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Developing Algebraic Thinking in the Earlier Grades: A Case Study of the U.S. *Investigations* Curriculum[1]

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Abstract In this paper we present a case study of a U.S. elementary school curriculum, *Investigations in Number, Data, and Space,* for the way it provides students with informal and formal algebraic experiences. We find that mathematical change is the unifying big idea of the algebra strand, and that other big ideas are patterns and relationships, representation, and modeling. We also find that its primary goals are aligned with two of the four NCTM algebra goals: *Understand Patterns, Relations, and Functions* and *Analyze Change in Various Contexts.* Finally, we find that its approach to achieving its goals is to foster mental processes that constitute the algebraic habit of thinking known as *Building Rules to Represent Functions*.

Introduction

Recommendations for algebra reform in the United States include increased attention to the study of algebra in the elementary grades in order to prepare students for more-sophisticated work in algebra at the middle and high school levels (NCTM, 1989, 1992, 1997, 2000). However, curriculum materials at the elementary level in the United States have historically given little attention to the study of algebra, and only recently have curriculum developers, educational researchers, teachers, and policy makers begun to investigate the kinds of mathematical experiences elementary students need to prepare them for the formal study of algebra at the later grades (Carpenter, Franke, & Levi, 2003; Driscoll, 1999; Kaput, 1998; Nemirovsky, Tierney, & Ogonowski, 1993; Noble, Nemirovsky, Wright, & Tierney, 2001; Tierney & Monk, in press).

In light of these current recommendations and increased attention to the study of algebra in the elementary grades, the development of algebraic concepts is being incorporated as a significant strand of study into elementary mathematics programs used in the United States. In this paper we present a case study of one such

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elementary school curriculum, *Investigations in Number, Data, and Space* (hereafter, *Investigations*), for the way it provides students with informal and formal algebraic experiences. First, we provide an overview of the curriculum. Then we report our analysis of the goals of the curriculum. Next we discuss in detail how the big ideas of algebra are developed within each grade level and across the grades. Finally, we reflect upon the development of algebraic processes across the curriculum.

An Overview of the Curriculum

Investigations is one of three elementary school reform-mathematics curricula developed with funding from the U.S. National Science Foundation (NSF). It was developed at the TERC research and development center in Cambridge, Massachusetts during the 1990s. In all, NSF supported the development of 13 reform-oriented comprehensive mathematics programs, three at the elementary school level, five at the middle school level, and five at the high school level. All 13 programs were designed specifically to help reform mathematics instruction by implementing the recommendations of the *Curriculum and Evaluation Standards for School Mathematics* developed by the National Council of Teachers of Mathematics (NCTM) in 1989. Mathematics reform is evident in the four general, overarching goals of the *Investigations* curriculum. They are (1) to offer students meaningful mathematical problems; (2) to emphasize depth in mathematical thinking rather than superficial exposure to a series of fragmented topics; (3) to communicate mathematics content and pedagogy to teachers; and (4) to substantially expand the pool of mathematically literate students. These goals are clearly aimed at reforming mathematics instruction in the United States' elementary schools. To achieve these goals, *Investigations* provides activities requiring all students to explore contextualized problems in depth; construct strategies and approaches based on knowledge and understanding of mathematical relationships; utilize a variety of tools (e.g., manipulatives, computers, calculators); and communicate mathematical reasoning through drawing, writing, and talking.

Investigations provides a complete mathematics program for grades K–5. Although it is a complete program, it looks quite different from a traditional elementary mathematics program. The program does not include student textbooks at any grade. The curriculum is presented through a set of grade-level teacher resource books called curriculum units. Reproducible student sheets are provided in an appendix to each teacher-resource curriculum unit. Each unit of instruction provides resources that enable the teacher to engage students in three to eight weeks of mathematical work. The units are organized as a series of investigations, each composed of one or more 1-hour class sessions. The complete program comprises 50 units, with the number of units at each grade level varying from six to eleven.

Investigations has an identifiable algebra strand. The *Content Overview* (Scott Foresman, undated) explicitly identifies six units that compose the "patterns, functions, and algebra" strand, (hereafter "algebra strand"). Each unit comprises 2-4 investigations. There are a total of 19 investigations in the algebra strand across the six grade levels. These units are:

- Kindergarten: *Pattern Trains and Hopscotch Paths* (Eston & Economopolous, 1998)
- Grade 1: *Building Number Sense* (Kliman & Russell, 1998)
- Grade 2: *Timelines and Rhythm Patterns* (Wright, Nemirovsky, & Tierney, 1998)
- Grade 3: *Up and Down the Number Line* (Tierney, Weinberg, & Nemirovsky, 1998)
- Grade 4: *Changes Over Time* (Tierney, Nemirovsky, & Weinberg, 1998)
- Grade 5: *Patterns of Change* (Tierney, Nemirovsky, Noble, & Clements, 1998)

Our examination of the *Investigations* curriculum focused on the six units that compose the algebra strand. Even though our analysis was based on a careful study of the six core algebra units, we also examined additional units as needed. The resulting case study is based on our analysis of three related dimensions of the curriculum's algebra strand - its goals, its content coverage, and its process coverage. In it, we report on how thoroughly the development of each of these interrelated dimensions of the *Investigations* curriculum incorporates features widely recognized as important to the teaching and learning of algebra.

Analysis of the Goals

Our purpose in analyzing the goals of the algebra strand of *Investigations* was to determine how closely they cohere to the vision of K-6 algebra laid out in NCTM's *Principles and Standards for School Mathematics*. The authors of *Investigations* do not specifically set mathematics learning goals for students and teachers to achieve. Instead, the authors introduce each investigation with a "Mathematical Emphasis" section comprising between 2 and 10 Mathematical Emphasis (ME) statements. For example, one of the ME statements that introduces Investigation 2 of the Kindergarten unit is "Predicting what comes next in a pattern," and one of the ME statements that introduces Investigation 3 of the 5th-grade unit is "Connecting the slope in a graph with rate of change." In all, the algebra strand has 108 such ME statements for the 19 investigations contained in the 6 units of the algebra strand.

The purpose of these ME statements is to tell the teacher "…what is most important for students to *learn about* [italics added] during the investigation," (Kliman & Russell, 1998, p. 1-19). Listing ME statements that describe the mathematics to be "learned about," rather than listing goal statements that specify mathematics to be "learned," reflects the overall philosophy of the *Investigations* curriculum. It conforms to the authors' long-term view of learning, namely that teachers should not expect all students to achieve scrupulously formulated learning goals within a predetermined time frame. The following excerpt aptly describes their view:

> Students gradually learn more and more about each idea over many years of schooling. Individual students will begin and end the unit with different levels of knowledge and skill, but all will gain greater knowledge of [the algebra concepts studied in this unit] and develop strategies for solving problems involving these ideas. (Eston & Economopoulos, 1998, p. 1-17).

Although the Mathematical Emphasis statements are not, strictly speaking, goal statements, they are nonetheless significant indicators of the curriculum’s learning goals. Therefore, we used them to measure the level of coherence between the goals of the algebra strand of *Investigations* and the goals of the NCTM algebra standard. Since many ME statements appear to address more than one expectation of the algebra standard, we decided to use the four goal statements of the algebra standard as our framework of analysis rather than the finer-grained set of NCTM expectations. Operationally, we compared the ME statements in the algebra strand to the statements of the NCTM algebra expectations in order to identify the NCTM algebra goal, if any, that each statement *primarily* addresses.

It should be indicated that some ME statements appear to address more than one goal. In these cases, rather than list multiple goals corresponding to each ME statement, we decided we would get a more-focused picture of the goals of the curriculum by choosing the goal that appears to be primarily addressed by the activities and contexts in the investigation.

Table 1 provides a quantitative measure of the extent to which the content goals of the algebra strand align with the goals of the NCTM algebra standard. It shows that 89/108 (82%) of the ME statements in the algebra strand address primarily NCTM Algebra standard goals. The remaining 19 statements do not have a primary focus to address goals of the Algebra Standard. Seventeen of the first-grade statements primarily address goals from the Number and Operations Standard, and two of the second-grade statements primarily address goals from the Measurement Standard.

The table also gives a profile of the relative emphasis placed by *Investigations* on specific NCTM algebra goals and expectations at each grade level of the curriculum. The data imply that analyzing change in various contexts (Goal 4) is the most emphasized goal of *Investigations’* algebra strand. In fact, more of the ME statements (40/108=37%) primarily address Goal 4 than any other NCTM algebra goal. Goal 4 clearly gives major impetus to the units of the grades 3, 4, and 5 algebra strand. Ninety-two percent (12/13) of the third-grade ME statements principally address Goal 4, as do 83% (19/23) of the fourth-grade statements and 47% (9/19) of the fifth-grade statements.

