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The Development of Algebra in the Elementary Mathematics Curriculum of V.V. Davydov

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Abstract: A comparison of the development of algebra in Davydov's elementary mathematics curriculum with the approach to algebra advocated by the National Council of Teachers of Mathematics in the US reveals striking differences. Rather than developing algebra as a generalization of number, Davydov's curriculum develops algebraic structure from the relationships between quantities such as length, area, volume, and weight. The arithmetic of the real numbers follows as a concrete application of these algebraic generalizations. The instructional approach, while similar to constructivist teaching methodology, emanates from a very different theoretical perspective, namely, the findings of Vygotsky and Luria that cognitive development is enabled by overcoming obstacles for which previous methods of solution prove inadequate. In a study in which the entire three-year elementary curriculum of Davydov was implemented in a US school setting, children using the curriculum developed the ability to solve algebraic problems normally not encountered until the secondary level in the US.

Introduction

Currently in the United States we are concerned about how to provide early algebra experiences for elementary school children that will prepare them for the formal study of algebra later at the secondary level. However, Russian psychologists and mathematicians, working from the psychological theory of L.S. Vygotsky, developed a very different elementary mathematics program. Rather than deferring the study of algebra to middle school or high school, they focus elementary school instruction on the order and algebraic structures of the real number system. One of the authors (Schmittau) directed a study of the elementary mathematics curriculum developed by V. V. Davydov, including the translation, editing, and preparation of the curriculum, and its implementation in a school setting in the northeastern United States. At the completion of the three-year curriculum (the duration of elementary school in Russia), the US children were able to analyze and solve algebra problems normally not encountered until the secondary level in the US. Examples illustrative of their work appear later in this paper.

In the elementary mathematics curriculum developed and researched by Davydov and his colleagues (Davydov, Gorbov, Mikulina, & Saveleva, 1999a, 2000a;

Davydov, Gorbov, Mikulina, Saveleva, & Tabachnikova, 2001), the study of algebra precedes the study of arithmetic. Algebra is developed from an exploration of quantitative relationships (i.e., *relationships between quantities*). The concept of number and actions on numbers (i.e., arithmetic), are also developed from relationships between quantities. But arithmetic is developed from a very different vantage point than it is in the US, enriched both by the level of generalization that is afforded by algebra and by the simultaneous development of ideas about relationships between quantities. Thus, while children in the US have pre-algebraic experiences that are numerical, Russian children studying Davydov's curriculum have *pre-numerical experiences that are algebraic.*

It is important to note also that Davydov's goals transcend the mathematical content of the program. In the spirit of Vygotsky, for whom learning leads development, Davydov's overarching goal is the children's *cognitive development*. Vygotsky distinguished between scientific or theoretical concepts, which he contrasted with spontaneous or everyday concepts. The latter are formed through children's everyday encounters with their environment, while the former, which transcend the empirical level of interaction, require a theoretical generalization and consequently, pedagogical mediation, for their appropriation.

Davydov (1972/1990), following Hegel, believed that keeping a child at the level of empirical approaches for too long hinders the development of theoretical thinking. Theoretical thinking involves both the apprehension of a concept at its most general and abstract level, and the application of generalized understandings to particular cases. The difference between theoretical and empirical generalization is illustrated by the diurnal cycle, which by appearances is caused by the sun rising and setting (Lektorsky, 1980/1984). This never happens, however; it is the earth's rotation on its axis that produces this apparent phenomenon. Nor can one attain to the theoretical or scientific understanding of the diurnal cycle by observing the sun's apparent path through the sky, that is, by its empirical characteristics. Theoretical and empirical generalizations are, in fact, qualitatively discontinuous (Davydov, 1972/1990).

It is important to remember that Vygotsky considered only theoretical or scientific concepts (which are not confined to the realm of natural science) to be true concepts. Thus, Davydov's curriculum has as its overriding goal the development of the ability to think theoretically, which then enables one to attain an understanding of mathematics concepts at their most abstract and generalized level. Mathematics concepts by their nature come under the designation of theoretical or scientific concepts (Schmittau, 1993a). Consequently, if mathematics is to be understood at its deepest structural level, children must learn to think theoretically, that is, to go beyond the empirical aspects of phenomena. Educators in the US, influenced by Piagetian "levels of development," have assumed that students were not capable of abstract thinking until their adolescent years. However, Russian psychologists working in the Vygotskian tradition, assert that from the age of about seven, children are capable of developing the ability to think theoretically. The curriculum is an important mediator in this process.

Another important mediator is the instructional pedagogy. When one of the authors researched the mathematical understandings of Russian children who had experienced Davydov's elementary mathematics curriculum (Schmittau, 1994, 2003), she noted the problem solving focus of the teaching methodology evidenced by the Russian elementary teachers who were using Davydov's curriculum. The curriculum itself (which Schmittau used as written in the implementation with US students) consists of nothing but a carefully developed sequence of problems, which children are expected to solve. The problems are not broken down into steps for the children, they are not given hints, and there is no didactic presentation of the material. There is nothing to read but one problem after another. The third grade curriculum, for example, consists of more than 900 problems. Teachers, in turn, present the children with these problems, and they do not affirm the correctness of solutions; rather the children must come to these conclusions from the mathematics itself. The children learn to argue their points of view without, however, becoming argumentative. While this instructional methodology bears some resemblance to "constructivist" teaching, the approaches emanate from very different theoretical perspectives. Vygotsky and Luria (1993) found in their studies of the development of primates, "primitive" peoples, and children, that cognitive development occurs when one is confronted with a problem for which previous methods of solution are inadequate. Hence, Davydov's curriculum is a series of very deliberately sequenced problems that require children to go beyond prior methods, or challenge them to look at prior methods in altogether new ways, in order to attain a complete theoretical understanding of concepts. More importantly, their consistent engagement with this process develops the ability to analyze problem situations at a theoretical rather than an empirical level, and thus to form *theoretical* rather than *empirical* generalizations, which is the distinguishing feature of Davydov's work.

Relationships between Quantities, Equations, and Symbolic Expressions

The grade 1 curriculum begins with an exploration of quantitative properties of real objects. Length, width, volume, area, and weight, are examples of "quantities." Children learn to isolate and identify a single quantitative property of an object (e.g., the surface area of a glass) and to abstract it from the object's spatial orientation. They learn to compare objects and to determine whether they are equal with respect to some quantitative property, and to use a line segment model to represent the results of their comparison of quantities. Initially they draw two equal line segments, "**II**," to indicate that two objects are equal with respect to some property and two unequal line segments to show that they are not equal. Children then learn to use an uppercase letter to represent a quantitative property of an object, and to represent equality and inequality relationships with the signs =, ≠, >, and <. For example, they compare the lengths of two boards, name the lengths A and B, and represent the relationship between them as $A=B$, $A\neq B$, $A>B$, or $A<B$. There is no reference to numbers during this work: "A" represents the unmeasured length of the board.

