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Teaching Algebra for Procedural and Structural Understanding

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Abstract

Reviews of the research on teaching and learning algebra have been primarily focused on procedural and structural aspects of the subject. Findings indicate two discernible themes: the accessibility of procedural over structural interpretations and the difficulty in obtaining a structural conception of algebra. Since the more advanced, structural paradigms are commonly evident in the content and teaching of algebra, many students fail to achieve an adequate level of understanding. Research studies on the integration of graphing technology within the curriculum provide evidence that this technology can function as an intellectual tool that enriches students' mathematical explorations, while facilitating and improving their mathematical understanding, problem-solving skills, and concept development.

This report aims to resolve the inherent procedural-structural conflict within algebraic instruction by describing a didactic approach to the teaching of algebra that minimizes the amount of symbolic manipulation by integrating graphing technology into the curriculum. The report concludes by illustrating how elements of algebraic understanding may be developed in a systematic and coherent way by using non-traditional and unfamiliar problems with students.

Introduction

In many parts of the world, traditional processes and curricular topics in mathematics are under attack by educational reformers. Algebra, the core subject of many mathematics programs, frequently receives a plethora of criticism. Kaput (1995) describes a first-year algebra course in the United States as: "...the manipulation of string of alphanumeric characters guided by various syntactical principles and conventions, occasionally interrupted by 'applications' in the form of short problems presented in brief chunks of highly stylized text" (p. 71). In its present form, others equate algebra to an arcane subject that filters large numbers of students from the study of advanced mathematics. Conceptual and understandings of many algebraic foundations, even among students who have successfully completed several courses of study, are frequently lacking.

According to the National Assessment of Educational Progress (NAEP, 1990), 83 percent of American secondary school seniors had reported having taken one year of algebra, while only 56% reported having taken two years. Also, the NAEP results reported that fewer than half of the secondary students showed a consistent grasp of simple algebraic notions and only five percent demonstrated an understanding of both geometry and algebra – subjects that are necessary for the study of advanced mathematical study.

This low performance for American students has prompted the call for "Algebra for All" and related reform efforts (Edgar, 1990). Still, after years of reform colleges and universities must maintain developmental mathematics centers to help students with a poor understanding of the fundamental basics of mathematics. In this struggle to ensure "algebra for all", we are learning that students who steer away from the study of algebra often eliminate themselves from some of the options for the future (Lott, 1998).

The value of problem-solving approaches to algebra, along with open-ended inquiry, have tended to dominate the studies of the content and methods of teaching algebra (Kirshner et. Al., 1999; Day & Jones, 1997; Rachlin, 1987). A growing research base has produced studies of students' understandings of functions (Brenner, et. al., 1997; Thompson, 1994; Eisenberg & Dreyfus, 1994; Breidenbach et. al., 1992; Markovits, Eylon, & Bruckheimer, 1988) and the different representations of functions Romberg, Carpenter, & Fennema, 1993; Yerushalmy, 1991; Wagner &Kieran, 1989). Yerushalmy and Schwartz (1993) emphasized common difficulties in the teaching and learning of algebra. They found that these difficulties resulted from: (1) those rooted in sequencing; (2) those resultant from concentrating on trivial cases, and (3) those due to inactive learning.

Investigations of algebraic misconceptions have also supported increased student exploration (Grouws, 1996). Kaurand Sharon (1994) hoted that, in general, first-year college students demonstrated a superficial understanding of the underlying concepts, failed to utilize proper mathematical language, and exhibited numerous misunderstandings of procedures and identities. The study's recommendation was for instructors to not rely solely on algebraic manipulation and the application of rules, but to treat algebra as a subject with more exploration and open-inquiry questions.

In general, reviews of the research on algebraic content, teaching, and learning have been directed toward the procedural aspects of algebra (e.g., verifying that an ordered pair satisfies a linear equation) as well as the structural aspects of algebra (e.g., performing operations on algebraic expressions to yield simplified expressions) (Foley, 1998; Sfard & Linchevski, 1994). Algebra, historically may have developed along procedural-structural cycles and some researchers (Kieran, 1992) have attempted to use this procedural-structural model of algebraic thinking to reconceptualize much of the existing education research. Thus, the reported findings indicate two discernible themes: the accessibility of procedural over structural interpretations and the difficulty in acquiring a structural conception of algebra.

A general conclusion that may emerge from a review of the research on algebra teaching and learning is that the majority of students do not develop any real understanding of the structural aspects of algebra. An examination of the content in most algebra books and curricular materials in the United States points out that they tend to not incorporate a procedural-structural perspective on students' learning of algebra. Studies on American algebra instructors have noted that they favor a structural approach in their teaching. As a result, the current research literature demonstrates that while both the teaching and the content of algebra emphasize structural paradigms, most students do not reach these goals.

