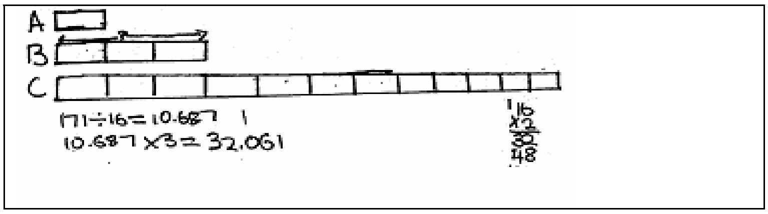

Figure 15. Relationship between B and C is multiplicative.

For Question 5, 11% of the pupils constructed models where the given relationship between the mathematics books and science books was reversed; that is, there were four times as many mathematics books as science books (see Figure 16). Perhaps these pupils read the sentence "some mathematics books and four times as many science books" as mathematics books four times science books. Also 5% of the pupils interpreted the phrase four times as many as four times more hence constructing a model which has five rectangles representing the number of science books (see Figure 17). Pupils also exhibited this misconception in Question 4 (see Figures 14a and 14b above).

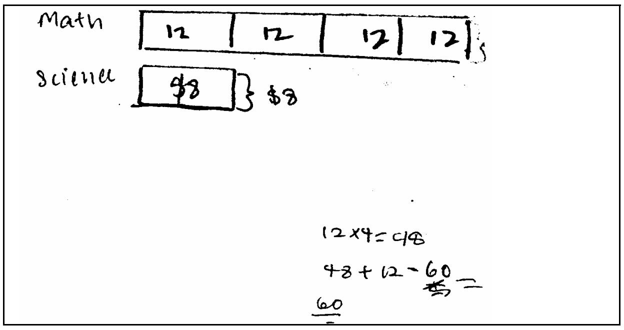

Figure 16. Reversal error

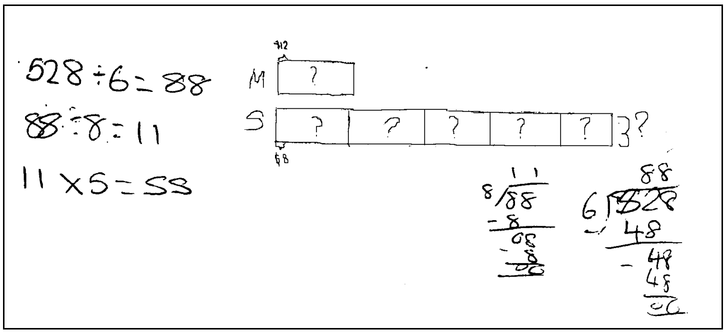

Figure 17. Multiplicative versus additive

Although pupils could use the model as a visual mnemonic to help them track the role of additional rectangles other than the unit rectangles, which represented the unknown, they were often confused by these very aids. For example, in Question 3 pupils who chose to use the cow as the generator drew rectangles with dotted lines as a reminder that these were to be subtracted from the related unknowns. However pupils' translated the information captured by the model incorrectly as they continued to treat these dotted rectangles as additions and continued to subtract these values from the total mass of the three animals (see Figures 18a and b). As the choice of the cow as generator was more complicated than using the dog, pupils may have confused themselves when they added to the dog and goat's rectangle so that there were now three unknown units of the same length but then failed to add the same amount to the total mass on the right of the bracket. Whether or not this failure was due to lack of understanding or a memory lapse is unclear. Perhaps pupils were more familiar with problems similar to Question 2 and continued to operate at that level, treating all additional rectangles as additions and hence should be subtracted from the total so that all the rectangles are the same size.