Thus, a letter denotes a specific quantitative property of an object. The teacher and the students initially point to objects when they use letters and the children identify the quantitative property (length, area, volume) of interest. This is in itself an exercise in abstraction, as the quantitative property in question must be distinguished from all other empirically detectable aspects of the object. However, the use of letters is also designed to move children forward in their level of abstraction. The line segment model helps them to focus on, and abstract the idea of an equality relationship, but it still retains some of the features of the represented situation: both the represented situation and the representation involve two objects that are equal or unequal with respect to a quantitative property. In contrast, letters do not have any apparent connection to the situation with objects. The use of such symbolism helps children to move away from the consideration of specific objects, and to focus on the relationship itself. Then they will be able to study its properties. Ideally they will begin to conceive of the situation with objects as a concrete illustration of an abstract idea, as opposed to viewing the relationship only as a property of the specific objects themselves.

The students then study the properties of equality and inequality relationships. This includes an investigation of the transitive, reflexive, and symmetric properties for both the relations of equality and inequality. Early in the first grade they deduce the order structure of the real numbers (cf. Schmittau, 1993b for a more extended discussion). These ideas are developed well before the introduction of equation solving in the curriculum.

Efforts to move children forward in terms of level of abstraction are evident in many of the problems involving the properties of equality and inequality relationships. In one problem, for example (Davydov et al., 1999a), the students are shown a picture of two balloons. The volume of one balloon is labeled L; this balloon is completely drawn. The other balloon of volume P is only partially drawn. The problem says: If $L=T$ and $T>P$, then L____P. The students are unable to directly compare the volumes (one is only partially drawn) so they have to make an inference about the relationship between L and P. Children learn that in some cases, by using information about equality relationships between quantities, they can logically deduce new relationships without having to measure or build the quantities. This rather quick movement from working with concrete objects to sometimes applying abstract ideas to something that cannot be confirmed empirically is part of the curricular approach. Children begin to understand that valid mathematical generalizations can be applied to a class of objects without having to verify each case independently. Algebraic representations serve to create problems that require children to apply concepts to cases that cannot be empirically verified.

The students subsequently explore the idea that we can change an inequality relationship into an equality relationship by adding something to the smaller amount or subtracting something from the larger amount. They learn to represent the actions of adding and subtracting quantities with the signs '+' and '–.' They also determine, through a carefully developed set of problems, that it is the difference between the larger and smaller quantity that must be added or subtracted, and they designate this difference with a letter. The children may be presented with two wooden rods of different lengths, K and D, with K>D. The class decides that the lengths can be made equal by adding something to the smaller rod. A third rod, of length A, where A is equal in length to the difference between K and D, is placed next to the rod of length D. The relationship between K and D is now represented with the notation: K>D (by A). The new quantity (D+A) is denoted by a new letter, V. The students may suggest writing V=D+A or K=D+A. The process of moving from an inequality to an equality relationship with the actions of addition or subtraction is represented symbolically by: If K>D (by A) then K-A=D and K=D+A.

Next students are introduced to the idea of going from an equality relationship to an inequality relationship and back to an equality relationship. This is a continuation of the study of the properties of equality and inequality relationships. The teacher may tell a story about a boy named Jamie who had two puppies that were littermates and always were given equal amounts of food. The teacher displays two bowls of nuggets and writes H=G. Then she explains that while the puppies were outside playing, Jamie's mother changed the amount of food in their bowls, and recorded the change as: H+C __ G. What did Jamie's mother do to the bowls of food? The students say that the mother added some food to one of the bowls. They also say that the ">" sign should be filled in for the blank. Would Jamie be happy with this situation? The mother remembered that Jamie always treated the puppies equally and changed the food levels as follows: H+C ___ G+C. Would Jamie be happier with this new statement? In another problem, the teacher presents two equal containers of water and writes K=Q. Some water is poured out of one container resulting in the following volume statement: K–T<Q. Some water is then poured out of the other container resulting in the following volume statement: K–T>Q–C. Why did this not produce an equality? (Davydov et al., 1999b). The students answer that an equal amount must be added or poured out of both containers to result in an equality. We have to conclude that volume C was not equal to volume T. Actually we can draw a stronger conclusion, that is, T<C.

Measurement and the Concept of Number

Up to this point, children have compared objects directly, or they have made logical deductions about equality and inequality relationships using some property of these relationships (e.g., the symmetric or transitive property). These ideas are then combined to develop the idea of measurement. The teacher chooses some unmovable objects in the classroom. It is impossible to compare their lengths directly. The children must discover a way to compare them using some third quantity, a length of string perhaps. They compare the length of one object, T, with a length of string, R, and conclude that T=R. Then they compare the length of the second object with the length of the same string and write H=R. Students conclude that T=H. They find that the string is a “helper” or “intermediary” - that is, a third quantity that we can use to compare two other quantities. Direct comparison is often impossible, but we can deduce a relationship on the basis of known relationships. Children learn to make or select an intermediary, and to use it to find or build a quantity equal to a given quantity.

After this, the children are introduced to the measurement of quantities. In one exercise (Davydov et al., 1999b), two students leave the room. The teacher shows the remaining children two paper strips. One is the quantity to be measured and the other is a much shorter strip, which will constitute the measuring unit. (A “measuring unit” is a quantity that is used to build or measure other quantities.) The students measure the longer strip with the shorter strip and use objects (tokens) to record how many measuring units they laid off on the strip. The strip that was measured is then hidden and the two students return to the room. With no words exchanged, they are given the shorter strip and the tokens; using these, the two students must build (i.e., make) the quantity that was measured. Then they check their work by comparing their new length with the original. In the next problem, two students again leave the room. The remaining students measure one strip with another, but this time they hide the measuring unit (the shorter strip). They show the two students the quantity that was measured and the tokens used (Davydov et al.). The two students must find the size of the measuring unit. Instead, they tend to repeat the prior exercise and use the quantity they are given as the measuring unit. This problem is designed to reveal to students the inadequacy of attempting to discern what is measured by means of a quantity and the tokens used, and the need for some univocal form of representation of the problem situation. After checking, the teacher asks why the result is not correct. She tells the class that they need a way to write what they have done so that no confusion will result. The children all invent some notation to record the results of their activity, as a way of avoiding such confusion in the future. They share what they have written and the class discusses each attempt. Finally they agree on a notation such as the following:

$$\overset{///}{U \rightarrow B}$$

U stands for the unit, B stands for the quantity that is built or measured, and the tally indicates how many times the unit is repeated to make up the quantity B.

A series of creative problems prompts the children to move from using tokens to using tally marks as a way of keeping track of the laying off of a measuring unit, and then to replacing tally marks with word labels. From this, the idea of the counting sequence is developed, as a tool for labeling how many units make up any particular quantity. Then the concept of number is developed as a relationship of quantities. Two notations are used to express this relationship, B = 5U or B/U = 5, meaning that there are 5 units in quantity B. Thus children learn to interpret a number as expressing a relationship between two quantities, a definition that can easily be extended to fractions later (cf. Morris, 2000; Schmittau, 1993b).

