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Short-Circuiting The Algebra

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Abstract

The indiscriminate use of algebraic techniques to solve mathematical problems on the basis that algebra provides general tools for problem solving has often led many teachers and students to uncritically produce long and tedious analytical proofs at the expense of intuitively shorter, elegant proofs. Using different problem-solving strategies, we will look at some of the benefits that can be derived in encouraging students to speak the non-algebraic language in mathematical problem-solving.

Introduction

A non-algebraic solution to a non-routine problem is generally more beautiful than its algebraic counterpart, often requiring a high degree of intuitive and inspirational intelligence. Solving a problem non-algebraically often helps promote creative thinking in mathematics, but failing to encourage students to produce non-algebraic proofs will only atrophy their creativity to come up with elegant solutions.

The Gifted Question

Using a modified question extracted from a Singapore Gifted Secondary School Mathematics Paper, in which only an algebraic solution to the problem was given, we will see that, with some insight, it can be solved intuitively, resulting in better and more elegant solutions.

By comparing the standard algebraic solution with solutions that short-circuit the algebra, we will discuss how each method is interesting in its own way.

Farmer Yan has goats, cows and chickens in his enclosure. There are twice as many chickens as there are cows. Altogether, there are 18 heads and 52 legs. How many goats are there in the enclosure? (MOE, 1995)

Let us use a few strategies to solve the above problem.

1. Algebraic Method

Most problem-solvers are likely to solve the problem algebraically – to let the number of goats be x, and then to form the desired equation. A typical solution would probably go as follows:

Let x be the number of cows and y be the number of goats.

Then $2x + x + y = 18$

$3x + y = 18 \quad (1)$

Also, $2(2x) + 4x + 4y = 52$

$8x + 4y = 52 \quad (2)$

$(1) \times 4$: $12x + 4y = 72 \quad (3)$

$(3) - (2)$: $4x = 20$

$x = 5$

Substituting $x = 5$ into (1), we have: $3(5) + y = 18$

$y = 3$

Therefore, there are 5 cows, 10 chickens, and 3 goats.

For experienced problem-solvers, expressing all three unknowns in terms of one variable, thus forming only one equation, would suffice to solve the question.

2. Guess-and-Check Method

There are 18 animals altogether. If 9 animals were goats and the other 9 were chickens and cows (i.e., 6 chickens and 3 cows), they would then have 60 legs.

Chickens	*Cows*	*Goats*	*Legs*
6	3	9	60

If we take a smaller number of goats, say 6, then we will have 8 chickens and 4 cows, yielding a total of 56 legs.

Chickens	*Cows*	*Goats*	*Legs*
8	4	6	56

Let us try 3 goats. We then have 10 chickens and 5 cows.

Chickens	*Cows*	*Goats*	*Legs*
10	5	3	52

This last combination yields the solution.
Hence, there are 10 chickens, 5 cows, and 3 goats.

3. Intuitive Method I

Let us look at one solution that short-circuits the algebra, which uses less trials and less guesswork, but more reasoning to solve the chicken-goat-cow problem.

If each chicken is standing on one leg, and each cow or goat is standing on its hind legs, only half of the legs would be used, i.e., $\frac{52}{2} = 26$ legs.

In the number 26, the head of a chicken is counted just *once*, but the head of a cow and the head of a goat are each counted *twice*.

Subtracting the number of heads (18) from 26, we are left with the number of cows and goats, i.e., $26 - 18 = 8$.

Thus, there are $(18 - 8) = 10$ chickens.

And, there are $\frac{10}{2} = 5$ cows, and $(8 - 5) = 3$ goats.

Hence, there are 3 goats.

4. Intuitive Method II

Another non-algebraic intuitive solution that uses a rather ingenious idea to solve the chicken-goat-cow problem is as follows:

> If all 18 animals had only 2 legs each, there would only be 36 legs.
>
> The extra (52 - 36), or 16 legs must come from either goats or cows.
>
> Since every two extra legs come from either goat or cow, we have $\frac{16}{2}$, or 8 four-legged animals.
>
> Number of chickens = 18 - 8 = 10.
>
> Number of cows = $\frac{10}{2}$ = 5.
>
> Number of goats = 18 – 10 5 = 3.
>
> Hence, there are 3 goats.

Clearly, the non-algebraic solution yields a more elegant solution than its algebraic counterpart.

5. Intuitive Method III

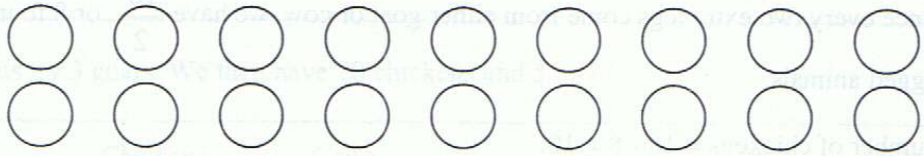
Let us look at one more non-algebraic intuitive solution that uses one step less than that of *Intuitive Method II* to solve the chicken-goat-cow problem.