Table 1
Count by grade level of the Mathematical Emphasis (ME) statements that primarily address the goals of the NCTM algebra standard

Grade Level **NCTM Algebra Standard Goal**	**K**	**1**	**2**	**3**	**4**	**5**	**Total**
Goal 1: Understand Patterns, Relations and Functions	21	1	4		1	10	**37** **34%**
Goal 2: Represent and Analyze Mathematical Situations and Structures Using Algebraic Symbols			1		2		**3** **3%**
Goal 3: Use Mathematical Models to Represent and Understand Quantitative Relationships		7		1	1		**9** **8%**
Goal 4: Analyze Change in Various Contexts				12	19	9	**40** **37%**
Does not primarily address any of the Algebra Standard Goals		17	2				**19** **18%**
Total	**21**	**25**	**7**	**13**	**23**	**19**	**108**

Understanding patterns, relations, and functions (Goal 1) is also an important goal across the algebra strand. In fact, 34% of the ME statements directly address Goal 1. This goal is clearly a major impetus behind the grades K, 2, and 5 algebra units. One hundred percent (21/21) of the ME statements in the Kindergarten unit align principally with Goal 1, as do 57% (5/7) of the second-grade statements and 53% (10/19) of the fifth-grade statements.

The other two goals of the NCTM Algebra standard do not appear to exert such strong influence on the *Investigations* curriculum. Data in Table 1 strongly intimate that learning to use algebraic symbols is simply not a goal of *Investigations*. NCTM Goal 2, "Represent and analyze mathematical situations and structures using algebraic symbols" is primarily addressed by only 3/108 (<3%) of the ME statements. Although elements of mathematical modeling are clearly present in the curriculum, modeling does not appear to be a primary goal of the curriculum since the number of Goal 3 ME statements (9) is small in comparison to number of Goal 1 ME statements (37) or Goal 4 ME statements (40).

In summary, based on Mathematical Emphasis statements, it appears that the major goals of the *Investigations* algebra strand are aligned with NCTM algebra Goals 1 and 4, but not Goals 2 and 3. By way of confirmation that Goals 1 and 4 are major influences on the algebra strand, it is interesting to note that the title of the capstone unit, “Patterns of Change,” clearly fuses the main concepts embodied in Goals 1 and 4. Furthermore, the 19 ME statements in this fifth-grade unit are evenly divided between those aligned with Goal 1 and those aligned with Goal 4.

Analysis of Big Ideas: The Centrality of Change

The goals of a curriculum are an important indicator of the concepts and procedures it intends to develop. However, to measure a curriculum’s potential effectiveness we need to analyze how the curriculum actually develops these concepts and procedures. Therefore, for the second part of our study we identified the "big ideas" of algebra that the developers chose to emphasize, and we traced how those big ideas are developed across the algebra strand.

Investigations’ implementation guides (Russell, Tierney, Mokros, Goodrow, & Murray, 1997; Russell, Economopoulos, Murray, Mokros, & Goodrow, 1998) identify change as one of four main mathematics content areas in the *Investigations* curriculum along with number, data, and space. The authors justify the study of change as a major content emphasis because of its importance in everyday life and its importance as a foundation for further study in mathematics:

> Change is one of the most pervasive aspects of our lives. We are constantly experiencing the flow of time and the changes that occur over time—motion, growth, and temperature, for example. One of the driving forces of mathematics is to understand and predict change (Russell et al., 1997, p. 8).

The authors also believe that the concept of change should be a qualitative unifying big idea in the algebra strand.

> Work in the Investigations curriculum emphasizes qualitative understanding of these ideas [of mathematical change] as students discuss the meaning of graphical and numerical patterns. The units strengthen the continuity between elementary school mathematics and “advanced” courses such as algebra and calculus by introducing important ideas about growth and change (*ibid.*, p. 8).

By recognizing the importance of studying change in the elementary curriculum, and by making it the centerpiece of its algebra strand, the curriculum serves at least two purposes. It not only helps students learn to deal with the abstract and largely neglected concept of change in everyday life, it also lays the foundation for advanced study of mathematics, for which change is a central and unifying concept (Steen, 1990).

We analyzed the *Investigations* algebra strand for its emphasis on change and on seven other widely-accepted “big ideas” of algebra: variables, structure, proportional reasoning, patterns and relationships, equations and equation solving, representation, and modeling. Our analysis confirmed that the concept of change is the central big idea in the algebra strand. Furthermore, we found that the big ideas of patterns and relationships, representation, and modeling also receive extensive attention. We found not only that mathematical change unifies all the units in the algebra strand, but also that change is the impetus behind the development of the big ideas of patterns and relationships, representation, and modeling. The big ideas of variables, proportional reasoning, structure, and equations and equation solving, on the other hand, are given neither formal nor extensive treatment in the algebra strand.

The following ordered list provides a snapshot of the change-related concepts that are developed across the algebra strand: repeating patterns, additive change, time, net change, change over time, growing patterns, and rate of change. The analysis that follows describes how the progression of these concepts across the algebra strand builds an increasingly formal and robust notion of the big idea of change and, in the process, establishes formal conceptions and connections to the other algebra big ideas of patterns and relationships, representation, and modeling.

Repeating Patterns: Absence of Change

The kindergarten unit, *Pattern Trains and Hopscotch Paths*, (Eston & Economopolous, 1998) is not overtly about change; it is overtly about patterns. To judge the content of the kindergarten unit outside the context of the entire curriculum might lead one to conclude that the kindergarten unit has little to do with change. This is because, somewhat paradoxically, the activities in kindergarten appear to be devoted mostly to helping students understand that the absence of change is a defining characteristic of patterns. Understanding that all patterns have an unchanging aspect is crucial to understanding the concept of pattern and change because mathematical change is neither random nor unpredictable. Indispensably in school mathematics, change has an unchanging component that makes it predictable. Because of this, students must learn to recognize the element of changelessness in any change they are studying.

The study of patterns in the *Investigations* curriculum helps students learn to coordinate the competing ideas of change and absence of change. In the kindergarten unit, repeating patterns (a,b,a,b,a,b…), as opposed to growing patterns (4, 6, 8, 10, …), are the main object of study. The unchanging aspect of pattern is more overt in repeating patterns than it is in growing patterns. Furthermore, because the changing aspects of repeating patterns tend to be disregarded, or at least downplayed, in the kindergarten algebra unit, we conclude that the kindergarten algebra unit is designed specifically to help students understand the unchanging aspect of pattern. A description of the kindergarten activities is given below to illustrate how the activities are designed to keep the changeless nature of pattern consistently at the forefront of the students' consciousness.

In the beginning of the kindergarten unit, students learn to look for repetition and regularity in the patterns they encounter. Students practice following their teacher's repeating pattern of body motions, they observe repeating patterns in children's literature, and they observe repeating patterns in an observation walk. They look for patterns in trains of colored interlocking cubes. Students are encouraged to look for repetition and regularity, and to compare their observations with one another. Based on their observations, they are also challenged to predict what comes next. At first the patterns are kept very simple. Teachers are encouraged to have students work with repeating patterns composed of two elements (e.g., ababab, abbabbabb) rather than three-element patterns which, they are warned, should be introduced only to the more advanced students. Students also construct "linear" patterns of their own with a variety of materials. (Here linear means the patterns are made with materials that are physically arranged in straight line "trains.") Students generate patterns by focusing on one of the attributes of a given material (e.g., its size, color, or shape). The ability to sort items by attributes is a related skill they practice and draw upon in these activities.

Once the students can recognize the repetition in patterns, the students make and extend repeating patterns in a pocket chart by arranging color tiles in the top row of pockets. They predict the colors that come next when the first four tiles are showing but the other six tiles are hidden behind blank cards. Students analyze the repeating aspect of the pattern by breaking down the 2-color pattern trains into repeating units of 2-cube cars (chunks). They are encouraged to categorize various types of patterns made with different materials. For example, the pattern: red tile, yellow tile, red tile, yellow tile is the same type as the pattern: blue cube, white cube, blue cube, white cube, but different than the pattern: red tile, yellow tile, yellow tile, red tile, yellow tile, yellow tile.

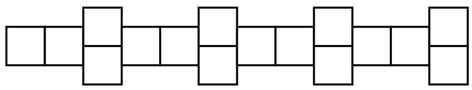

Figure 1. Hopscotch Pattern

Until this point in the unit, students have created patterns of objects by varying only attributes like shape, color, or size (e.g., white cube, white cube, blue cube, white cube, white cube, blue cube, …). A hopscotch activity near the end of the unit expands the students' concept of pattern by showing that they can form patterns by varying the position of objects in two dimensions. Unlike the colored cubes and tiles, objects in the hopscotch activity are squares that are indistinguishable except for their positions relative to one another (see Figure 1). The students build life-size hopscotch patterns, play on them, and then represent them with paper squares glued on strips.

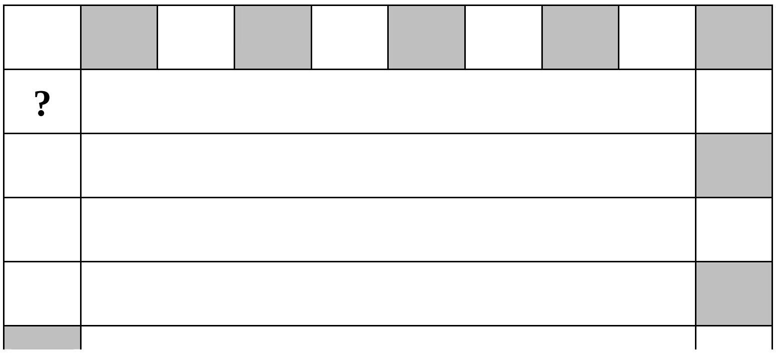

Figure 2. Students predict what color square should go in the last empty space.