The children learn that we can compare quantities by comparing the numerical values that have been assigned to the quantities, as long as they have been measured with the same unit. The previously developed ideas about equality and inequality relationships are then extended to the numerical values of quantities with the help of the number line. (The idea of a number line is carefully developed prior to this topic.) For example, the children may be presented with two containers of different shapes. One container of volume H has 6 measuring units, U, of water. The amount of water in the second container, G, is unknown. Students measure volume G using the measuring unit U, and write G=4U, H=6U, 6U>4U by ___? They make a number line with an arc between the beginning of the number line and 4. Another arc is made between 4 and 6, and is marked with a question mark. The children use the number line to determine the answer and fill in the blank with 2U. Next the teacher asks students how to make 6U>4U into an equality. She says that the students can use addition or subtraction. Some students work with the number line, some with the formula, and some with objects. The students determine that if 6U>4U by 2U, then 6U=4U+2U and 6U–2U=4U. Work such as this is an extension of their previous work with quantities. Then they learned to represent the move from an inequality to an equality relationship with the notation: If A>B (by C), then A=B+C and A–C=B. The idea and the notation are applied and extended here to the numerical values of the quantities. To move from an inequality relationship to an equality relationship, we still have to subtract the difference from the larger quantity or add the difference to the smaller quantity. However, now children describe the quantities in terms of a number of units, whereas prior to this, they were simply designated with letters.

These ideas are also applied to purely numerical statements—for example, 4 + __ = 9. Although this looks like a typical first grade problem, the meaning that the children give to the equation is affected by the prior work with quantities. The number 4 is interpreted as the numerical value of the smaller quantity, and 9 as the numerical value of the larger quantity. There is an inequality relationship between these quantities. In order to move from an inequality to the desired equality relationship, we have to add the difference to the smaller quantity. Children draw an arc between the beginning of the number line and 4 to show the quantity whose numerical value is 4; this is the smaller quantity. They draw another arc between 4 and 9. This length is the difference - the quantity that must be added to the smaller quantity in order to move to an equality relationship.

This kind of interpretation of statements of equality is evident throughout the curriculum: Children interpret statements of equality in terms of relationships between quantities, and in terms of actions on quantities or arithmetic actions on the numerical values of measured quantities. The meanings of the indicated actions and relationships are derived from work with quantities, abstracted, and applied to work with numbers.

In summary, the first portion of grade 1 develops children's understanding of an equality and inequality relationship, the properties of these relationships, how to move from one relationship to the other by adding or subtracting, how to represent these relationships and actions with literal symbolism, and the relationship between the literal formulas A=B–C and A+C=B. Children also learn to interpret a number as expressing a relationship between two quantities, an interpretation that will extend to fractional numbers and eventually irrationals. Ideas about quantitative relationships are initially developed by working with unmeasured quantities, and are then extended to measured quantities which generate numbers as their measures.

Part-Whole Relationships

The next topic involving quantitative relationships is part-whole relationships. The curriculum first develops the idea that a quantity may be composed of parts. Children may be shown three pattern block parallelograms. They label their areas P, R, and M. Next they put the three parallelograms together to form a hexagon.

Students name the area of the hexagon Y. The children then represent the situation with a schematic that indicates that Y is the whole and P, R, and M are the parts:

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    Y
  / | \
 P  M  R
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The schematic also suggests the action of composing the whole from its parts.

Additional problems may be used to introduce the idea of decomposing a whole into parts (moving in the reverse direction along the schematic). After many problems that involve identifying part-whole relationships among concrete objects, students analyze story texts in terms of part-whole relationships. For example, texts such as the following may be presented for analysis: A bowl contained K apples. T apples were eaten by the children after school. M apples were left. Another text might read: There were V kg of carrots in a bag. W kg were added, after which C kg of carrots were in the bag. The children are asked to analyze the relationships between the quantities in the stories and make a drawing to model them. They find it helpful to draw the "/\" schematic. K is the whole, and T and M are parts in the first story; in the second, C is the whole and V and W are the parts.

Next the children learn to interpret a literal formula of the form A+B=H as representing the composing or the building of the whole from parts. For example, children may be presented with two strips of paper, having lengths they label as R and A. On the board the teacher draws a "/\" schematic with Y as the whole and R and A as the parts. She asks the students how they can find Y. The students decide that they must add R and A to find Y. This building or finding of the whole is represented by the formula Y=R+A. There is a similar treatment of the idea of finding a part. The teacher has some volume of water, H, in a glass. She draws the "/\" schematic with H as the whole and C and M as the parts. She says that the part C is known. The students must find M. How can they find it? The students conclude that they must pour out C, and M will be left. They write the formula M=H–C to describe this action.

These interpretations of literal formulas are compatible with the prior interpretation of the formulas in the treatment of equality and inequality relationships. For example, for the last two problems, since Y>R by A, Y=R+A, and since M<H by C, M=H–C. Overlaid on this, however, is the interpretation of the quantities involved in a part-whole relationship (e.g., H, M, and C) in terms of parts and wholes. In the second case, for example, M is the smaller quantity, but now it is also a part. H is the larger quantity, but now it is also the whole. C is the quantity that is equal to the difference between the smaller and the larger quantities, but it is also the other part.

Additional problems develop the children's ability to find a missing part or a missing whole, and to interpret and write literal formulas that describe the required actions. Children might be given a large cardboard square (of area A) comprised of two small right triangles (having areas B and C), and a larger right triangle (of area D). A "/\" schematic is drawn with A as the whole and B, C, and D as the parts. The teacher writes on the board: A–B–C=? Students first guess and then check their answer (D) using the triangles. Then the students try different ways of removing (subtracting) the triangles, such as A–D–B=__, and A-C–B=__. Next students are presented with literal formulas or schematics, and asked to make up story texts for the formulas or schematics. The following problem is illustrative. A schematic may show *d* as the whole and *s* and *a* as the parts. Students have to complete the formulas and to write a story text for each formula. For the formula __ - __ = __, for example, *d* must be filled in for the first blank, and the parts *s* and *a* must be filled in for the remaining blanks in any way (e.g., $d – s = a$). Children learned to interpret a formula of this form as representing the finding of a part. Thus, they might write the following story text for the formula $d – s = a$: "There were *d* birds in the tree. *s* birds flew away. *a* birds were left."

Then a part-whole relation is presented in which one of the parts is unknown. For example, the children may be presented with a rod labeled B, and two other rods labeled A and T. A "/\" schematic is drawn on the board with the whole labeled "B" and "8," and the parts labeled "A?" and "T?". The children are asked what numbers they might get for A and T if they measure the length of each. A child or the teacher may suggest a length of 9 for A. The children argue that a part cannot be bigger than the whole. The students suggest several numerical values for A and T. They find that if A has a length of 5, then T's length must be 3. If the length of A is 6, then the length of T must be 2, and so forth. The teacher points out that the children suggested many values for A. However, once they had chosen a value for A, the children determine that there was only one possible value for T. These types of problems develop the idea that, if there is a part-whole relationship between three quantities, and two of the quantities are known and the third is unknown, then it is not necessary to measure or build the unknown quantity; it can be figured out "in our heads." Children begin to adopt the convention of designating a missing quantity (i.e., an unknown part or whole) with the letter *x*.