> If all 18 animals had 4 legs each, there would be 72 legs.
>
> The extra (72 - 52), or 20 legs must come from the chickens, each contributing an excess of 2 legs.
>
> Thus, we have $\frac{20}{2}$, or 10 chickens.
>
> Therefore, there are $\frac{10}{2}$, or 5 cows, and (18 - 10 - 5), or 3 goats.
>
> Hence, there are 3 goats.

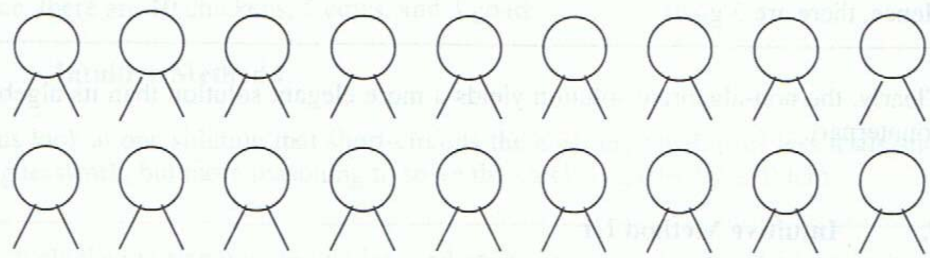
6. Draw-a-Diagram Method

The draw-a-diagram strategy, which requires the problem-solver to draw pictorial representation of the problem, is relatively easier to understand, especially among young children, as compared to the previous strategies used earlier.

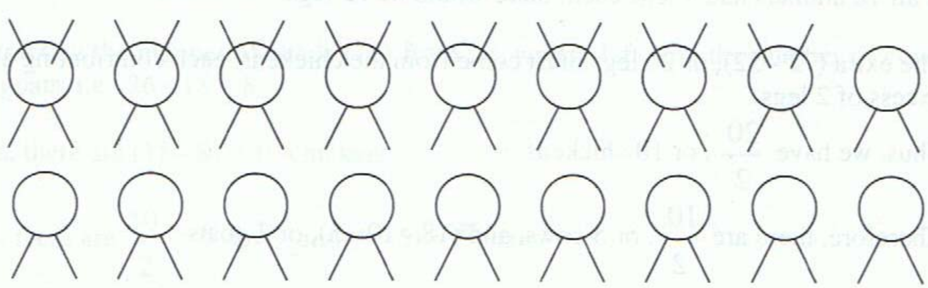
1. We draw 18 circles to represent the total number of chickens, cows, and goats. We also note that each animal has at least 2 legs (a chicken has 2 legs, and both cows and goats have each 4 legs).



2. Next, we assign 2 legs to each animal, represented by a circle, by drawing two lines to each circle. Altogether, we allocate 18 pairs of legs to the 18 animals.



3. The remaining unassigned legs belong to the cows and the goats. By assigning two more legs to each animal until all the legs have been taken up, we observe that only 8 animals have 4 legs each.



4. The above pictogram shows there are (18 - 8) = 10 chickens.

Since the number of chickens is twice the number of cows, therefore, the number of cows is $\frac{10}{2} = 5$.

Therefore, the number of goats is (18 - 10 - 5) = 3.

Hence, there are 3 goats.

Discussion

Which Method is Better?

Looking back at the six strategies to the chicken-goat-cow problem, we observe that each approach has its merits and demerits – each is interesting in its own way. While there exist other strategies to tackle the above problem, which approach to use will depend on the problem-solver's maturity and mastery of basic skills (Fong, 1999).

Algebraic Method

The advantage of the algebraic method is that it works equally well for large numbers as for small ones. However, algebraic solutions rarely spark any bright idea, just a little proficiency in the use of algebraic language.

For beginners who use the algebraic method, the question might be redundantly translated into a system of two equations with three unknowns. Students should be encouraged to formulate a problem with the least number of variables so that the solution may then be more direct. One such solution may be as follows:

Let the number of cows be x.

Then the number of chickens and goats will be $2x$ and $18 - 3x$, respectively.

Thus, $4x + 2(2x) + 4(18 - 3x) = 52$.

On simplifying, $x = 5$.

Number of goats = $18 - 3(5) = 3$.

While the algebraic method remains the most commonly used method among school children, solving a problem algebraically often limits the technique to a mere symbol manipulation activity, and may over time nurture inflexibility among its practitioners. Algebra, which is rich in structure, is weak in meaning.

