The Mathematics Educator
1999, Vol. 4, No.2, 83-90

A Multimedia Approach to Visualize Projectile Motion with Air Resistance

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ABSTRACT

Consider a projectile fired from the ground level with an initial velocity v ft/sec, at an angle θ with the horizontal. Using Newton's second law in physics, one can show that in the absence of air resistance, the projectile follows the path of a parabola. However, this is not quite true if the air resistance is taken into account. The mathematical models describing the equations of motion for the projectiles with air resistance are usually complicated. However, such equations can be analysed using today's fast computers, combined with powerful computer algebra systems (CAS). This paper describes how to use the multimedia capabilities of the CAS *Mathematica* to visualize the projectile motion with air resistance. As a by-product of the paper, we will also observe the role of a CAS as an experimentation or a conjecture-forming tool in contemporary mathematics education and research.

INTRODUCTION

Suppose that a projectile is thrown from a point O on the ground with an initial velocity v ft/sec, at an angle θ with the horizontal. Consider a coordinate system OXY in the plane of the motion, with O as the origin, the horizontal line OX as the positive direction of the X-axis, and the vertical line OY as the positive direction of the Y-axis. Let (x, y) be the position of the projectile after t seconds, with respect to this coordinate system. Newton's second law of motion implies the following equations of motion in the absence of air resistance:

$$x = v \operatorname{Cos}\theta \, t \qquad (1.1)$$

$$y = v \operatorname{Sin}\theta \, t - g t^2 / 2 \qquad (1.2)$$

In the above, g is the acceleration due to gravity, which is equal to 32 ft/sec ^{2}. One can eliminate t between the equations (1.1) and (1.2) and denote y by $y(x)$ to obtain

$$y(x) = x\,\mathrm{Tan}\theta - g\,x^2 / (2\,v^2 \mathrm{Cos}^2\theta) \tag{1.3}$$

The equation (1.3) is a quadratic function with a negative quantity as the coefficient of x^2. Therefore, its graph is a parabola opening down. Hence it follows that, in the absence of air resistance, the path of a projectile is a parabola.

However, when the air resistance is taken into account, the above equations no longer hold. We will use the popular model, which assumes that at any time t, the frictional force due to air is directly proportional to the velocity of the projectile (Symon 1971). Even though not considered in this paper, one can refer to (de Alwis 1995a) for a more general model. Denoting the mass of the projectile by m, and the constant of proportionality by b, one can obtain the following equations of motion:

$$x = \frac{m\,v\,\mathrm{Cos}\theta}{b}\left(1 - e^{-\frac{bt}{m}}\right) \tag{1.4}$$

$$y = \left(\frac{m^2 g}{b^2} + \frac{m\,v\,\mathrm{Sin}\,\theta}{b}\right)\left(1 - e^{-\frac{bt}{m}}\right) - \frac{m\,g\,t}{b} \tag{1.5}$$

Eliminating t between the equations (1.4) and (1.5) and denoting y by $y_{air}(x)$ yields the following equation of the trajectory of the projectile with air resistance:

$$y_{air}(x) = \left(\frac{m g}{b\,v\,\mathrm{Cos}\theta} + \mathrm{Tan}\theta\right)x - \frac{m^2 g}{b^2}\ln\left(\frac{m\,v\,\mathrm{Cos}\theta}{m\,v\,\mathrm{Cos}\theta - b\,x}\right) \tag{1.6}$$

The equation (1.6) does not represent the graph of a parabola, in general. Therefore, when the air resistance is taken into account, the path of the projectile is not a parabola.

A computer algebra system (CAS) such as *Mathematica* is quite useful in analyzing the equations (1.1)-(1.6). Some good references on *Mathematica* are (Gray & Glynn 1991) and (Wolfram 1991). For the uses of *Mathematica* as a powerful problem solving, pattern recognition, and a conjecture-forming tool, the reader can refer to (de Alwis 1993, 1995b, and 1998).

The following two examples show how to use *Mathematica* to perform some basic calculations involving equations (1.1)-(1.6).

Example 1.1 A particle is fired from the origin with an initial velocity of 40 ft/sec at an angle of 35° with the horizontal. Find the position of the particle after 1.3 seconds, in the absence of air resistance.