The students may also be shown cards on a board that say, for example, $x – K = C$. On the reverse side of the *x* card is the number 19, on the reverse side of the *K* card is the number 8, and the *C* card 11. The students know that if they know two quantities they can then find the third using their heads (i.e., performing actions on objects is not necessary). They must solve for *x*. From their prior work, they know that to find a missing whole, they should add the parts. When students determine that $x = C + K$, they can display the numbers for cards K and C by turning them over. Now the equation looks like this: x=11+8. Turning over the x card enables the students to check their work.

Other problems develop the idea that for a given part-whole relationship involving three quantities, three different equations can be written because each of the three quantities can be the unknown. Given an equality such as K+C=D, children may be asked to place x wherever it may represent a quantity. Knowing that any of the quantities can be an unknown, the children note that they can make three separate equations, viz., x+C=D, K+x=D, and K+C=x. For each of the equations, the children must choose some numerical values for the known quantities. For each set of values the children come up with, they have to solve for the third quantity. For example, for the equation x+C=D, a child might suggest making C equal to 6, and D equal to 8. Thus the equation becomes x+6=8. Next children solve the equation. The missing quantity is a part. To solve for a part, the other part must be subtracted from the whole. Thus the plan for the solution is x=8–6. Then the children perform the calculation: x=2. Such problems follow easily from previous work with quantities.

To summarize, children first learn to write literal formulas that describe part-whole relationships. They learn to write literal formulas that describe the actions that need to be performed to find a missing whole and a missing part. Next, literal formulas are converted into literal equations by making one of the quantities unknown. Children solve the equations by using the previously developed ideas about finding a missing part or whole: They concluded in their prior work that to build or find a whole, they should add the parts, and to find a part, they should subtract the other parts from the whole.

Word Problems

The students also determine that by making one of the quantities in a story text unknown, the story text can be converted into a story problem. The question of the story problem asks the solver to find the missing quantity. Children learn to analyze and solve story problems by applying all of their understandings about part-whole relationships. A story problem such as the following is illustrative: Mark has some baseball cards. His brother gives him another 7 cards. Now Mark has 11 cards. How many cards did Mark have originally? Children may be asked to determine which of the following equations describes the problem, and then to solve the equation.

$h + r = t$	$h + r = t$	$h + r = t$
11 7 x	x 7 11	7 x 11

The equation x+7=11 denotes the *actions performed on the quantities*. Thus the second equation is correct. In order to analyze the part-whole relationship in the problem, children draw a "/\" schematic. This enables them to identify the missing quantity and determine the appropriate *arithmetic action* for solving the problem. Since the missing quantity is a part, they need to subtract. Thus x=11–7. Finally, the children may be asked to write story problems for the remaining equations, 11+7=x and 7+x=11.

Children solve many addition and subtraction word problems by analyzing the relationships between quantities in the problem situation. For example, they may be given a problem such as the following: *m* children were skating when some more children came to the rink to skate. Then there were *t* children on the rink. How many children came to skate? The children perform both a *quantitative* and *arithmetic* analysis similar to that of the preceding problem. They write the equation that describes the actions in the problem: $m+x=t$. And they write the equation $x=t–m$ as the solution.

Children are also asked to analyze equations. For example, they may be asked to make the schematics for equations such as the following and to decide which schematic differs from all the others: $a+x=n$, $x–a=n$, $x+a=n$, and $n–x=a$. In another example of this type of analysis, children may be asked to determine for which of the following equations $x=c–n$ is the solution plan: $x–c=n$, $c+x=n$, or $n+x=c$. Students make a schematic for each of the equations as a means of analyzing the problem.

As part of the work of learning to write an equation that allows us to find a missing quantity without measuring or building the quantity, the children study another idea. They are given a problem (Davydov et al., 1999a, p. 148) with the formula B+C=H, and a picture depicting some water in a barrel, to which more was added, and then there was a volume H in the barrel. The symbols *x*, 15, and 42 under the letters indicate that these are the measured volumes of the quantities B, C, and H. The children are trying to find an unknown part, so they should subtract in their actions with numbers: x=42–15. In two subsequent problems, the literal formulas, K+T=H and P+Π=A (with 18, *x*, and 34, and 19, 14, and *x* appearing below the terms in the first and second formula, respectively), again indicate that water was added each time. However, one equation, x=34–18, calls for subtraction, and the other equation x=19+14 calls for addition. The text points out that in every case water was *added*, and questions why sometimes we might add and sometimes subtract in our actions with numbers. The students respond that it depends on whether we are trying to find a whole or a part. A conclusion is drawn: *Actions with numbers may not match actions with objects*. To find the right action with numbers the children find it helpful to use a schematic. If they are trying to find a whole, they know that they must combine the parts (add). If they are trying to find a part, they need to subtract the known part(s) from the whole.

Following these problems, another set of problems is given (Davydov et al., 1999a, p. 149), in which three literal formulas indicate that, for a piece of wire, a length was cut off each time. However, by varying the position of *x* in the literal formula, the unknown quantity is first a whole, then a part, and another part. The children are instructed to write equations of the form *x*=______ for each of the three problems; these involve addition, subtraction, and subtraction respectively, despite the fact that all of the situations with objects involved cutting off or subtracting a piece of the wire. Again, the text queries children as to why this is the case. Such problems illustrate the level of analysis required by this curriculum. First there is the level of conceptual analysis required to design such a curriculum, and then there is the manner in which children are challenged to closely attend to and analyze every aspect of the conceptual content of a problem situation.

Other problems develop the idea that three different story problems can be written for a single story text that involves a part-whole relationship between three quantities. For example, children may be told to make up a story about muffins using the formula $c–b=n$. They might write: There were *c* muffins. John ate *b* muffins and there were *n* left. Next they may be asked to make up three problems from their story using the numbers 12 and 4, and substituting *x* for first *c*, then *b*, and finally *n*, in the equation c-$b=n$. Hence, the students should make *c* the unknown in their first story problem, *b* the unknown in the second story problem, and *n* the unknown in the third. In addition, they should substitute 12 and 4 for the known amounts in an appropriate way. Thus, for the first story problem, they might write, "There were some muffins. John ate 4 muffins and there were 12 left. How many muffins were there to begin with?" They complete the equation for each of their stories. Then they solve the equations.

Varying the position of *x* in an equation like $c–b=n$, and asking children to create a story for each equation, develops the idea that the particular relationship between the quantities, and the number of quantities that are involved, together constrain the number and types of problems that can be written. Such problems also reflect the goal of developing theoretical comprehension. Some very general and abstract ideas about relationships between quantities, and about the nature of mathematics problems are also being developed.

Other Important Algebraic Ideas

The remainder of the grade 1-3 curriculum will not be described in such detail. Instead examples from the remainder of the curriculum will be used to show how these grade 1 ideas are extended to develop certain "big algebraic ideas" in the remainder of the curriculum. (Equally important, the examples illustrate how important algebraic concepts and representations – letters, equations, and relationships between quantities - are used to develop other mathematical ideas.)