Recent reform policies, national documents, and vision statements encourage the infusion of technology into the existing curriculum, and in particular, the use of graphing calculator-supported explorations as contexts for mathematics instruction. The recommendations of the National Council of Teachers of Mathematics (2000), the Mathematical Sciences Education Board (1991), and the Mathematics Association of America (1991) all support this notion. The underlying assumption is that with the availability of technology, there is the potential of engaging students more actively in the processes of mathematics instruction.

In recent years, several studies and research syntheses on the impact of the use of technology on students' cognitive outcomes have provided evidence supporting the recommendations made by the mathematics and mathematics education communities (Hatfield & Bitter, 1994; Kaput, 1992; Hoyles, 1991). Another line of research has investigated the development of the fundamental algebraic concepts of function and variable while exploring problems where much of the symbolic-manipulation and graphing has been relegated to technology. Zbiek (1988) and Heid (2000, 1995, 1990) explored technologically-enhanced approaches to algebra while placing the decision-making to the students by engaging them in the analysis of realistic problems.

Others (Demana, Schoen & Waits, 1993) have found that by using graphing technology, the curriculum can become enhanced with the addition of visualization to standard algebraic techniques. In particular, it has been shown that students who use graphing calculators have been found to solve more complex algebraic problems, even problems that did not have an algebraic solution. The findings have provided evidence that the use of graphing technology as an intellectual tool has enriched students' mathematical explorations, while facilitating and improving their mathematical understanding, problem-solving skills, and concept development.

In spite of the calls for the integration of graphing technology into mathematics teaching, and current research validating its effectiveness on student learning, in most cases algebra is still being taught in a formal and symbolic-manipulative manner. Publications on the use of algebra with technology in mathematics classrooms have varied from outright recipes that students must follow step-by-step to open-ended activities that encourage inquiry, conjecturing, and generalizing. Both variations may be beneficial for students at certain times, but in encouraging students to enhance their algebraic thinking, the types of activities described in this report are especially well facilitated through the use of graphing technology.

Activities

In the first activity, students are asked to enter four functions into their calculators: $y_1 = x^2 + 5$; $y_2 = .2x^2 - 9$; $y_3 = 4 - .5x^2$; $y_4 = x^2 - 20x + .85$. The screen is set for a range of $X_{min} = -9$, $X_{max} = 9$, $Y_{min} = -6$, and $Y_{max} = 6$. When the display is shown, graphs of only three of the functions are shown. Students are asked to reflect on the graphs and to answer the questions similar to the following:

1. What type of figures should be displayed on the graph?
2. How many parabolas are displayed in the graphs?
3. Which of the equations do not correspond to points in the graph?
4. What range of values for the graph screen would clearly display all four parabolas?

A follow-up activity would be to ask the students to clear the calculator's screen, and graph $y = x^2 - 4x + 6$ by using the calculator's function drawing feature (Draw F). Then the screen will display several parabolas. Pertinent questions to ask are:

1. Why did all the parabolas appear on the screen?
2. Which parabola is the one that was intended to appear?
3. What should have been done to display only the graph of the intended parabola?

These types of questions require students to reflect on the problems more intently as they present new and unfamiliar aspects of algebra to them. Students must draw on their prior knowledge and graphing calculator skills to respond to these types of questions. Similar types of activities are worthwhile in helping to reinforce students' understanding of other types of algebraic graphs such as rational expressions, piecewise functions, and transformations.

An alternative, to as well as a check for, algebraic solutions to equations and systems of equations can be performed with graphing technology. For example, students can be given a simple linear equation such as $x - 6 = -3x + 2$. They then can readily obtain a graphical solution and a check for their algebraic solution(s) by simply graphing as separate functions the left and right sides of the equation, and determining (by using trace and zoom features) the value of x for which the two expressions are equal.

After finding this value for x, students would be able to perform transformations on the equation and see whether the transformed functions still intersect at the same point. Problems of this nature are exploratory and instructive in that they serve as an aid for students to better comprehend which transforms preserve the equivalence of equations and why.

Another example of how technology can be utilized to provide an alternate graphical solution and check on an algebraic solution is to ask the students to find the approximate solutions of a function such as $2x^3 - 6x^2 + 4x + 1 = 0$.

Once the function has been graphed, the students can use the calculator's zoom capability to trace over parts of the graph to display the independent and dependent variables and to obtain estimates of the zeros of the given polynomial.

Multiple representations of systems of equations (e.g. graphical and tabular) can improve students' development and articulation of algebraic thinking. An insightful exercise would be to have students solve the following system of equations: $-3x - 2y + 4 = 0$ and $5x + 4y - 6 = 0$ by using five methods. These methods are: (1) symbolic manipulation; (2) tracing along either line to determine the point of intersection; (3) zooming in on the point of intersection; (4) zooming in on the solution using a calculator's tables feature; and (5) calculating the solution using an automatic solver or intersect option. After using each of these five methods, students can gain further insights by writing about their preferred methods and the reasons for their choices.