An activity called Pattern Borders provides a link between linear patterns and two-dimensional patterns. Students make linear patterns that form rectangular borders. They focus on what happens to the pattern when it turns a corner. They also notice the relationship between the last color and first color placed in their border to determine whether or not their border makes a continuous pattern (see Figure 2). They also must predict what comes next and what is a hidden element further along.

In summary, we can see how important attributes of repeating patterns are systematically varied to provide a stable initial concept of pattern. Students learn to recognize patterns made of various materials and configurations. They categorize a variety of patterns. They make patterns of their own using different materials. They vary either the attributes of the objects or their positions in space. They vary the number of elements in the pattern, identify the repeating unit, and predict the next element or an element further along. In short, the authors have carefully sequenced a host of activities specifically designed to help students construct a stable concept of pattern. Therefore, it is conspicuous that all but one of the patterns in the kindergarten unit are repeating patterns. A compelling explanation for this obvious omission of growing patterns is that the intent is to teach students about the unchanging aspect of mathematical change. Repeating patterns are especially well-suited to do so, but growing patterns are not.

Coordinating Change and the Absence of Change

To fully understand pattern, students must be able to coordinate the unchanging aspect of pattern with the changing aspect of pattern. The final activity in the kindergarten unit provides students with a glimpse of growing patterns, a pattern type that will form an important part of their future study of patterns. It is the only activity in the unit that uses growing patterns, as opposed to repeating patterns. In this activity, students make staircase patterns out of interlocking cubes, record them on grid paper, and analyze them. The students are asked to describe a (recursive) rule for how the staircase pattern grows (or shrinks). This activity forces students to focus on both the changing aspect of the pattern (each successive tower in the staircase is different in size) as well as the unchanging aspect of the pattern (the difference in size of successive towers is always 1 cube). The inclusion of a growing pattern at this point in the curriculum is noteworthy because growing patterns do not appear again until fifth grade, when they are used to study rate of change.

In the first grade unit, *Building Number Sense* (Kliman & Russell, 1998), the curriculum returns students' attention back to repeating patterns. However, the activities are now designed to help students coordinate the unchanging aspect of repeating patterns with the changing aspect. Also, the activities in the first grade unit mark the first time students focus on how changes in one pattern relate to changes in another, an important aspect of mathematical change. This is because the study of function crucially involves overt attention to the way that change in one quantity produces change in another.

In an activity called "Clapping Patterns," the students act out repeating clapping patterns and represent them using drawings, numbers, or objects. For example, students can represent the clap-clap-knees-knees pattern with interlocking cubes as blue-blue-white-white, blue-blue-white-white, … or on the Hundred Number Wall Chart (see Figure 3). In the hundred-chart representation the students insert transparent red markers on top of every third and fourth number for two or three rows. Then they discuss what they notice.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30

Figure 3. A representation of a clapping pattern on the hundred chart

By representing a repeating pattern with both colors and numbers on the hundred chart the discussion of a repeating pattern can be orchestrated to highlight both the changing and unchanging aspects of pattern. When students predict which numbers beyond the fourth row should be highlighted in red, they have to mentally coordinate the unchanging aspect that every third and fourth number will be red (representing knees), with the changing aspect that the numbers themselves will be different. The use of the hundred-chart representation in this activity is significant because it helps students focus informally on the functional nature of patterns. By coordinating the changing and unchanging aspects of the pattern the students come to realize that, given a number (and thus its position on the chart), they can predict whether it represents clap or knee. Conversely, they can discuss which numbers can possibly represent claps or knees. By way of contrast, the cube representation of the pattern does not emphasize the functional aspects of the pattern because all the white cubes, for example, look the same regardless of their position in the train.

Additive Change

Students' understanding of mathematical change should be grounded in their experiences with change in everyday situations. The challenge for educators is to help students learn to use mathematics to represent in a precise way the changes they experience every day. The authors of *Investigations* attempt to meet this challenge by posing everyday problem situations in which change is an underlying concept. In the second half of the first grade algebra unit the authors pose combining and separating problems, which can be solved using addition and subtraction. Without formally referring to the concept of change, first grade students learn to model and solve everyday problems in which they combine an original amount and a change amount. They also learn to solve everyday problems where they remove a change quantity from an original amount. Ultimately students need to realize that these change situations can be modeled and solved using the operations of addition and subtraction. However, during these early activities, the operations of addition and subtraction are not the focus of the unit. Students devise their own methods of solving the problems based on an analysis of their semantic structures (combining or separating), rather than according to the arithmetic operation that can be used to solve them (addition or subtraction).

As previously discussed, an overriding philosophy in *Investigations* is that it is essential to slowly build conceptual understanding based on student's existing intuitive notions, and that it is counter-productive to teach conventional representations and procedures before a solid foundation of understanding is in place. The approach taken in the first grade unit clearly reflects this philosophy. In the unit, students are encouraged to develop their own models to solve problems involving additive change. For example, the following typical first-grade problem encourages students to reflect on their problem-solving strategies.

> ***Apples and Oranges***
> *I went to the store to buy some fruit.*
> *I bought 5 apples and 4 oranges.*
> *How many pieces of fruit did I buy?*
>
> *Don't write only the answer.*
> *Show how you solved the problem.*
> *Use words, pictures, or numbers.* (*ibid*., p. 209)

Because the problem statement tells students to “Use words, pictures, or numbers” to report how they solved the problem, the students do not feel obliged to model the problem with a number sentence. The result is that each student feels free to create a personally meaningful model they can use to solve the problem. This can be empowering for students because the models they develop tend to elicit valid problem solving strategies that are grounded in students’ prior experiences in similar real-life situations, rather than on misunderstood properties of addition and subtraction or mindless recall of addition and subtraction facts. Put another way, the models that students devise generally are based on the semantic structures of the problems, which students understand, rather than on the underlying mathematical structure of addition, which many do not yet understand. The intent is that over time, as students compare their strategies for solving these types of problems, they eventually will understand that combining problems and separating problems can be modeled with addition and subtraction sentences.

One way that this philosophy is advanced in the curriculum is via “Teacher Notes” sprinkled throughout the algebra units. For example, one of the Teacher Notes in this first grade unit, suggests that the teachers should give their students the opportunity to develop their own strategies for solving these problems—strategies that generally fall into three categories: direct modeling, counting strategies, and numerical reasoning. Clearly, the ultimate goal of the curriculum is the formal development of the third of these, numerical reasoning. However, as the Note goes on to say, the use of numerical reasoning develops gradually over the elementary years:

> Many first graders will need to continue counting by 1's for most problems. As they build their understanding of number combinations and number relationships over the next year or two, as well as their ability to visualize the structure of a problem as a whole, they will begin developing more flexible strategies. (*ibid.*, p. 135)

Teachers are encouraged to devote considerable attention to helping students explain their problem solving strategies in writing and with pictures and diagrams. It is of special interest that, although equation notation (e.g., 4+3=7) is introduced in the first grade unit, the students are not required to use the notation themselves. The following excerpt from one of the Teacher Notes makes this clear:

> While first graders should become familiar with standard equation notation, it is not essential that they use it themselves at this level. By the end of second grade, students can be guided to use equation notation correctly, but first grade is too early to do that for most students. (*ibid*., p.124)

This first grade unit does not formally introduce students to the concept of change. For example, none of the problems in the unit explicitly uses the word “change.” To keep the teachers mindful of the centrality of change in this unit and throughout the algebra strand, in yet another Teacher Note, the authors discuss two different problem structures within each of the combining and separating categories: change unknown or outcome unknown. They hasten to add that, for the most part in the first grade curriculum, students work on just two problem types: combining with unknown outcomes and separating with unknown outcomes. Combining and separating problems with unknown change are, for all practical purposes, postponed until later in the curriculum.

Time

Time is an illusive concept for many students. Research has shown that students' inability to conceptualize time as a quantity stands in the way of their ability to understand rates of change like speed (Thompson, 1994). The focus of the grade 2 algebra unit, *Timelines and Rhythm Patterns*, (Wright, Nemirovsky, & Tierney, 1998) is the exploration of the mathematical aspects of time – sequence, duration, and cycles. Its purpose is to continue to lay the foundation for the mathematics of change, which is studied in increasingly explicit and formal ways in grades 3, 4, and 5.

The students begin their study of time by analyzing timelines. At first, the emphasis is on how timelines show sequences of time (left of, right of) rather than on how timelines show duration (length of). They create timelines of their lives in which the emphasis is on representing time by a scale with tick marks at one-year intervals, and on marking and finding places on the timelines corresponding to different life events, even if they are between the whole number tick marks.