For example, many ideas about equations are developed in grade 2 by building on ideas about part-whole relationships. For instance, the teacher presents two containers of water, of volume A and volume B. The water from the second container holding volume B is poured into the first container. Now there is a new volume of water in this container, the whole. This action is described by the formula A+B=C. The teacher then says that she had obtained volume B by pouring together two volumes K and E that were in two containers which she shows the students. The children formulate volume B as K+E=B. Now the teacher reconstructs the experiment: She pours volume B out of volume C. Then she pours the water of volume B into volume K and volume E. She asks whether it was necessary to pour together volumes K and E first in order to make volume C. The students discuss the question and conclude that the answer is no. The students discover that they can make volume C in the following ways: A+K+E=C, or A+B=C (Davydov et al., 2000b). They conclude that a quantity can be added as a whole or as several parts.

A subsequent problem may present a situation in which three children have wooden rods originally *k* cm long. One girl, Mary, added a 6 cm Cuisenaire rod to her original rod, took a break, and then added a 4 cm rod. Altogether the wooden rod length was now 36 cm. This action is represented by the equation k+6+4=36. The other children added to their rods differently. The equations that represent their work are k+10=36 and k+5+5=36. Who produced the longest wooden rod? The students conclude that everybody added Cuisenaire rods of the same total length. They might work with rulers or Cuisenaire rods and see that the length of all the rods changed by 10 cm. The difference is that one of the children added that as the whole and the others as two parts. The class may then be asked to find one more way to add the same length: $k + \square + \square = 36$. The idea that an amount can be added as a whole or as parts is applied to the problem of solving equations and performing calculations.

Similarly, the children learn that an amount can be subtracted in parts or as a whole. There are three containers on the table with volumes A, K, and C. The class uses water to determine the relationship between the volumes. It is K+C=A. Now the teacher puts out one more container B (larger than A). The teacher pours out volume A from volume B. This action is represented with the formula B–A=T. Now the teacher asks whether the formula B–K=T is true. The students conclude that it is not, and suggest using another letter instead of T, writing, for example, B–K=G. Next the teacher asks the students to determine other ways to get volume T without using container A. Their solution is to pour out K, then to pour out C: B–K–C=T.

After the discussion, the students check this answer in an experiment with the water. Now is B–C–K=T also true? The students get the answer to this question through an experiment trying to explain their actions. Indeed, C and K together equal A. The conclusion is made that a quantity can be subtracted as a whole or as several parts (Davydov et al., 2000b).

Children can now be asked to find the correct numbers in a problem such as the following:

$$m - 54 - 10 = 82$$
$$m - 44 = \underline{\quad\quad}$$
$$m - 40 - 24 = \underline{\quad\quad}$$

The students may make some incorrect deductions. They may decide, for example, that the answer to the second problem is the same as the first because 54–10=44. The class discusses why this is or is not correct. After considerable argument they conclude that 54 and 10 are both parts and that it is not correct to take away a part from a part. This problem again illustrates how children interpret equations. Equations, even when they include numbers, are interpreted in terms of actions on quantities, and relationships among quantities. Children also understand how arithmetic actions are related to actions on quantities.

In grade 2, other ideas about equations are also developed by building on ideas about part-whole relationships. As part of the study of multiplication, there is a lengthy study of wholes that consist of equal parts, wholes that consist of unequal parts, and wholes that consist of both equal and unequal parts, and these ideas are related to equation writing and solving. For example, students may be asked to make equations for schematics such as:

```
    54              47
   /  \            /  \
  /\    x        9   / | \
 8  8                x  x  x
```

In these schematics, the unknowns are parts and the whole has been divided into equal and unequal parts. Students write these equations as: $(8 \bullet 2) + x = 54$ and $(x \bullet 3) + 9 = 47$. As indicated in the schematics, the "8" in the expression $8 \bullet 2$ is the number value of the part, and 2 is the number of parts. Similarly in the expression, $x \bullet 3$, x is the number value of the part, and 3 is the number of parts. (Students also develop the commutative property, and consequently know that $x \bullet 3$ may be represented as $3 \bullet x$.) Thus, part-whole ideas are used to develop more sophisticated equations. (Equations involving multiplications and divisions are introduced in grade 2, and the development is similar to the development of equations involving addition and subtraction.)

Other problems, such as the following, require the identification of different types of quantities in an equation. Second graders, for example, may be asked to identify the whole and the parts as well as the products and factors in equations such as: (a•4) – d = m, k + (t÷2) = m, (v÷6) – 18 = w, (m-2) = b – 7, and (k•2) + t =m. The point of problems of this sort is to help children to look past the surface features of equations and attend to the underlying structure: What are the parts and whole? What are the products and factors? Students solve complicated equations by analyzing the underlying structure of quantitative relationships, and then applying their knowledge about how to solve for parts and wholes, and for products and factors.

Second graders are introduced to two–step word problems that involve both a difference comparison and a part-whole relationship. For example, students might be told that a cyclist traveled 25 miles in the morning and 9 miles less than his morning distance in the afternoon. They must determine how many miles he traveled that day. This is a common approach in the curriculum: After studying basic relationships between quantities in depth and comparing them (e.g., part-whole relationships and the comparison of two quantities in terms of a difference), the basic quantity relationships are combined to produce more complex problems. Hence, students must extend their analysis to the quantitative relations present in these more complex situations.

With regard to the use of literal symbolic designations, the problems above have shown a progression from using uppercase letters as specific quantities, to using lower case letters as unspecified numbers (e.g., *a* and *b* in equations such as $a–b=x$) and unknown numbers (x in equations such as $a–b=x$). In many grade 1 and 2 problems, letters that represent unspecified numbers are replaced by several different numerical values. For example, students may suggest several numerical values for *a* and *b* in an equation such as a-b=x. For each of the values that the class suggests, the students have to find the corresponding numerical value for x. This helps to develop the idea that letters like *a* and *b* can be replaced by different numerical values. There may be constraints on the possible substitutions for *a* and *b*, however. For example, the whole must be bigger than the parts and a given numerical value may not make sense for a given contextualized situation. These kinds of activities prompt students to begin to interpret letters like *a* and *b* as standing for different numerical values. The idea that a letter can stand for any number is then developed in grades 1 and 2. For example, in grade 2, the teacher and the children work together to derive the associative law that $a•(b•c) = (a•b)•c$, in which the letters stand for “any number.” In grade 3 (Davydov, Gorbov, Mikulina, Saveleva, & Tabachnikova, 2001), letters are used to denote variables. Children use letters to denote unspecified values, unknown values, and variables, even within the same problem.