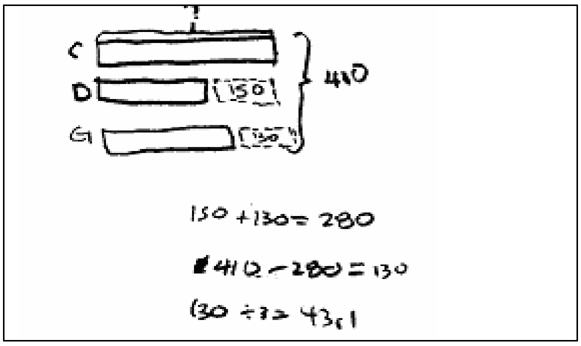

Figure 18a. Visual mnemonic could be confusing

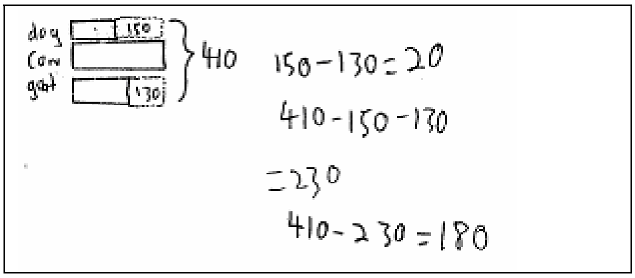

Figure 18b. Visual mnemonic could be confusing

Pupils found relationships involving fractions challenging. For example in Question 4, 19% of the pupils drew models where the relationship between A and B was correct and drew rectangles where C was a fraction of B. The rectangle representing C suggests that pupils understood that C was less than B but they were unable to determine the exact relationship. Figures 19a and 19b show two examples where pupils had difficulties determining the exact relationship between B and C.

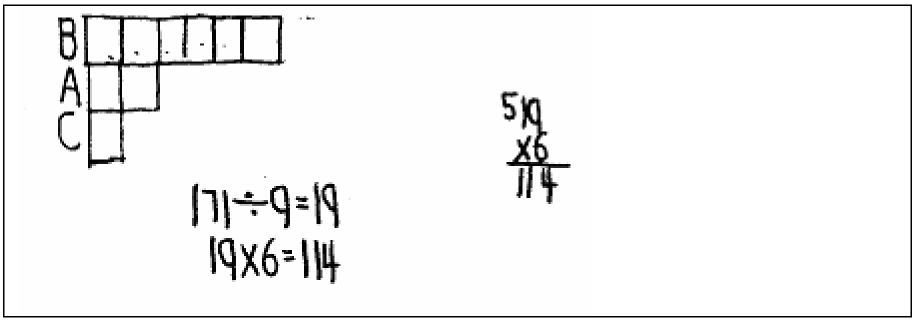

Figure 19a. Difficulties computing the precise relationship between B and C

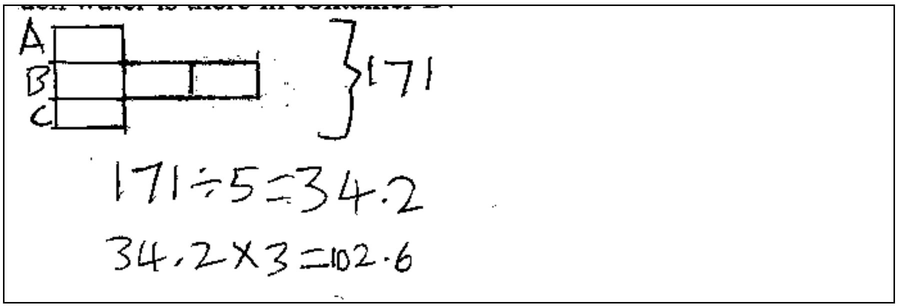

Figure 19b. Difficulties computing the precise relationship between B and C

The low rate of response for Question 5 suggests that the pupils found this question to be most challenging. There are two possible reasons. First more information is provided for the two unknowns in the question than it was for Questions 2 – 4. While the unknowns in Questions 2 – 5 were either linked to each other by comparative or proportional relationships those in Question 5 had additional information, namely the costs of the books were given. While most pupils drew one rectangle to represent the number of mathematics books and four for the number of science books, they were unclear what to do with the cost of the mathematics and science books. Second, the function of the rectangles representing unknowns is no longer the case for Question 5. Because the rectangles represent the cost of the books, the size of the rectangles representing the number of mathematics books and science books should be different, which is a deviation from what pupils were doing for the other questions. However although they identified the rectangles with the cost of the books, the size of the rectangles remained the same. Pupils' drawings suggest that pupils did not use the rectangles meaningfully; rather the rectangles were just a mnemonic aid and the model a heuristic.