With respect to duration, an important concept to learn informally is that the length of a segment on the timeline is proportional to the length of the time it represents. To help reinforce this idea, students make a timeline for a special day. They are introduced to the convention of using a horizontal line to communicate in a visual way the length of time it takes to do an activity. The students also act out a timeline of a special day while the teacher counts the hours out loud at five-second intervals. For example the teacher says "6:00," and the students in their seats act as if they're sleeping. Five seconds later the teacher says "7:00," and the students stand up and pretend they are getting dressed. In this way students not only learn to represent times of different duration using different-length segments, they also actually experience the relative sense of time that the segments represent.

In the unit, students imitate rhythms and invent their own rhythm patterns, paying close attention to the way the pattern of the rhythms cycle over time. Students learn that keeping track of cycles is an aspect of measuring time that is related to repeating patterns. The activities are designed to help lay an intuitive foundation for symbolic representations in algebra. Students learn that written symbols can be used to represent rhythmic activity, including pauses, by reading and interpreting codes and by eventually inventing their own ways to represent rhythm patterns. For example, they could use **XX** •++ **XX** •++ to represent the rhythm pattern: clap-clap (short pause) tap-stamp-stamp (long pause) clap-clap (short pause) tap-stamp-stamp. In order to appreciate the need to standardize their use of symbols, the students agree upon a common set of symbols and test their understanding by playing a game called "Guess My Rhythms." In a culminating activity, students look at traditional notation that shows how people represent time in music. In order to learn important aspects of representation, students compare their own symbols to the conventional symbols used in the sheet music for a well-known song. They also use their own notation to represent the rhythm patterns for the song.

The activities in this second-grade unit prepare students to eventually understand and represent the concept of speed, a difficult abstraction that requires the fusion of a stable concept of time with a stable concept of change of position. The unit is designed to establish the foundation for understanding and graphically representing the time aspect of speed apart from its change aspect. In third grade, however, students undertake a formal study of change that avoids explicit attention to the concept of time.

Net Change

Although change is the unifying big idea of the algebra strand in grades K, 1, and 2, the concept of change is only implicit in the activities of those grades, and the study of change is informal. Explicit and increasingly formal study of mathematical change begins in the third-grade unit, *Up and Down the Number Line* (Tierney, Weinberg, & Nemirovsky, 1998). This unit develops the concept of net mathematical change in conjunction with the introduction of positive and negative numbers. Rather than using the familiar approach of developing the algebraic concept of net change after students have learned addition and subtraction of integers, the authors take the opposite, less common approach. They use students' intuitions about change in a concrete situation not only to develop formal notions of generalized change but also to develop intuitive notions about integers and their operations.

In this third-grade unit, the concept of net change is introduced in a contextualized activity that examines changes in the position of a fantasy skyscraper's elevator, which can go "up and up forever," and "down and down forever." Students label the floors above and below ground level (…,B3,B2,B1,0,1,2,3…) on a cutout skyscraper. Then, working in pairs, the students create a table of net changes in the following way. One student (the elevator operator) sticks post-it notes labeled "Start" and "End" on two floors of the skyscraper while the other student (the recorder) writes the starting and ending floor in the first two columns of a 3-column chart. Then the operator moves a counter from the starting to the ending floor while both students count the number of floors traveled. Finally, the recorder writes the number of floors moved in the "net change" column. Typically, the teacher needs to remind students that the numbers in the net change column must include a "+" or "–" to show both how many floors and in what direction they moved.

Once students are able to find net change given a starting and an ending floor, the unit provides a variety of activities that broaden and strengthen the basic concept of net change. For example, in one task students find the net change that results from as many as 30 individual changes. By developing strategies for combining large numbers of changes, students learn important properties of net change. One strategy for computing net change is to rearrange and combine opposite changes to reduce the number of changes that must be forward- or backwards-counted (or added or subtracted). The use of this strategy teaches students at least two important properties of change: (1) net change is the same regardless of the order in which the individual changes are combined, and (2) positive change cancels (undoes) negative change.

In another activity, special "change" buttons on the skyscraper elevator are introduced as a vehicle to get students to find missing changes. In this activity, students are asked to figure out a sequence of change buttons to press in order to stop at 3 or 4 or 5 particular floors before ending up at a specific final destination floor. Other problems (see the third example, below) ask students to find the starting floor when they know the ending floor and the sequence of change buttons pushed to get there.

The highlight of the unit is an inventive set of activities that introduces students to some profound aspects of the graphical representations of change. Students devise their own ways to represent elevator trips graphically. By trying to interpret one another's graphs, they learn how to represent changes in direction on a graph, how to determine net change from a graph in two different ways, and how to interpret different overall shapes of graphs.

The way that these and other topics are developed in this unit provides instructive examples of *Investigations'* focus on formalizing the big idea of change while simultaneously laying an informal foundation for future formal work on a different big idea or concept. We discuss four such examples next.

The first example concerns connections between addition of integers and change. Throughout this unit students consistently use positive and negative numbers to represent the answers to net change problems. Students are encouraged to check one another's answers. Some students recognize they can check an answer by adding the net change number to the starting floor number. Other students begin at the starting floor on the skyscrapers and count out the net change to see if they finish at the ending floor. The materials instruct the teacher to encourage (but not require) students to check their answers using both methods. This approach focuses students' attention on formalizing their intuitive concept of change, while at the same time establishing intuitive notions about the addition of integers. It must have been tempting for the authors to include at this point in the curriculum activities designed to establish proficiency with addition and subtraction of integers. Nonetheless, the authors resisted this temptation. Undoubtedly they did so because they believe that the concept of net change both enhances and is enhanced by an informal understanding of integer addition. However, a formal treatment of integer addition in the unit would serve to distract from the main goal, namely to develop the concept of net change.

The second example concerns connections between algebraic properties of integers and change. Students are given starting and ending floors and asked to enter the net changes in a table designed to show special properties of net change (see Figure 4). In the table, students see that each trip starts on either the second floor above ground or the second floor below ground. Also, they see that each trip ends on either the fifth floor above ground or the fifth floor below ground. By examining the table they notice that although the net changes are different, they all have an absolute value of 3 (the difference between 5 and 2) or 7 (the sum of 5 and 2). Students are asked: “What patterns do you see?” “Why do you think some of the trips have the same net change?” Students also are asked to construct different pairs of starting and ending floor numbers that produce the same net change of, for example, +7. As in the first example, the purpose is to teach formal ideas about net change, not formal ideas about integer subtraction. In fact, the use of B5 rather than –5 in the table actually discourages a formal treatment of integer subtraction. Instead, students are expected to develop some *informal* intuitions about subtraction of integers that can form the basis for formal learning of the concept in middle school.

Trips	**Start-Floor**	**End Floor**	**Net Change**
A	B2	5	+7
B	B2	B5	-3
C	2	B5	-7
D	2	5	+3

Trips	**Start Floor**	**End Floor**	**Net Change**
E	5	B2	-7
F	B5	B2	+3
G	B5	2	+7
H	5	2	-3

Figure 4. Net Change Tables

The third example concerns connections between equation solving and change. Students are asked to solve problems like the following:

> I got into my elevator and pushed the +2 button. After this I pushed the –3 button. When I got out of my elevator, I was on floor 1 at the library. What floor did I start on?” (*ibid*., p. 24)

An accompanying commentary provides guidance for the teacher. It states that on this type of problem, which gives various changes and the ending floor but no starting floor, students with strong understanding generally use one of three approaches: (1) Some students use an undoing strategy by working backwards starting at the ending floor, then adding (or counting forward) what was subtracted and subtracting (or counting backward) what was added. (2) Other students use another type of undoing strategy by finding the net change and adding its opposite. (3) Still others use trial, error, and adjust. The commentary also cites common incorrect strategies, indicating that missing start problems can be quite challenging for students. Nonetheless, teachers are advised to allow students to come up with their own strategies for solving them and not to be concerned if the students are not successful. This is presumably because the purpose of these problems is to formally teach students about inherent properties of net change, and to introduce informal notions of equation solving. The purpose clearly is not to teach students how to solve equations.

The fourth example concerns connections between graphs and change. In a session called “Plus and Minus Graphs,” students make graphs of changes that follow one of several sequences of positive and negative changes. For example, the teacher begins by putting the sequence of symbols + + – – 0 on the board, and says “Let’s say this is all the information we have about where the elevator goes. We don’t know the exact floors the elevator stops on, but this sequence of pluses, minuses, and zeroes gives us important information.” By considering a variety of possible graphs corresponding to the sequence (see Figure 5), students discover that all the graphs have a similar shape even though the net change can be positive, negative, or zero. They also learn that anytime there is a change from + to – the graph reaches a peak. However, when there are two peaks (e.g., + + – + + – –) they cannot know if the highest point is after the second or the fourth plus.