Proportional Reasoning

Consider the following grade 3 level problem: “At 8:00 in the morning a motor vessel (ship) left a pier. At the same time from another pier a motor boat moved toward the vessel. The ship traveled at a speed of 35 km/hr, and the motor boat at 40 km/hr. The distance between the piers is 150 km. When did they meet? How many kilometers farther than the ship did the boat travel?” (Davydov et al., 2001). A US student’s solution is shown in Figure 1. The student solved the problem by organizing the information in a table, first entering the specific values of the boats’ speeds and the total distance between them (filling in the 35km/hr, 40km/hr, and 150 km cells in the table). The child then determined that the times of travel of the two vessels would be equal, and designated this with a vertical “=” sign between the two times, then entered an *x* for each, since this was the unknown quantity that must be found. The next entries were the 35•*x* and 40•*x* for the two distances traveled.

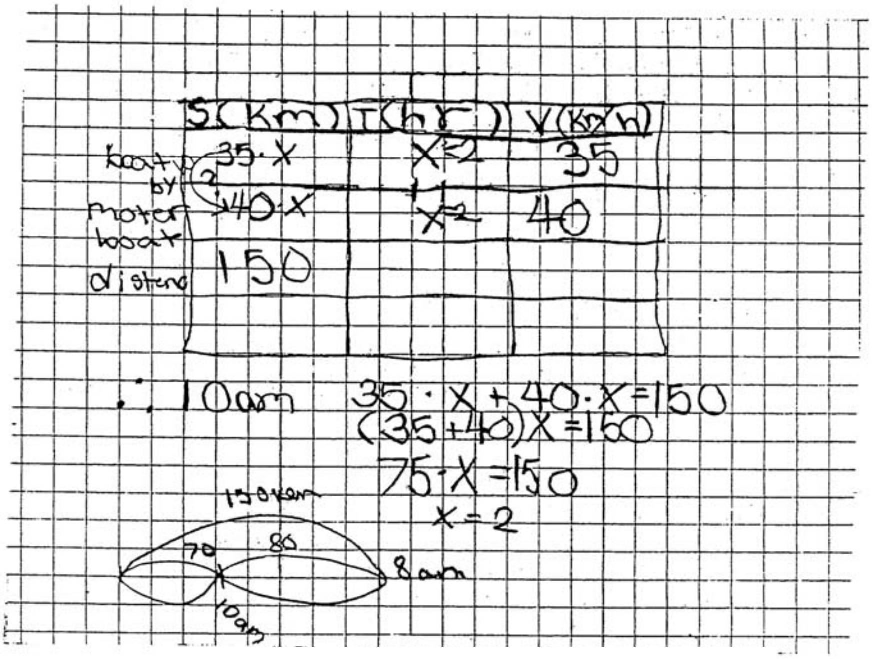

Figure 1. Solution involving proportional reasoning

The child then reasoned that 35•*x* + 40•*x* = 150 since the sum of the parts must equal the whole or total distance covered. The student applied the distributive property and solved for the time, and this solution appears below the table. Knowing that to solve for a factor, the product must be divided by the other factor, *x* = 150÷75 = 2. Finally, the part-whole relationship between the distances traveled was represented in a schematic, and the child concluded that the boats would meet at 10:00 A.M.

The child also drew an arrow from 35•*x* to 40•*x* and labeled the arrow "by ?" This notation shows the comparison of two quantities in terms of a difference; an arrow labeled 'by *x*' means that the bigger quantity is bigger by *x* units than the smaller quantity, and the arrow is always drawn from the smaller quantity to the bigger one. The child is indicating that the value of '?' will be the answer to the second question, "How many kilometers farther than the ship did the boat travel?"

A problem such as this is a typical algebra problem. An analysis of the requirements of the problem can help the reader to understand the conceptual analysis that underlies Davydov's curriculum. That is, the following ideas are required to solve this type of problem, and each of the ideas is developed in the curriculum prior to the introduction of such problems.

1. *Quantity*: Children have to understand the idea of quantity. In this problem, there are some quantities - some distances (lengths), times, and speeds.
2. *Measuring Unit*: Children have to understand the idea of a measuring unit. This is also an idea about quantities. There are two measuring units in this problem, hour and kilometer, and a more complicated measuring unit, km/hr. Students have to understand the relationships among measuring units in a system of measuring units, and the relationship between measuring units that is involved in a rate of change (such as km/hr). This is necessary in general for working with denominate numbers.
3. *Relationships between quantities*: In many mathematical problems, children have to be able to think about, and to represent relationships between quantities. In this problem, there is a part-whole relationship between the distances. Explicit recognition and denotation of the part-whole relationship between these quantities will help students to solve the problem. The child represented the part-whole relationship with a schematic. There is also an equality relationship between the times of travel that children have to recognize in order to solve the problem. The child also represented this relationship in the table. Students have to understand the idea of a changing quantity, and the idea that there can be different kinds of relationships between changing quantities. In this case, they have to understand that there is a proportional relationship between the changing quantities S and T. Tables are used to represent these types of

relationships. There is also a comparison of two quantities in terms of a difference, and the child represented this with the arrow marked "by ?." Finally, there is another quantitative relationship in the problem involving the measuring units for S and T.

4. *The meaning of numbers*: In this problem, children have to attend to both the numbers and the measuring units. They should understand what a number means; for example, the various numerical values for S and T are a result of the units with which they were measured. Students have to distinguish between the quantities in this problem (distance, time, speed), and the known and unknown numerical values of these quantities. They also have to be able to relate ideas about numbers and ideas about quantities. For example, children have to understand that the unknown numerical values of the quantities can be figured out without measuring or building the actual quantities, by using the relationships between the quantities involved and the known numerical values of the other quantities.
5. *Proportional relationships*: In this problem, students have to understand that there are some quantities that change during the process of motion: Distances change and times change. Students have to understand that, for each of the boats, there is a proportional relationship between S and T. In order to understand proportional relationships, children must understand the idea of comparing two quantities in terms of a multiple. The Davydov curriculum treats this idea in depth in grades 2 and 3. The idea of a changing quantity and the idea of a proportional relationship are developed in grade 3.
6. *The speed of uniform motion*: Students have to understand that the speed of uniform motion is a constant. The work with measuring units in grades 1 and 2 is built upon in grade 3 to develop the idea that speed shows how many measuring units of S (distance) correspond to one measuring unit of T (time). The children are used to thinking about measuring units and ratios that express the relationship between quantities, so they are prepared to think about this new type of relationship between two measuring units.
7. *Letters and variables*: Students have to be able to use a letter to designate the unknown time, and to designate variables. They have to understand the concept of a variable.
8. *Equation and equation solving*: Children have to be able to formulate an equation that shows the relationship between the quantities. They have to be able to solve the equation.

This analysis is intended to illustrate how the curriculum, beginning in grade 1, carefully lays the groundwork for more advanced problem solving. Complicated problems like this require many component understandings. Davydov and his colleagues identified foundational concepts, such as quantity, direct and indirect measurement, units, and part-whole relationships, that underlie many mathematical ideas. The children's understanding of these conceptual antecedents is thoroughly developed. As new topics are introduced, these foundational concepts are the constituent understandings that are needed to understand the new topic. Students are never confronted with a complex topic such as proportional reasoning unless all the component ideas are carefully developed well before its introduction.