Solving various types of equations, inequalities, and systems of equations has been the focus of much of the traditional algebra curriculum. In many cases, students are taught to view the type of equation to be solved and then to apply an appropriate technique such as factoring or using the quadratic formula for quadratics, factoring (if possible) for cubics and higher order polynomials, and using logarithms for exponential equations. However, when students are asked to consider an example such as $3x = x^{10}$, these traditional algebraic methods are not helpful. In the absence of a symbolic procedure, graphical and numerical procedures appear to be the only way to investigate this problem. A student could treat the left and right sides of the equations as separate functions ($y = 3x$ and $y = x^{10}$) and then search for intersection points of the two graphs. The zoom and trace features on the graphing calculator would also be helpful features in solving this problem.

Other motivating activities that may enhance the algebraic thinking of students involve the graphical verification of a variety of algebraic and trigonometric identities. For example, students can readily verify the factoring formula for the difference of cubes by entering the left and right sides as separate functions, graphing the functions, and checking that the graphs coincide. A specific example would be to graph $y_1 = x^3 - 64$, $y_2 = (x-4)(x^2+4x+16)$, and $y_3 = y_1 - y_2$ with a viewing window of $X_{min} = -5$, $X_{max} = 5$, $Y_{min} = -150$, and $Y_{max} = 100$.

In a similar manner, graphing $y_1 = \tan x$, $y_2 = \sin x/\cos x$, and $y_3 = y_1 - y_2$ with a viewing screen of $X_{min} = -2\pi$, $X_{max} = 2\pi$, $Y_{min} = -3$, and $Y_{max} = 3$, provides a graphical confirmation of this trigonometric identity. By using the same values for the viewing screen, students can justify which of the following are valid identities: $\sin^2 x + \cos^2 x = 1$; $1 + \tan^2 x = \sec^2 x$; $\csc^2 x - 1 = 1 + \cot^2 x$; and $\sin x \bullet \sec x = 1$.

In attempting such confirmations, students are provided with the opportunity to review their knowledge of basic trigonometric identities while verifying the identities without the use of lengthy algebraic proofs. The experience gained through these types of verifications serves to sharpen thinking skills and provides a visual reinforcement of their algebraic thought processes.

Nonlinear systems of equations present a new avenue for a multitude of interesting applications, many of which would be beyond the algebraic understanding of even many first-year college students without the use of the capabilities of the graphing calculator. For example, consider the two outermost planets in the solar system, Pluto and Neptune. Their orbits can be represented by the Cartesian equations $y = \pm(0.968)\sqrt{(1465 + 19.68x - x^2)}$ and $y = \pm\sqrt{(904 + 0.54x - x^2)}$, respectively, where x and y are expressed in astronomical units (A.U.), the distance from the earth to the sun. When given these equations, students can easily enter them into their graphing calculators, graph them, and view their point of intersection. When a viewing window of $X_{min} = -50$, $X_{max} = 50$, $Y_{min} = -75$, and $Y_{max} = 75$ is used, a graph of what appears to be the planets' orbits is shown. On closer inspection, the orbits appear to intersect. Activities such as this one can help students further advance their knowledge of nonlinear systems by determining the actual point of intersection. Further study can be initiated by asking students to research the motions of the planets, write a report that indicates whether Pluto and Neptune will ever collide, and give supporting facts and reasons for their findings.

Conclusion

The current research literature contains abundant studies that indicate that much of the present algebra curriculum consists of practice in symbolic manipulation. While such skills are important, students are apt to view these skills accordingly if they are presented as interesting applications and other situations that lead to algebraic understanding.

The research supports the enhancement of algebraic thinking through open-ended and exploratory activities that focus on applications, alternative representations of functions, and the use of technology. In addition, there is evidence that misconceptions in algebraic thinking including misunderstanding of identities and language inadequacies may be aided through explorations and inquiries that accentuate procedural rather than structural conceptions.

With the current emphasis on changing the presentation methods for the teaching of algebra, it is hoped that teachers will promote the integration of algebra with current technologies to give students new avenue for interesting algebraic explorations. Problems and activities such as the ones described in this report, when incorporated into the existing curricula, can give students many opportunities to reflect on the concepts and processes inherent to successful algebraic thinking. The didactic approach to teaching algebra presented in these illustrations includes explorations that use the light of current technologies to achieve a new vision for the teaching of algebra. In order to make the vision a reality in mathematics classrooms, teachers need to rethink their current practices and to teach within a classroom culture where mathematical attention is directed and sustained toward both procedural and structural understandings.

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