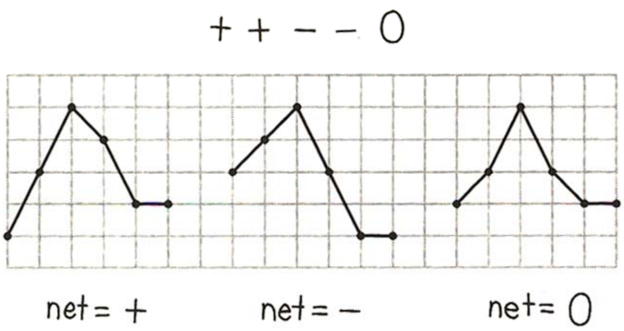

Figure 5. Graphs of a Plus and Minus Sequence

Our intent in offering these four examples is to demonstrate how the activities in *Investigations* consistently establish connections between the big idea of change and other important big ideas. Clearly, the primary reason these activities are in this third-grade unit is to establish a formal concept of net change. However, a secondary aim is to establish connections to informally-developed mathematical ideas that can eventually serve as the foundation for the formal development of advanced concepts such as integers, rates, slope, limits, derivatives, extrema, etc.

Changes Over Time

In the fourth-grade unit, *Changes Over Time* (Tierney, Nemirovsky, & Weinberg, 1998), students learn to coordinate the concepts of time and change. This coordination is accomplished principally by activities in which students learn to graph a variety of changes over time. In previous units, students' graphs focused on comparing things that take place at the same time. Even the third-grade graphs of elevator trips up and down the fantasy skyscraper largely ignored time. Successive heights of the elevator were shown on most graphs from left to right as if the elevator was simultaneously in different elevator shafts. In this way, the students' attention was focused on the change dimension of the graphs, i.e. the vertical axis. Thus, students didn't have to explicitly consider the complicating dimension of time, despite its implicit presence along the horizontal axis of their graphs. These activities nonetheless laid an intuitive foundation for the explicit coordination of change and time, a difficult task that is tackled in the fourth grade unit.

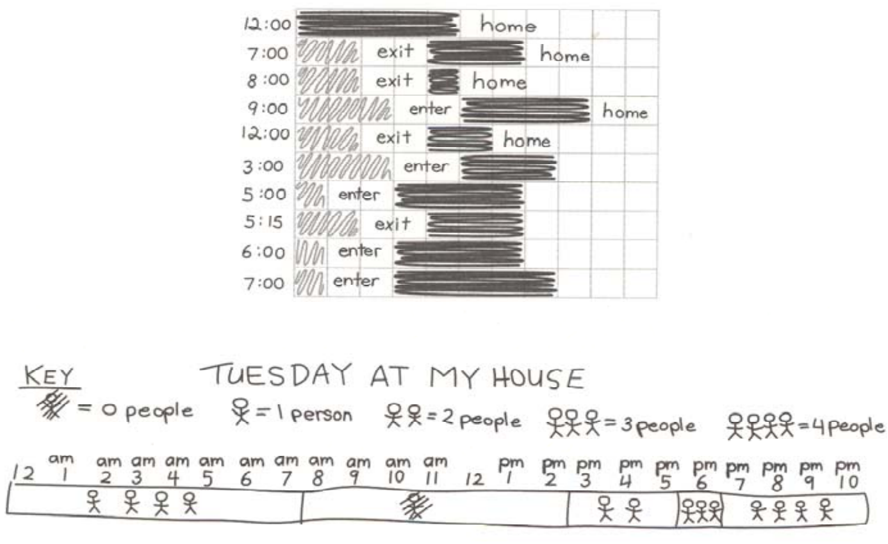

Figure 6. Graphs of Student Representations Showing Change Over Time

In the first activity of the fourth-grade unit students devise their own ways to represent changes over time for different situations such as people going in or out of their houses. Students first collect data on the number of people in their homes over the course of a typical day and then create a graphical representation of the changes in population. Once they have completed their graphs, students are paired and put into groups of four. Students in each pair exchange their graphs with the students in the other pair in their group. The pairs are asked to interpret each other's graphs and to plan how to revise them to make them clearer. Figure 6, found on page 15 of the unit, shows examples of graphs teachers can expect to receive from their students.

One purpose of discussing how to make graphs clearer is for the students to appreciate conventional graphs when they are introduced later in the unit. For example, the top graph in Figure 6 explicitly shows the relationship between time, changes in the number of people, and the total population. The bottom graph clearly shows the relationship between time and total population, but it does not explicitly show the changes in population. Discussion of this difference highlights not only important distinctions between change quantities and total quantities, but also how to find change from graphs of total population vs. time. Another fruitful discussion of these two graphs might center on the different ways they represent time. The bottom graph enumerates every hour between midnight and 10 p.m. The top graph shows only the times when people go in or out. Appropriately guided discussions can help students realize that each method has advantages and drawbacks, for example, although showing exact times gives more accurate information, it does not show an overall pattern of population change over time.

As was the case in the third-grade unit on net change, intuitive equation-solving activities are embedded in the study of changes over time. The fourth-grade “missing start” problems are structurally identical to the third-grade “missing start” problems, although they are semantically different. In third grade, the students are asked to find the start floor given an ending floor and a list of the elevator change buttons pushed. In fourth grade, students are asked to find the number of people at home in the beginning, given the number at home at the end and the numbers of people that went in and out. For example, students are asked: *Three people go out and one person returns home. Then there are four people at home. How many were home at the beginning*? The teacher commentary suggests that the students be alerted that this type of problem is a “backward problem,” that can be translated symbolically as ? – 3 + 1 = 4. As in third grade, however, students are not taught a method to find the answer. Instead students are expected to devise their own strategies to solve the problems. Examples of the three strategies students typically use (see the previous section on Net Change) are reiterated. In addition it is suggested that students can use their graphs to solve this problem.

The text also suggests that students themselves should make up a number of similar problems for the class to solve. The teacher is coached, however, on how to teach the students to write "missing start" problems: *Pick a time on your graph to start. Mark the place, but don't tell how many people are at home then. Just tell how many went in and out* (ibid., p. 25).

Clearly, the fourth-grade treatment of equations is only slightly more advanced than the third-grade treatment. The problems themselves are similar, and although there is an expectation that students learn to write equations that model the problems (without the use of a letter to represent a variable), there is no expectation that all students learn to solve them.

> It is much more difficult to find the beginning number than the end number. Don't worry if many students can't do these problems after this brief introduction. Return to these problems from time to time after the unit is over. (ibid., p. 26)

As before, this statement indicates that the purpose of these problems is to teach students about inherent properties of net change, not to teach them how to solve equations. It also indicates, however, that the authors feel it is important early on to informally introduce the process of undoing and to revisit it throughout the curriculum. In spite of the suggestion to return to these problems from time to time, formal equation solving is not covered in this curriculum. We believe that it is the curriculum's good intention to have "undoing" introduced in this informal way to prepare students' later formal equation-solving activities. However, full realization of this good intention in the *Investigations* curriculum would require some modifications to the algebra strand. See the postscript section for a discussion of the revisions planned for the algebra strand in the new edition of *Investigations*. It does not appear that formal equation solving will be a goal in the new edition of the curriculum.

Arguably, *Ways to Show Change Over Time* is the most noteworthy investigation of the unit. During the first two days of the investigation, the teacher leads discussions about how to interpret line graphs that show change over time. Graphical representations of change are interpreted from two points of view: point-based and interval-based. For interval-based interpretations, students compare the shapes of graphs that show change over time. What does a line going high up mean? What does a line going down low mean? Students learn to distinguish between representations of something that can change and representations that show change. For example, students discuss the relation between the picture of a wheelchair racer going uphill and the speed vs. time graph of a wheelchair racer going up a hill. On the third day of the investigation, students learn to make conventional line graphs. The teacher leads a demonstration/discussion about how to draw line graphs using the Cartesian coordinate system, as well as about the benefits of using this conventional approach.

Next the students graph the height over time of plants they planted before the unit began. By this time, each pair of students has a week's worth of height data to graph. Once the students finish their graphs and discuss them, they extend their graphs by drawing what they think the graph will look like in one more week. As the week unfolds, their predicted graph serves as a catalyst to help the students think about the subtle aspects of representing growth over time with a line graph. They learn to recognize the shape of the graph when their plants were growing fast or slow. They also learn to answer questions like "Is it going faster or slower than you predicted?" "Where does the graph show the fastest growth?" "Explain how the shape of the graph tells you where the plant was growing more slowly." "How can you tell where the plant was speeding up?" Unlike the expectations for equation solving, students are expected to become proficient in interpreting graphs of changes over time. To help them achieve proficiency, the curriculum provides many activities where the students match graphs with tables and with stories. Students also write stories based on graphs or draw graphs that describe stories like:

> *Your plant was growing quickly for a few days. Then you dropped it and the top of it broke off. It stopped growing for a while before it started growing again*. (*ibid*., p.92)

Growing Patterns and Rate of Change

The fifth-grade algebra unit, *Patterns of Change* (Tierney, Nemirovsky, Noble, & Clements, 1998) is the capstone unit in the algebra strand, and speed is its culminating concept. The unit provides experiences describing, representing, and comparing rates of change. The activities build on concepts established during the previous five units' work studying informal and formal change: i.e. on concepts of change and changelessness in patterns (Grades K-1), on strategies for modeling additive change (Grade 1), on concepts of time (Grade 2) and net change (Grade 3), and on graphical representations of change over time (Grade 4).