Solution The given data implies that $v = 40$ ft/sec, $\theta = 35^\circ$ and $t = 1.3$ seconds. In this system of units, $g = 32$ ft/sec^2. The equations (1.1) and (1.2) imply that $x = 40 \times \text{Cos}(35^\circ) \times 1.3$, and $y = 40 \times \text{Sin}(35^\circ) \times 1.3 - 32(1.3)^2/2$. One can use *Mathematica* to evaluate these two expressions. The input lines are given below:

```
0*Cos[35 Degree]*1.3

0*Sin[35 Degree]*1.3 - 32(1.3)^2/2
```

In order to evaluate the above, click anywhere on the command lines, and press **"Shift-Enter"**. The outputs are approximately 42.60 and 2.79. This means that $x \approx 42.60$ ft and $y \approx 2.79$ ft. Therefore, in the absence of air resistance, the approximate coordinates of the particle are $(42.60, 2.79)$. •

Example 1.2. A particle weighing 0.5 lb is fired from the origin $(0,0)$ with an initial velocity of 40 ft/sec, at an angle of 35° with the horizontal. Find the position of the particle after 1.3 seconds, if the air resistance is taken into account. Assume that $b = 0.2$.

Solution Use $v = 40$ ft/sec, $\theta = 35^\circ$, $t = 1.3$ seconds, $g = 32$ ft/sec^2, $m = 0.5$ and $b = 0.2$. The equations (1.4) and (1.5) imply that $x = 0.5 \times 40 \times \text{Cos}(35^\circ)(1 - e^{-0.2(1.3)/0.5})/0.2$, and $y = (0.5^2 \times 32/0.2^2 + 0.5 \times 40 \times \text{Sin}(35^\circ)/0.2)(1 - e^{-0.2(1.3)/0.5}) - 0.5(32)(1.3)/0.2$. In order to evaluate these, use the following *Mathematica* input lines:

```
0.5*40*Cos[35 Degree](1 - Exp[-0.2*1.3/0.5])/0.2
(0.5^2*32/0.2^2+0.5*40*Sin[35 Degree]/0.2)(1 - Exp[-0.2*1.3/0.5]) -
0.5*32*1.3/0.2
```

The outputs are $x \approx 33.21$ ft, and $y \approx 0.35$ ft. Therefore, in the presence of air resistance, the approximate coordinates of the particle are $(33.21, 0.35)$. •

In the following sections, we will show how to use *Mathematica* to carry out simulated experiments on projectile motion with and without air resistance. These experiments lead to conjectures. We will also discuss how to make the simulation into a "*Mathematica* movie" by using the multimedia capabilities of *Mathematica*. What we have used was *Mathematica* version 3.0, running on a *Windows* 95 platform.

A Simulation Of The Projectile Motion With And Without Air Resistance

We have written a *Mathematica* program, based on equations (1.1)-(1.6), to simulate the projectile motion with and without air resistance. The program uses input values for v, θ, m, b and g. For any time t, the program calculates the coordinates of the position of the projectile without and with air resistance, via the equations (1.1), (1.2), (1.4) and (1.5). The *Mathematica* command **"ParametricPlot"** was used to plot the trajectory of each projectile, without and with air resistance.

Example 2.1 Before executing the Program 2.1 given below, one must assign values for v, theta, m and b. (see the first line of the program). For example, let us assign the values v = 60, theta = 30 Degree, m = 10, and b = 2. One must be careful when assigning a value for theta. Make sure to type "30 Degree" rather than just "30". Then click on any line of the program, and press **"Shift-Enter"**. Group the resulting sequence of graphs into a single cell and double click on it to start the animation (Wolfram 1991). When run on a color monitor, one can observe the motion of two projectiles, one in solid red (without air resistance) and the other in dashed blue (with air resistance). On the top of each graph, one can see the current time and the position of each projectile. Not only that, accompanied with every movement of the projectile, one can also hear an interesting sound. A student with a sharp ear can notice that this sound also changes with time. This sound was created internally in *Mathematica* using the command **"SampledSoundList"** (see the lines 15 and 16 of Program 2.1). One can also add sounds externally using any sound source. At the very bottom, the maximum heights (h and h[air] without and with air, respectively) and the ranges (r and r[air] without and with air respectively) of the projectiles are displayed. One can make this even more interesting. One can export a few frames of this animation to a graphics program such as *Canvas*. Within *Canvas*, one can start editing these frames. One can also import images from popular clip art packages and then do the final editing within *Canvas*. Then export the final edited frames back into *Mathematica*. In this way, the instructor can make "*Mathematica* movies" to make a good impact on teaching! Some frames of the animation are given below:

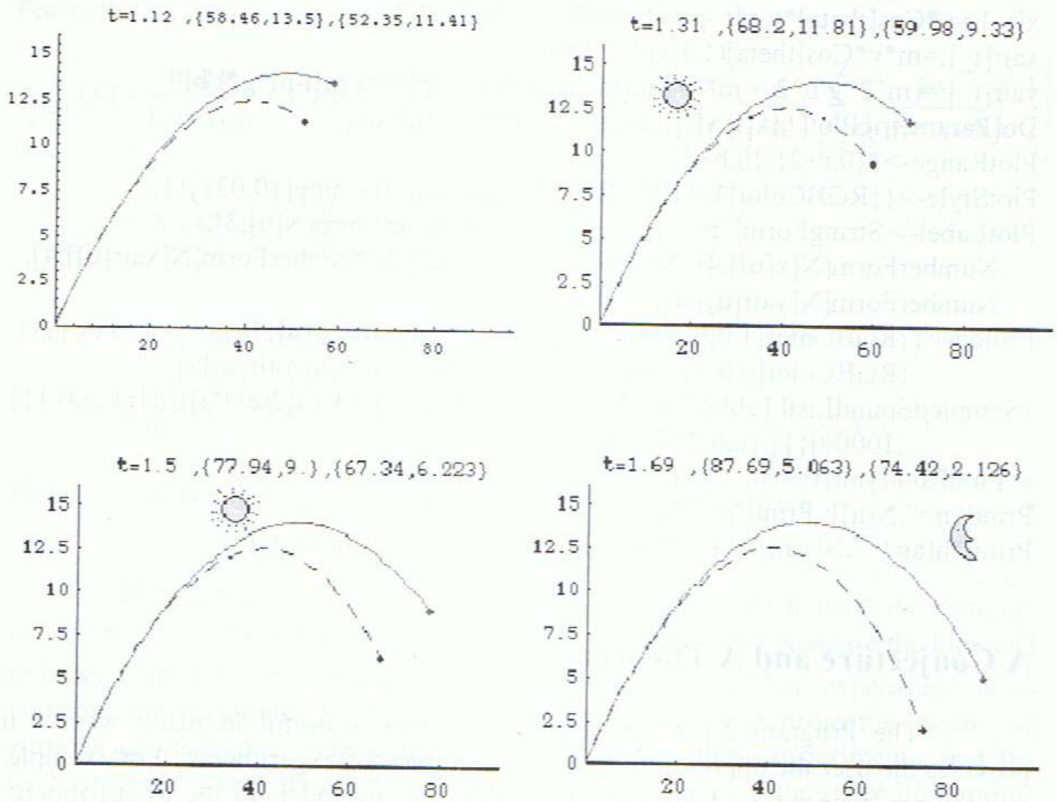

Figure 2.1 : Projectile Motion with and without Air Resistance

Here is the *Mathematica* program that we used to create the simulation of projectiles with and without air resistance.