The reader will recall that in introducing word problems, children were first presented with story texts without any questions, required to analyze the relationships between the quantities in the story text, and then later asked to formulate a question to convert the story text into a problem. So too, in the beginning of grade three, students are presented with texts that refer to quantities and specific values of the quantities. They are now asked to simply organize the information presented into tables according to various categorizations (such as, by weight or volume, for example). Later, questions can be formulated that turn these texts about specific quantitative values into problems in which some value must be found. Children find it useful to represent the quantitative information in tabular form, and to analyze the relationships between quantities to solve the problem.

Next children study various "processes," such as work, moving, buying-selling, and compiling a total from parts. These processes provide examples for studying quantities that change. Through problems, the class comes to the following conclusions: During the process of work, time and the amount of work change. During the process of moving, time and distance change. In the process of buying-selling, the cost and the amount of the goods change. In the process of compiling a total from parts, the total and the number of parts change.

The next idea that is developed is that there can be different kinds of relationships between these changing quantities. The curriculum distinguishes between a uniform process (in which the variables are in direct proportion) and a non-uniform process. Children can create tables of each type of process, using weights, or lengths, or volumes, for example. Or they may be presented with a table of values and asked to determine whether the process that generated the values was or was not uniform.

The children then solve many problems, similar to the following, which involve proportional relationships: Over a seven-day period, a restaurant used 28 gallons of milk. How many days will 48 gallons of milk last, if customers drink it at the same rate? The teacher asks the students to name the variables in the process, which they identify as number of days and volume. The students build a table for this process, and solve the problem.

The speed of processes is also introduced. Finally there are multi-step problems that combine part-whole relationships, proportional relationships, and multiple and/or difference comparisons—i.e., most of the ideas involving quantitative relationships from grades two and three. Problems such as the following are given: Amy bought 12 packs of gum and 5 bags of candy for a party. Packs of gum cost C dollars each, and a bag of candy costs three times as much as a pack of gum. What did she pay for all the candy and gum?

In summary, component concepts, skills, and methods of analysis that are needed to solve proportional reasoning problems are built up gradually. For example, children need to learn to pick out the variables in such problems, and there are numerous problems that develop just this ability prior to the introduction of proportional relationships. However, proportional reasoning also builds on the foundation of two years work, including extensive work with multiple comparison. In grade 3, by combining proportional relationships and other basic quantitative relationships within a single problem, the curriculum develops the ability to analyze and solve complex problems that require multiple steps.

How Algebraic Representations and Schematics Serve to Accomplish the Curriculum Goals

In a traditional elementary school curriculum, there are representations for numbers and actions on numbers, but there are very few representations for ideas about quantities. Because of the focus on quantitative relationships in the Russian curriculum, there are various ways to represent quantities, relationships between quantities, and actions on quantities, as well as representations for numbers and actions on numbers. There is some overlap in the representations for relationships between quantities and numerical relationships, and for actions on quantities and actions on numbers. For example, the "/\" schematic is used to represent part-whole relationships between quantities and between numbers. The symbol "+" is used to denote the action of adding quantities and the action of adding numbers. The use of the same representations helps to extend ideas from quantities to the numerical values of the quantities. For example, if we add parts A and B we obtain the whole quantity C; then by adding the numerical values of the parts we obtain the numerical value of the whole.

The relationships between quantities that are represented include equality and inequality relationships, measurement relationships, part-whole relationships, the comparison of quantities in terms of a difference and a multiple, and proportional relationships. The actions on quantities that are represented include building or measuring a quantity with other quantities, separating a quantity into parts, composing a whole from its parts, adding two quantities to make another quantity, subtracting one quantity from another, making a quantity bigger or smaller by adding or subtracting some amount, and making a quantity n times bigger or n times smaller.

The algebraic representations and the schematics are essential to accomplishing the curriculum goals. First, they are designed to focus a child’s attention on actions (e.g., the decomposition of a whole), to help the child to keep an action in mind even after the action has been completed, and to enable him/her to describe and study it. Second, they draw attention to the relationships between different types of actions. For example, children write “If C<P by B, then C=P–B and C+B=P.” The notation indicates that we can move from an inequality to an equality relationship by adding or subtracting the difference, and that addition and subtraction are related actions. Schematics also develop ideas about reversible actions. For example, the unit/quantity/tally representation for measurement of quantities also applies to building them, and the “/\” schematic serves to unify related actions of composing of the whole from parts and the decomposition of the whole into parts (cf. Schmittau, 1993b, in press). Understandings about reversible actions are essential to algebraic reasoning.

Third, the representations are designed to help children abstract relationships between quantities. Fourth, they are used to develop children’s ability to distinguish and analyze quantitative relationships, and to apprehend the differences between various types of relationships between quantities. Fifth, children use schematics to solve equations, and to model what type of quantity the unknown is (e.g., the larger quantity in a difference comparison), and this helps them to identify the action that they need to perform in order to solve for the unknown. Sixth, the representations help children to relate ideas about quantities and ideas about numbers.

More generally, the representations are used to help achieve the goal of developing theoretical comprehension. Schematics provide models of abstract relationships, allowing the child to look at the relationship, to work with it, to examine its properties, and to pick it out in various contexts. For example, children are often presented with an algebraic representation (e.g., an equation) or a schematic: The representation itself depicts some kind of mathematical action or relationship that is meaningful to the children. Then they are asked to create a story problem or to find a concrete example that illustrates this structure. This type of problem would be very difficult to give to children without the help of the algebraic representations and the schematics. Such modeling also enables children to extend existing understandings to analyze new material—another goal of developing theoretical comprehension. It also helps children to form connections across mathematical ideas and to unite various topics into a single schema.

Modeling also enables children to engage in a very general analysis of mathematical problems. In addition, schematics and algebraic symbolism are used to prove general laws, such as the associative law and the distributive property.

Algebraic representations are also used to develop children's ability to plan. Children have to write a plan for a problem's solution in the form of an algebraic equation, for example.

Children are also given problems involving geometric representations and only literal designations of length and area. Figure 2 presents a US student's solution to such a problem. Initially this problem consists of the two rectangles, together with the area a and lengths c, p, and m (Davydov et al., 2001). It is necessary to find the area (designated by "?") of the rectangle on the right. The student first determined the length of the rectangle on the left to be $a \div c$, and designated this as a part of the length of the rectangle on the right. The child found the length of the rectangle on the right by adding $a \div c + m$, and then multiplied $(a \div c + m) \bullet p$ to obtain the required area.

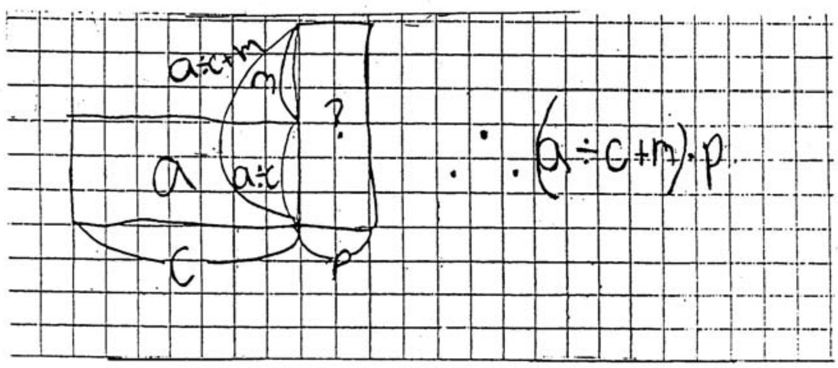

Figure 2. Solution involving geometric representation and literal designation.