The students begin the unit by exploring growing patterns of tiles and analyzing them. They examine, in turn, one linear pattern (2, 4, 6, 8, …), two quadratic patterns (1, 4, 9, 16, … and 1, 3, 6, 10, …), and one exponential pattern (2, 6, 14, 30, …). Their focus is on two important aspects of the patterns: "step size" and "total so far." They compare geometric, graphical, and tabular representations of the patterns, noting how each version shows the number of new tiles added at each step (step size) and the number of tiles all together ("total so far"). The activities help students to recognize key qualitative relationships between rate of change (step size) and the resulting accumulation ("total so far"). For example, students learn to recognize that:

- In the linear tile pattern, the step sizes do not change, but the "total so far" increases at a constant rate.
- In both quadratic tile patterns, the step sizes increase at a constant (additive) rate, but the "total so far" grows at an increasing (additive) rate.
- In the exponential tile pattern, the step sizes increase by a constant (multiplicative) rate, and the "total so far" –2 grows at the same (multiplicative) rate.

It should be emphasized that the concept of rate of change, including speed, is developed by equating it to step size, rather than by derived ratios. The level of generalization students are expected to exhibit differs depending on the individual, and is left to the professional judgement of the teacher. All students are expected to come up with a general rule, but not necessarily an algebraic equation, for both the step size and total of the linear pattern. A Teacher Note states, "Encourage [students] to write the general rules they find in any informal way that makes sense. Do not push them to use *n* at this time" (*ibid*., p. 12). Later, when the students compare all four patterns to one another, the teacher is discouraged from requiring students to figure out general rules for the "total so far" in the non-linear patterns. The teacher commentary suggests that the general rules for the "total so far" in the quadratic patterns are much more difficult to figure out than the general rules for the step size, and only the more advanced students should be expected to find them. Throughout the unit, students are encouraged to represent even simple general rules in a mostly informal way. Formal use of variables to represent rules occurs infrequently, if at all.

The mathematical emphasis in the second investigation is on using graphs and tables (but not rules) to explore relationships among time, speed, and distance. The activities build on the first investigation's work with growing patterns by applying the general ideas of step, step size, and "total so far" to notions of time, speed, and distance respectively. In this investigation, students plan and act out trips of changing speed along a straight path. Then the students invent and discuss their own methods of representing the trips (see Figure 7). The commentary warns that the students' representations will likely be non-standard, and encourages teachers to analyze the students' work to determine which variables they explicitly represent as well as the way they represent them. For example, in Figure 7 the columns of the student's table are labeled "steps" and "speed." The student does not have columns for time, step size, or distance. However, the "4 sec." entry in the steps column shows that the student has an intuitive notion that the number of steps is somehow related to time. Also, the width of the steps in the drawing shows step size, and their accumulation shows total distance. These last two are a rather direct reproduction of the physical situation, and are not particularly informative. However, the fact that the student shows speed by the height of the steps, implies that the student thinks only vaguely of speed as a variable related to distance and time. From this, the teacher might infer that this student understands the basic building blocks of the concept of speed, and is ready to benefit from upcoming activities that integrate these difficult ideas of time, distance, and speed.

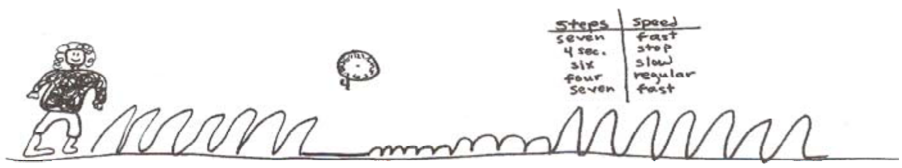

Figure 7. A student's representation of changing speeds during a trip

Besides its obvious diagnostic purpose, a goal of the foregoing activity is to prepare students to appreciate the power and elegance of conventional graphs and number tables, which they use extensively in subsequent sessions. The students act out a variety of fast and slow trips along a track, dropping beanbags at equal time intervals. They make tables and diagrams showing where the beanbags land. The emphasis is on understanding the inverse relation between speed and the number of beanbags (a measure of time) dropped over a given distance. Importantly, the students learn to represent the trip with tables that show accumulated measures of time and distance, as opposed to non-cumulative measures, which is their wont. Eventually, the students are weaned away from the need to actually act out trips and drop beanbags. Now the activities are based upon motion stories that the students must envision in their heads. They are asked to imagine where beanbags might fall and to mark them on a number line representing the track. Then they make tables corresponding to the motion stories. Teachers are encouraged to use this activity as a checkpoint to see how well students understand using tables showing position at regular time intervals to show speed. The investigation ends with tasks that strengthen the connections among motion stories, tables, and graphs of distance versus time. They match the motion stories, graphs of distance versus time, and tables that describe the same trip. They also create tables, write motion stories, and draw distance versus time graphs of the same trip.

In the third and final investigation in the fifth-grade unit, students use a computer software program called *Trips*, which is provided with the curriculum. Students investigate changes each second in the positions of computerized runners on a track over periods of ten or more seconds. The *Trips* software shows a boy and girl running a race along parallel tracks (number lines). The students can vary the starting points of the boy and girl, as well as their speeds and direction. To make the boy or girl go faster or slower, students change the size of their steps. Students can stop the trips at any time. They can also observe the action one step at a time. After running a trip, students can click on the graph or table window to see a graph or table of the trip. After a preliminary orientation to the software, the students are given three motion stories (e.g., "The girl starts behind the boy, but she passes the boy and gets to the tree first"). Students act out the stories, either physically along meter sticks or virtually on the computer, by varying the starting positions and step sizes until the resulting trips match the stories.

The investigation, and indeed the algebra strand itself, culminates with a set of tasks designed to crystallize the concepts of speed, time, and distance they have been developing. For these tasks, students compare and contrast graphs of step size versus time with graphs of position versus time. For example, one task describes a trip and presents a collection of position versus time graphs and step size versus time graphs. Students must choose the position graph and the step size graph that correspond to the trip, and then explain their reasoning (see Figure 8). By working on these tasks and discussing them, students construct a robust, though qualitative, concept of rate of change and its graphs. For example, some of the features that students learn by tasks like the one shown in Figure 8 are the following:

- The steeper (less steeply sloped) the line on the position graph, the higher (shorter) the bar on the step-size graph and the faster (slower) the motion.
- A horizontal line on the position graph corresponds to a bar of zero height on the step-size graph and a stop in the motion.
- When the line on the position graph goes up (down), the step size bars are above (below) the horizontal axis, and the motion is forward (backward).

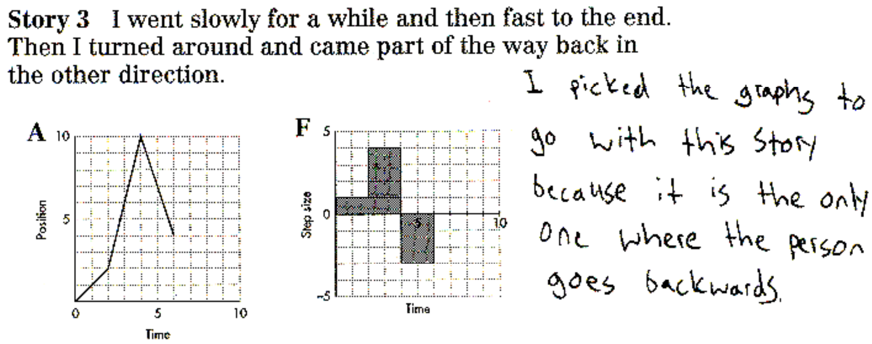

Figure 8. A student’s response to one of the culminating tasks in Patterns of Change

The development in this unit is in stark contrast to a quantitative development of distance, time and speed, which typically consists of students using the formula d=rt to solve a myriad of word problems they only marginally understand. The qualitative approach in *Investigations* is aimed at helping students understand the relationships among these sophisticated concepts, and away from teaching them to use the formula d=rt. In fact, and somewhat surprisingly, the formula d=rt is not even formally presented in the unit.

Analysis of the Development of Algebraic Processes: Habits of Thinking

In this section, we address the following the question, what mental processes are fostered in the algebra strand to bring about the achievement of these goals and big ideas? To answer the question, we analyzed how *Investigations* develops mental processes called algebraic habits of thinking. We relied on a framework developed by Driscoll (1999, 2001) which posits that people who use algebra to solve problems bring three habits of thinking into play: Doing-Undoing, Building Rules to Represent Functions, and Abstracting from Computation. In this section, we consider the development of each of these habits in turn.

Building Rules to Represent Functions

Critical to algebraic thinking is the capacity to recognize patterns, organize data, and represent situations by well-defined functional rules. In Driscoll's framework, he identified this capacity as a habit of thinking called *Building Rules to Represent Functions*, which is characterized by seven features:

- Organizing Information: In ways useful for uncovering patterns and the rules that define patterns.
- Predicting Patterns: Noticing a rule at work and trying to predict how it works.
- Chunking the Information: Looking for repeating chunks of information that reveal how a pattern works.
- Describing a Rule: Describing the steps of a rule without using specific inputs.
- Different Representations: Wondering what different information about a situation or problem may be given by different representations, then trying the different representations.
- Describing Change: Describing change in a process or relationship.
- Justifying a Rule: Justifying why a rule works for "any number."