Program 2.1

```
v=?;  theta=? ;  m=?;  b=?;  g=32;
r=v^2*Sin[2*theta]/g;  h=v^2*(Sin[theta])^2/(2*g);  T=v*Sin[theta]/g;
Tair=m*Log[1+b*v*Sin[theta]/(m*g)]/b;
x[t_]:=v*Cos[theta]*t; y[t_]:=v*Sin[theta]*t-g*t^2/2;
xair[t_]:=m*v*Cos[theta](1-Exp[-b*t/m])/b;
yair[t_]:=(m^2*g/b^2 + m*v*Sin[theta]/b)*(1-Exp[-b*t/m])-m*g*t/b;
Do[ParametricPlot[{{x[t],y[t]},{xair[t],yair[t]}},{t,0,u},
PlotRange->{{0,r+2},{0,h+2}},
PlotStyle->{{RGBColor[1,0,0]},{RGBColor[0,0,1],Dashing[{0.03}]}},
PlotLabel-> StringForm["t=``,{``,``},{``,``}",NumberForm[N[u],3],
   NumberForm[N[x[u]],4], NumberForm[N[y[u]],4],NumberForm[N[xair[u]],4],
   NumberForm[N[yair[u]],4]],
Prolog->{{RGBColor[1,0,0],PointSize[1/60],Point[{x[u],y[u]}]},
          {RGBColor[0,0,1],PointSize[1/60],Point[{xair[u],yair[u]}]},
{SampledSoundList[Table[100*Sin[Cos[100+Cos[100+Cos[2000*i]]]],{i,0,u,0.1}]
          ,10000]}}],{u,0,2*T,2*T/10}];
s=FindRoot[yair[t]==0,{t,2*T}];
Print["h=",N[h]]; Print["r=",N[r]];
Print["h[air]=",N[yair[t]/.t->Tair]];  Print["r[air]=",N[xair[t]/.s]]
```

A Conjecture and A Theorem

The Program 2.1 of the previous section is useful in many ways. It provides the user the opportunity to carry out numerous experiments. For example, one can run the program for various values for v, θ, m and b to see what happens. Each time it seems as if the solid red graph lies above the dashed blue graph. Recall that the solid red graph is the trajectory of the projectile without air resistance while dashed blue graph is the trajectory of the projectile with air resistance. This is quite natural to expect, since the air resistance impedes the motion. Therefore, one can form the conjecture that the trajectory of the projectile with air resistance always lies below that of the projectile without air resistance. How can one mathematically verify the conjecture using the equations (1.1)-(1.6) ? The answer is provided below.

Theorem 3.1 For any $x < mv\text{Cos}\theta / b$, $y(x) > y_{air}(x)$.

Proof The key idea is to expand the logarithm term of equation (1.6) in a Taylor series (Larson & Hostetler 1998). One obtains, for $bx/(mv\text{Cos}\theta)<1$,

$$y_{air}(x) = \left(\frac{mg}{bv\text{Cos}\theta} + \text{Tan}\theta\right)x - \frac{m^2 g}{b^2}\sum_{i=1}^{\infty}\left(\frac{bx}{mv\text{Cos}(\theta)}\right)^i \frac{1}{i}$$

For such x,

$$y_{air}(x) = \frac{mgx}{bv\text{Cos}\theta} + x\,\text{Tan}\theta - \frac{m^2 g}{b^2}\left\{\frac{bx}{mv\text{Cos}\theta} + \frac{b^2x^2}{2m^2v^2\text{Cos}^2\theta} + \sum_{i=3}^{\infty}\left(\frac{bx}{mv\text{Cos}\theta}\right)^i \frac{1}{i}\right\}$$

$$= x\,\text{Tan}\theta - \frac{gx^2}{2v^2\text{Cos}^2\theta} - \frac{m^2 g}{b^2}\sum_{i=3}^{\infty}\left(\frac{bx}{mv\text{Cos}\theta}\right)^i \frac{1}{i}$$

Therefore, the equation (1.3) implies that

$$y_{air}(x) = y(x) - \frac{m^2 g}{b^2}\sum_{i=3}^{\infty}\left(\frac{bx}{mv\text{Cos}\theta}\right)^i \frac{1}{i} \quad \text{for} \quad x < mv\text{Cos}\theta/b$$

Hence, it follows that $y(x) > y_{air}(x)$ for any $x < mv\text{Cos}\theta / b$. •

Here are some more experiments that one can perform using the Program 2.1: One can fix the values for v, θ, b (and of course g) and compare the blue and red trajectories for increasing values of mass m. Another experiment is to compare the maximum heights and the ranges of the two projectiles with and without air resistance. The conjectures formed from these experiments, and the theorems so obtained are not included in this paper because of the space limitations.

In summary, this paper illustrates novel ways of teaching projectile motion with air resistance, using the multimedia capabilities of a modern CAS. Similar methods can be used in teaching other mathematics/science topics as well. Even though we have used *Mathematica*, one can also use other CAS such as *Maple* and *Derive*. The multimedia approach we have discussed serves more than just a medium for visualizátion: It also enables the user to experiment freely to come up with his or her own conjectures. The student might be able to discover important theorems in the process!

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