Guess-and-Check Method

The solution by guessing-and-checking lends itself easily to the given numbers. More successive trials would be needed for larger or more complicated numbers. The guess-and-check strategy is a valid heuristic often ignored by teachers who discourage students from using it, because it allegedly promotes wild guessing among its users.

Far from being mere guesswork, the guess-and-check strategy helps develop number sense – a feeling for what makes a good guess different from a bad guess, especially when faced with unfamiliar problems, whose algorithms we do not know.

In the long run, the guess-and-check heuristic makes us become risk-takers; it brings with it an intuitive favour to problem solving.

But blindly embracing the strategy can take hours. We need to know whether a potential solution path is worth pursuing; otherwise, we need to consider alternative solution paths.

Intuitive Methods

Because of its intuitive approach, many students generally find difficulty understanding a non-algebraic solution, often showing surprise that the method is much shorter than its algebraic counterpart. Many students simply lack the experience and maturity to appreciate the elegance of an intuitive solution.

While the intuitive methods involve less trials and less guesswork, they require more reasoning. The solution in all three cases is ingenious, requiring a clear intuitive grasp of the problem.

Generalisation

To better appreciate the bright ideas from intuitive solutions, Polya (1981) used the generalisation approach to explain them. Ironically, here we use an algebraic argument to show why a non-algebraic method works.

(i) **Intuitive Method I**

Let x be the number of cows and y be the number of goats.

If h stands for the number of heads, and l for the number of legs, then

$$2x + x + y = h$$
$$3x + y = h \qquad (4)$$

Also, $2\,(2x) + 4x + 4y = l$
$$8x + 4y = l \qquad (5)$$

(4) x 2: $6x + 2y = 2h \qquad (6)$

(5) – (6): $2x + 2y = l - 2h$
$$x + y = \frac{l}{2} - h \qquad (7)$$

Translating equation (7), this means:

The number of cows and goats equals one half of the number of legs, less the number of heads. This is what we did in ***Intuitive Method I***.

(ii) **Intuitive Method II**

From equation (7), $x + y = \frac{l}{2} - h$.

Re-arranging, $x + y = \frac{l - 2h}{2}$.

This equation can be translated as:

The number of four-legged animals (cows and goats) equals half the difference between the number of legs and twice the number of heads. This is what we did in ***Intuitive Method II***.

(iii) **Intuitive Method III**

From equation (4), $3x + y = h$.

Re-arranging, $y = h - 3x$. (8)

From equation (5), $8x + 4y = l$.

Substituting (8) into (5), we have $8x + 4(h - 3x) = l$.

Simplifying, $2x = \frac{4h - l}{2}$. (9)

Translating equation (9), this means:

The number of chickens ($2x$) equals half the difference between the number of legs and four times the number of heads. This is what we did in ***Intuitive Method III***.

While an ingenious solution brings an element of surprise, we need to bear in mind that bright ideas are rare – we need a lot of luck to conceive one (Polya, 1981). And, the challenge is to look for the most elegant solution among those that short-circuit the algebra.

Draw Diagram Method

This strategy, which requires students to pictorially represent the situation, is particularly suitable for Upper Primary School children. As long as children have mastered the Counting-On Strategy (Fong, 1999), and are aware which animals have 2 legs or 4 legs, the approach may even be taught to children at lower age groups.

The Draw-a-Diagram Strategy is useful and powerful as a problem-solving heuristic - generally for solving problems set at a higher level. A routine algebraic problem at the secondary level may be re-classified as a non-routine problem at the primary level if a slightly modified strategy is used (Fong, 1999).

However, when the numbers used in the question are rather big, thus requiring many pictograms, the Draw-a-Diagram Strategy might have to be sacrificed for less time-consuming methods.

Conclusion

While algebra remains undeniably a generally powerful problem-solving technique to solve mathematical problems, an uncritical exposure to algebraic techniques may blind us from exploring other alternative methods of solution that short-circuit the algebra – many of which are often shorter and more elegant than their algebraic counterpart.

We saw how an elementary problem like the chicken-goat-cow problem lends itself to different strategies, thus enabling us to look at a problem from different angles – there are more than one way to skin a mathematical cat.

Moreover, far from merely indulging in algebraic symbolism, we get an intimate understanding of the problem when the method of solution short-circuits the algebra. While we need more time and effort to look for intuitive solutions, the benefits and experience gained far outweigh the disadvantages.

Short-circuiting the algebra in a problem often encourages the following:

- Enhances creative thinking in mathematics.
- Awakens students' awareness that algebraic thinking could be independent of algebraic symbolism.
- Trains the mind to look for intuitive solutions that generally require a higher degree of inspiration.
- Requires a clear intuitive grasp of the problem situation.
- Enhances problems at a higher level to be set at a lower level using intuitive methods or modified strategies.

Let us remember that an elegant, non-algebraic solution is worth more than a thousand algebraic routines.

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