Essentially, these features are mental processes that students use to acquire a robust concept of function and to understand change. Since *Investigations'* goals are closely aligned with Goal 1 (functions) and Goal 4 (change) of NCTM's algebra standard, it is important that the curriculum foster these seven features of the *Building Rules to Represent Functions* habit of thinking.

Let us briefly recall how the big ideas of the algebra strand are developed. The kindergarten unit of the algebra strand teaches very young students to recognize and represent simple repeating patterns. The activities in that unit begin a carefully designed succession of K-5 activities that help students attain goals consistent with the two NCTM algebra standard goals identified above. As we have seen, these activities consistently require students to discover ideas, invent representations, and design procedures based upon their existing mathematical intuitions. In the process students organize information, predict patterns, chunk information, describe rules, compare representations, and describe change—all features of the "building rules" habit of mind described by Driscoll.

We believe, therefore, that the algebra strand in *Investigations* is specifically designed to help students develop ways of thinking consistent with the habit of mind "Building Rules to Represent Functions." In this regard, the intent of the curriculum is primarily to help students establish a predisposition to qualitatively and informally look for, describe, use, and make sense of the underlying properties of patterns and functions, especially the property of change. Significantly, it is not the intent of the curriculum that students develop the ability to formally represent functions with algebraic symbols. That is, the primary goals of the curriculum are not aligned with the second goal of the NCTM algebra standard: *Represent and analyze mathematical situations and structures using algebraic symbols*. It is likely that this variance from NCTM is a deliberate attempt to forestall the predisposition of many teachers to introduce symbols before students are ready to understand them. The *Investigations* authors consistently postpone the use of the abstract symbolism of algebra, perhaps believing that the delay will help students to develop more flexible and robust mental habits. If successful, it is expected that in middle school and beyond these powerful habits of thinking will provide a foundation that facilitates the easy acquisition of formal algebraic capabilities.

Abstracting from Computation

Driscoll (2001) characterized abstracting from computation as the capacity to think about computations independently of particular numbers. Essentially, the A*bstracting from Computation* habit of mind affords students the ability to abstract regularities of the number system from the computations they perform. Since abstraction is one of the most evident characteristics of algebra, it is reasonable to conclude that algebraic thinking indispensably involves this ability to think about computations freed from the particular numbers being calculated.

Unlike the *Building Rules* habit of thinking, *Abstracting from Computation* does not appear to be a major priority in *Investigations'* algebra strand. Take, for example, the Grade 1 unit *Building Number Sense*. The unit belongs to the Number and Operations strand as well as to the Patterns, Functions, and Algebra strand. As we have seen, the activities in the unit are designed to develop addition, subtraction, and number sense. In light of recent research on the teaching of algebra, it is not surprising to see a unit cross-listed as both a number and operations unit and as an algebra unit. Carpenter, Franke, and Levi (2003) argue that a goal of instruction in arithmetic should be to help students develop "…ways of thinking about arithmetic that are more consistent with the ways that students have to think to learn algebra successfully" (p. 2). These ways of thinking correspond to the features of the *Abstracting from Computation* habit of mind identified by Driscoll.

One of the six features posited by Driscoll of *Abstracting from Computation* is *Computational Shortcuts: Looking for shortcuts in computation based on an understanding of how operations work*. Students can use this line of thinking to intuitively generalize underlying algebraic properties (e.g., equality, associativity, commutativity, distributivity) in arithmetic procedures, and implicitly use these properties to solve a variety of problems, including story problems. For example, most students who complete the first-grade unit realize that 9+4 and 10+3 give the same result. Some of them are able implicitly to use the properties of addition and subtraction to compute 9+4 by taking 1 away from 4 and adding it to 9. However, another of the features of Abstracting from Computation is *Justifying Shortcuts*: *Using generalizations about operations to justify computational shortcuts*. Therefore, if *Abstracting from Computation* were a goal of the curriculum, it would provide activities to help students reflect on the commutative and associative properties that justify this approach. However, none of the activities in the algebra strand seem to be designed specifically to do so. In fact, it seems likely that the first grade unit is considered part of the algebra strand primarily because it continues to build students' notions about pattern and change, not because it is intended to develop an *Abstracting from Computation* habit of thinking.

Nonetheless, there are many activities in the unit that can help students develop the habit of mind *abstracting from computation*. Students play games and solve problems about how numbers can be combined and separated to yield other numbers. They represent numbers with dot patterns; they count sets of objects; they find the total of two or more numbers; they compare numbers; they learn to read, write, and sequence numbers up to 100. All of these activities help students intuitively develop the *Abstracting from Computation* habit of mind. The question is, however, “Is *Abstracting from Computation* a priority that helps shape the activities in the unit?”

There are many reasons to believe that *abstracting from computation* is not explicitly intended. Here are three of them: (1) The *Building Number Sense* unit, which is part of both the number and the algebra strands, does not include activities explicitly designed to help students generalize the properties or operations of arithmetic. (2) Instead, the activities in the *Building Number Sense* unit are devoted to representing patterns and understanding change in addition and subtraction – features of the *Building Rules* habit of thinking. (3) As we have seen, priority is not given in any unit of the algebra strand to using variables to represent rules or procedures – an integral component of each of the following features of *Abstracting from Computation*:

- Generalizing beyond examples: Going beyond a few examples to create generalized expressions, describe sets of numbers, or either state or conjecture the conditions under which particular mathematics statements are valid;
- Equivalent Expressions: Recognizing equivalence between expressions;
- Symbolic Expressions: Expressing generalizations about operations symbolically.

As a consequence, we have concluded that *abstracting from computation* is not a true priority in the *Investigations* algebra strand, although the work with computation inevitably helps build an intuitive foundation for its development. It appears that the revisions being planned for the new edition of *Investigations* will be much more purposeful about establishing this habit of mind (see the Postscript section). Preliminary indications are that algebraic connections will be integrated at each grade level into the number and operations strand.

Doing-Undoing

Reversibility, that is, having the ability to undo mathematical processes as well as to do them, is an important component of effective algebraic thinking. In effect, it involves the capacity not only to use a process to achieve a goal, but also to understand the process well enough to work backward from the ending point. Possessing a *Doing-Undoing* habit of mind makes it natural to analyze mathematical tasks backwards and forwards. It forms the intellectual foundation for a large collection of mathematical activities, including equation solving, factoring, inverse functions, anti-derivatives, etc.

Driscoll (2001) identified two features of the *Doing-Undoing* habit of thinking. They are: (1) *Input from output: Finding input from output, or initial conditions from a solution*, and (2) *Working backward: Working the steps of a rule or procedure backward.* We have seen that many activities in the *Investigations* algebra strand establish informal intuitions about how operations undo one another and about how to work backwards to solve problems. An intuitive foundation for *doing-undoing* begins in kindergarten when students learn about the repeating aspects of pattern, which can be traversed forwards or backwards. This foundation is strengthened in first grade. Students represent repeating patterns on the hundred chart and gain flexible intuitions about the reversibility of functional rules. Also in first grade students learn intuitively that combining and separating are opposite actions both involving a change quantity (ultimately to be modeled by the inverse operations of addition and subtraction).

In third grade, students investigate properties of addition and subtraction by considering the net change in position of an elevator in a fantasy skyscraper. This begins the explicit study of change as a mathematical entity and starts the process of formalizing students' heretofore-intuitive notions of change. Important to the doing-undoing habit of thinking, the third-grade development of net change entails serious consideration of the undoing effect of negative change on positive change. It introduces positive and negative numbers, and broaches the relationship between positive and negative change and addition and subtraction.

Change is also the foundation for equation solving, an important undoing process in algebra. In third grade, students are encouraged to develop their own strategies to solve "equations," which are presented as problematic situations involving stories of elevator trips that describe the ending floor and various floor changes, but no starting floor. Rather than teaching equation writing and equation-solving techniques, the students are asked to invent their own ways to solve these problems. This philosophy is carried over to fourth and fifth grades, but now the students are taught to represent equations (using question marks in lieu of variables). As before, however, they are challenged to solve the equations but are not taught how.

These succession of activities in the algebra strand are clearly aimed at providing a way of thinking about *doing-undoing*, rather than establishing its more formal aspects, like rules for adding and subtracting integers or procedures for solving equations. The result (and intent) is a firm but informal foundation for qualitative understanding of important algebraic processes that involve *doing-undoing*.

Conclusion

In this case study, we have identified the goals, the big ideas, and the algebraic processes that determine the nature and scope of the algebra strand in the *Investigations* curriculum. We have found that mathematical change is the unifying big idea of the algebra strand, that its principle goals are aligned with the NCTM goals *Understand Patterns, Relations, and Functions* and *Analyze Change In Its Various Contexts*, and that its approach to achieving its goals is to foster mental processes that constitute the algebraic habit of thinking known as *Building Rules to Represent Functions*.