By depriving children of context cues in the form of numerical designations for quantities such as length and area in problems such as those found in Figure 2, they must understand the underlying relationships in order to solve the problems. Denied empirical data, they must have a theoretical understanding of the mathematical relationships in question.

Finally, algebraic representations are used to change the nature of arithmetic instruction. They help change the focus from computation, to relationships between quantities, the concept of number, relationships between concepts having to do with both quantity relationships and number, and solution processes. Algebraic representations are used to develop concepts about computation and arithmetic laws at a more general and abstract level, and help students to view a set of numerical problems as a class of problems that involves the same relationships between quantities.

Process Coverage

The curriculum develops children's ability to think in a variety of ways that foster algebraic performance. First, it develops theoretical thinking, which according to Vygotsky comprises the essence of algebra (Vygotsky, 1986). For example, the children develop a habit of searching out relationships among quantities across contextualized situations, and learn to solve an equation by attending to its underlying structure. The curriculum develops children's capacity for analysis and generalization. Their ability to interpret a letter as "any number" allows the teacher to introduce children to the kind of general argument that is the hallmark of algebraic justification and proof. The curriculum helps them to understand the nature of algebraic argument, and the possibility of making claims about infinite sets of objects in mathematics (Morris, 1999; Morris & Sloutsky, 1998). They grow accustomed to searching for "the general." Second, the students have learned to expect algebraic expressions and equations to have meaning. Their understandings of algebraic representations have been grounded in the work with real quantities, actions on quantities, and models of quantitative relationships. The students have learned about the structures and principles that govern the manipulation of algebraic symbols. When they leave grade 3, they have conceptual understandings that allow them to extend their knowledge to new contexts and problems, and the ability to analyze and model mathematical situations they have not previously encountered. They also have the confidence to attempt to do so (Morris & Sloutsky, 1998; Schmittau, 1994, 2003).

Comparing the Algebra-related Goals of Davydov's Curriculum with NCTM'S *Principles and Standards for School Mathematics* Algebra Standard

How do the algebra-related goals of the Russian curriculum compare with the four algebra goals specified in the National Council of Teachers of Mathematics's (NCTM) *Principles and Standards for School Mathematics*? The Algebra Standard states that instructional programs from pre-kindergarten through grade 12 should enable all students to -

- understand patterns, relations, and functions
- represent and analyze mathematical situations and structures using algebraic symbols
- use mathematical models to represent and understand quantitative relationships
- analyze change in various contexts.

The Russian curriculum also emphasizes relationships among quantities, ways of representing mathematical relationships, and the analysis of change. However, the interpretation of what these goals mean, and beliefs about the ways in which these goals are to be achieved are different.

The Algebra Standard suggests that when children demonstrate a situation like "Gary has 4 apples, and Becky has 5 more" by arranging counters, they are doing beginning work with modeling quantitative relationships (p. 39). In the Russian curriculum, "using mathematical models to represent and understand quantitative relationships" means making a model of the *relationships* between the quantities in the problem, not a model of the sets of objects (e.g., a pile of 4 counters and a pile of 9 counters). The kind of analysis of relationships between quantities that the students do with schematics and equations not only engenders a meaning for equations, but enables students to reason algebraically in a way that a focus on just the numerical values and actions on the sets of objects may not.

The NCTM Algebra Standard emphasizes systematic experience with patterns in arrangements of objects, shapes, and numbers, and predicting what comes next in an arrangement, as the basis for understanding relationships between variables and the idea of a function. It emphasizes various ways of thinking such as building rules to represent functions, and abstracting from computation. The *Standards* also stresses students' recognition and description of solution processes as the basis for developing the idea of a relationship between varying quantities. Algebraic symbols are used to represent these discovered regularities or the reasoning processes that led to their discovery. For example, when students are computing how much they have to pay for seven balloons if one balloon costs 20 cents, they might, by studying their own reasoning processes, realize that they can create a formula that works for any number of balloons, namely $C=20\bullet B$, where C is the cost and B is the number of balloons.

The Algebra Standard stresses the child's discovery of relationships, mathematical generalizations, and patterns involving numbers and geometric objects. The inductive discovery aspect of the Algebra Standard is absent in the Russian curriculum, nor is there any work with patterns. The Davydov curriculum engages children in a focused analysis and description of the quantitative world as the starting point for developing algebraic thought. The manipulation of quantities (e.g., building and measuring quantities) is the starting point for developing ideas about mathematical relationships and generalizations. Special kinds of actions on quantities are designed to focus attention on their relationships. For example, the action of decomposing and composing a quantity is at the core of the idea of a part-whole relationship, and activities engage children in performing these kinds of actions on objects. Children also model the relationship that the action is supposed to reveal. The schematics and algebraic representations employed for this purpose enable the child to abstract and represent the properties of the relationship, as opposed to letting the child induce the idea on her own with no systematic way to represent the same relationship across problems. There is no guarantee, for example, that when a child induces a pattern involving a relationship between numbers or geometric objects that she understands the exact nature of the relationship, that she can spot it or use it again, or that she understands the properties of the relationship (cf. Morris, 1999; Morris & Sloutsky, 1998).

The Algebra Standard also suggests that algebra should build on students' experiences with number. The Russian curriculum does not use experience with numbers as the basis for developing algebra, but instead uses relationships between quantities as the foundation. For example, it does not teach children to solve equations by thinking about "doing and undoing" numerical operations, but instead teaches them to solve equations by thinking of them in terms of relationships between quantities. Moreover, it does not use children's informal understandings of counting numbers as the basis for developing the concept of number, because these informal understandings do not extend to fractional numbers. Instead it develops understandings about measuring units, and relationships between measuring units and other quantities that will allow students to easily extend their idea of number to fractional numbers (cf. Morris 2000).

Conclusion

The differences described above reflect fundamental differences in the bases for developing algebraic understandings, and divergent suppositions about the precursors of algebraic thought.

We have described in considerable detail the manner in which the goals of theoretical thinking, the establishment of the concept of number on the activity of measurement rather than counting, the development of algebraic understandings through direct work with quantities, and the representational modeling of mathematical actions and relations, are met in the mathematics curriculum of Davydov. By teaching algebra in the early grades, the elementary and secondary curricula in mathematics are brought into alignment. The elementary grades are not consigned to computation while reserving the abstract understanding of mathematics for the upper grades. The differences with the US curriculum are sufficiently profound as to constitute a paradigm shift. We will not repeat them here. The learning paradigms in which the US and Davydov curricula are grounded (constructivist and Vygotskian) are also fundamentally different, but their comparison is beyond the scope of this paper.

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