Discussion

This study analysed how Primary 5 pupils used the model method to solve arithmetic and algebraic-word problems. Because of its very visual nature, the models drawn by pupils allowed us to infer the nature of the difficulties these pupils had with word problems and to consider implications these findings have on the teaching of word problems in primary classrooms.

Those who used the model method successfully to solve word problems could be classified into two groups – those who were efficient in the use of the models and those who observed rules. Efficient solvers saw the structure inherent in the model and were more efficient in their solutions. For example in Question 1, rather than finding the individual enrolments for each school, such pupils multiplied the enrolment of Dunearn Primary and added the difference of the two other schools to this total. Pupils who observed the rules first found the enrolment for the individual schools and then summed them. In Question 3, efficient problem solvers used the mass of the cow as the generator because they knew that was what the question wanted. Those who observed the rules chose the generator according to the sequence in which they first appeared. However for those who used the cow as the generator followed the rule to solve Question 2 as this was the more efficient route to the solution. While these pupils should not be faulted for their choice, nevertheless their solution strategies could be enhanced. To encourage pupils to be more efficient, teachers could offer, for discussion, two sets of correct solutions for pupils to compare and contrast. Through the discussion pupils may learn how they can be more efficient and economical in their solutions.

To further pupils' understanding of a generator, they could be given correct solutions to a selected problem but which uses different generators (as in Question 3). The task then would be to construct the appropriate mathematical sentences with generators different from those they were given. For example, if they were given a response sheet where the dog was the generator, their task would be to construct a model which uses the cow or the goat as the generator. Pupils would then compare and contrast how the choice of generators affected the solution paths. This may help enhance pupils' awareness that there are more direct paths to solve similar problems.

This study also showed that pupils had difficulties working with multiplicative relationships. Clement's (1982) pioneering work with university engineering students in America showed these students had difficulties providing the correct equation for the "students and professors problem":

> Write an equation using the variables S and P to represent the following statement: “There are six times as many students as professors at this university.” Use S for the number of students and P for the number of professors.

Clement’s study found that 68% of those who answered incorrectly represented the problem as $6S = P$. The explanation for these errors was that students tended to transliterate the information presented in the text – six students to every professor – became $6S = P$. MacGregor’s (1991) work with secondary students in Australia also showed similar misconceptions. However further research by MacGregor found that students continued to translate wrongly data presented in tabular or diagrammatic form. It seemed that students related the bigger value with the letter S. But the finding from this study is the reverse of these studies as pupils drew diagrams that were the reverse of the information provided. It would seem that the misconception exists even in visual form where pupils were not required to construct algebraic equations to model the situation. Wollman (1983) and MacGregor (1990) separately conducted teaching experiments in America and Australia respectively. Part of the teaching required students to express verbally, the different ways of describing the existing relationship. Besides asking pupils to draw models to represent problems, perhaps Singapore teachers could provide pupils with different types of models representing various situations and pupils are asked to describe the relationships that exist between the different unknowns.

This study also found that when pupils had difficulties understanding the demands of the questions they used the numbers as they appeared in the questions ignoring existing relationships. The above teaching experiment could also be used to help such pupils. Models representing situations such as those in Questions 1, 2 and 3 could be presented to pupils who then have to offer problems to fit the given models and they had to use the comparative terms to describe the relationships suggested by the rectangles.

Although the model method offers pupils a chance to circumvent operating with fractions directly, even so pupils had to utilise their knowledge of fractions to draw the models accurately. This study showed that pupils had difficulties working with relationships with three variables and where one relationship involved fractions. Methods used by pupils who successfully solved Question 4 (see Figure 9) offer a possible means to help solve similar problems. From these pupils’ work, it was important to identify the one whole and then to work out the part-whole relationships based on this one whole. It is possible to draw the appropriate model once the whole and the part-wholes were identified.

In conclusion, the model method is a powerful aid to solving word problems when pupils have no knowledge of formal algebra. The model method offers mathematics teachers a useful teaching aid as its very visual nature allows teachers to identify some of the difficulties pupils have with word problems. Teachers could capitalise on the information provided by pupils' models to help successful pupils and those who have difficulties with word problems to improve their skills with such word problems.

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