Based on our analysis, three reform-oriented principles appear to undergird and direct the trajectory of the activities included in *Investigations'* algebra strand: (1) Students must construct for themselves new knowledge and understandings based on what they already know and believe. (2) Understanding occurs as the result of actively constructing new knowledge that is connected to existing knowledge: the more numerous the connections to existing knowledge, the greater the understanding. (3) A rush to symbolism is counterproductive to the learning process.

As might be expected, these reform principles appear to have a profound effect on the choice of content and the development of activities in the curriculum. For example, we believe that the curriculum's consistent use of a problem-solving approach to teach new concepts and procedures is an artifact of Principle (1). In fact, we have cited numerous instances in which the curriculum delays formal treatment of mathematical content until it can enrich and refine a foundation of basic mathematical intuitions that children have informally learned through their daily interactions. Students do extensive work combining and separating everyday items in first grade, but formal introduction to the operations of addition and subtraction is postponed until grade 2. Students develop their own strategies to solve missing start problems during extensive problem solving sessions in grades 3 and 4, but formal equation solving is not taught anywhere in the curriculum. A variety of tasks in the third grade unit uses integers to model net change, but formal procedures for adding and subtracting integers are not taught in *Investigations*. In fifth grade, the entire series of activities is designed to characterize rate of change, including speed, as being the same thing as the step size (rather than a ratio) despite the obvious limitations it imposes.

Similarly, principle (3) undoubtedly accounts for *Investigations'* practice of postponing symbolic representation until the students have had significant experience using and discussing the informal representations they have developed on their own. For example, in first grade, students develop their own ways of representing numbers and of finding combinations, but writing formal addition equations (3+7=10) is postponed until grade 2. In second grade students design their own timelines before they are introduced to standard methods of representing time along a number line. In third grade students label the floors of the fantasy skyscraper, B2, B1, 0, 1, 2, (rather than –1, –2, –1, 0, 1, 2) even though they use negative numbers to represent change in the downward direction. Also in third grade, despite the fact that students invent their own methods to solve missing start problems, formally representing the problems with equations is postponed until grade 4. Even then, a question mark is used as a variable to represent the missing start in the equation. In fourth grade, students spend considerable time devising and analyzing their own ways to represent change over time, even though those methods are eventually abandoned in favor of more standard methods. In fifth grade students come up with rules that describe the generalized change in various growing patterns, but they do not use variables to express their rules. Also in fifth grade, students work extensively to analyze the relation between distance, speed, and time, but they are not introduced to the formula d=rt, nor is speed represented as a ratio.

Even major characteristics of the overall structure of the algebra strand can be explained by the three reform principles. For example, we have made the case that the concept of change is the central big idea in the algebra strand, and that it is also the foundation for establishing formal conceptions and connections to the other big ideas of the algebra strand, namely patterns and relationships, representation, and modeling. Structuring the strand in this way is arguably a consequence of the authors' adherence to principle (2) above.

What may not be as obvious are the reasons for choosing not to develop the other commonly accepted big ideas of algebra, namely, variables, proportional reasoning, structure, and equation solving. Once again, we believe the answer can be found in principles (1) and (3). The argument can be made that it is because of principle (3) that the curriculum does not progress to the use of variables and to formal symbolic work in algebra, like equation solving. Postponing the development of proportional reasoning, on the other hand, is more a consequence of principle (2). Since students' intuitions of relative change are generally poorly developed, even by fifth grade, it appears the curriculum developers have decided that it is wiser to postpone the development of proportional reasoning until the students are intellectually more mature. As we have noted in the algebra strand, which is built on the concept of change, the concept of rate of change, including speed, is not developed through the use of ratios. Finally, the reasons for not developing the big idea of algebraic structure are probably related to the reasons for the curriculum's neglect of the *Abstracting from Computation* habit of mind. As we have noted, this seems to be an oversight by the curriculum developers since the revisions being planned for the new edition of *Investigations* will be much more purposeful about establishing an *Abstracting from Computation* habit of mind.

The principles we have discussed, which shape the goals and development of the *Investigations* algebra strand, are grounded in the epistemological foundations of the reform movement in school mathematics, of which *Investigations* is a case. Because of this, we expect that many of the characteristics of the algebra strand of *Investigations* that we have discussed are illustrative of other reform curricula.

Postscript: Revision of the Investigations Curriculum

A revision of the algebra strand in the *Investigations in Number, Data and Space* curriculum began in 2001 with funding from the National Science Foundation. One of the intents of the revision is to strengthen the integration of algebraic thinking throughout the curriculum. This is being accomplished in two ways. First, the plan is to include separate curriculum units at each grade level for the patterns and functions strand. The current algebra units are being redesigned to include a more cohesive and coherent scope and sequence across the grades. This will include the development of new units on pattern and function at grades 1, 2, and 3 and significant revisions at each of the other grade levels. Second, algebraic connections will be integrated at each grade level into the number and operations strand building on the work of Carpenter, Franke, and Levi (2003) and Schifter (1999). An algebra essay is planned for inclusion in each of the number and operation units that will highlight the specific algebraic connections in the unit and a graphic marker will identify specific activities throughout the curriculum where a teacher might expect an opportunity for discussion of early algebraic ideas. For example, a draft essay discusses how students use computational short cuts based on regularities they have noticed in the number system. This is an opportunity for students to verbalize their reasoning with an eye toward formulation of generalizations, and teachers are provided with background information on how the ideas are related to future work with algebra and algebraic notation. Although the planned revision will include increased attention to students' ability to write equations, methods of solving the resulting equations will not be a priority.

References

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Driscoll, M. (1999). Fostering algebraic thinking: A guide for teachers grades 6-10. Portsmouth, NH: Heinemann.

Driscoll, M. (2001). *The Fostering algebraic thinking toolkit: A guide for staff development.* Portsmouth, NH: Heinemann.

Eston, R., & Economopoulos, K. (1998). *Pattern trains and hopscotch paths*. Menlo Park, CA: Dale Seymour Publications.

Kaput, J. J. (1998). Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K-12 curriculum. In National Council of Teachers of Mathematics & Mathematical Sciences Education Board (Eds.), *The nature and role of algebra in the K-14 curriculum: Proceedings of a National Symposium* (pp. 25–26). Washington, DC: National Research Council, National Academy Press.

Kliman, M., & Russell, S. J. (1998). *Building number sense: The number system*. Menlo Park, CA: Dale Seymour Publications.

Mokros, J., Russell, S. J., & Economopoulos, K. (1995). *Beyond arithmetic: Changing mathematics in the elementary classroom.* Palo Alto, CA: Dale Seymour Publications.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1992). *Algebra for the twenty-first century: Proceedings of the August 1992 conference*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1997). *A framework for constructing a vision of algebra: A discussion document*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Nemirovsky, R., Tierney, C., & Ogonowski, M. (1993). *Children, additive change, and calculus* (Report No. TERC-WP-2-93). Cambridge, MA: TERC. (ERIC Document Reproduction Service No. ED365536)

Noble, T., Nemirovsky, R., Wright, T., & Tierney, C. (2001). Experiencing change: The mathematics of change in multiple environments. *Journal for Research in Mathematics Education, 32*. 85-108.

Russell, S. J., Economopoulos, K., Murray, M., Mokros, J., & Goodrow, A. (1998). *Implementing the investigations in number, data, and space curriculum: Grades K, 1, and 2.* Menlo Park, CA: Dale Seymour Publications.

Russell, S. J., Tierney, C., Mokros, J., Goodrow, A., & Murray, M. (1997). *Implementing the investigations in number, data, and space curriculum: Grades 3, 4, and 5*. Menlo Park, CA: Dale Seymour Publications.

Scott Foresman (undated). *Investigations in number, data, and space: Content overview*. Chicago: Author.

Schifter, D. (1999). Reasoning about operations: Early algebraic thinking in grades K-6. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12* (pp. 62-81). Reston, VA: National Council of Teachers of Mathematics.

Steen, L. (1990). *On the shoulders of giants*. Washington, DC: National Academy Press.

TERC (1998). *Investigations in number, data, and space.* Menlo Park, CA: Dale Seymour Publications.

Thompson, P. W. (1994). The development of the concept of speed and its relation to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of calculus.* Albany, NY: SUNY Press.

Tierney, C., & Monk, S. (in press). Children's reasoning about change over time. In J. Kaput (Ed.), *Employing children's natural powers to build algebraic reasoning in the context of elementary mathematics*. Available March 16, 2003 from http://www.simcalc.umassd.edu/NewWebsite/EABookChapters.html.

Tierney, C., Nemirovsky, R., & Weinberg, A. S. (1998). *Changes over time*. Menlo Park, CA: Dale Seymour Publications.

Tierney, C., Nemirovsky, R., Noble, T., & Clements, D. (1998). *Patterns of change*. Menlo Park, CA: Dale Seymour Publications.

Tierney, C., Weinberg, A. S., & Nemirovsky, R. (1998). *Up and down the number line*. Menlo Park, CA: Dale Seymour Publications.

Wright, T., Nemirovsky, R., & Tierney, C. (1998). *Timelines and rhythm patterns: Representing time*. Menlo Park, CA: Dale Seymour Publications.

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