The Mathematics Educator
1999, Vol. 4, No.2, 1-21

Assessing Students' Thinking in Modeling Probability Contexts

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Abstract

This study investigated how students and adults use probability-generating devices (colored bears, dice, spinners, colored disks) to model different contextual tasks. Seven Midwestern USA students in grades 2 through post-secondary were interviewed to assess their probabilistic reasoning as they identified models for one-dimensional and two-dimensional tasks. The results showed that all but the second grade student were able to use correspondence between sample space elements in the context and in the generator to identify and justify their models. Even prior to instruction students exhibited intuitive notions of correspondence that played a major role in enabling them to model. Students who demonstrated stronger links between modeling and theoretical probability were also able to model two-dimensional contexts and make more connections among equivalent generators. A key implication of this study is that students' intuitive knowledge of correspondence is a powerful concept on which to build instruction in experimental probability.

Introduction

Worldwide reform movements in school mathematics (e.g., Australian Education Council [AEC], 1991, 1994; National Council of Teachers of Mathematics [NCTM], 1989) have brought renewed emphasis to the study of probability at all levels of the curriculum. More specifically, this emphasis incorporates an expectation that even young children will develop greater understanding of probability through experiences involving modeling and experimentation (NCTM, 1989).

Although there has been substantial research on elementary through college students' probabilistic thinking (Fischbein, Nello & Marino, 1991; Green, 1991; Jones, Langrall, Thornton, & Mogill, 1997; Piaget & Inhelder, 1951/1975; Shaughnessy, 1992), little of this research has focused on probability modeling or the use of probability generators to produce probabilities (Truran, 1996; Truran &

Ritson, 1997). Given the importance of research-based knowledge of students' thinking in informing instruction (Fennema et al., 1996), there is a need to expand the research base on students' thinking in probability modeling.

Aims of the Research, Definitions, and Conceptual Framework

This study seeks to address the void in research-based knowledge of students' thinking in experimental probability by investigating elementary, middle, high school, and university students' thinking on a series of contextual tasks that incorporate probability modeling. In particular, the study asked the following questions:

(a) What is the nature of the probability reasoning used by students with varying mathematical experience when they are asked to identify probability generators to model a contextual task?

(b) Can students recognize and explain when two different probability generators are equivalent and are they able to identify more than one probability generator to model a contextual problem?

Definitions

In this study, probability generator refers to a random device that produces a specific probability distribution. For example, a die when rolled, is a probability generator that produces a finite rectangular distribution in which all of the six die outcomes are equally likely.

Modeling a contextual problem occurs when a student selects a probability generator whose sample space outcomes and their probabilities can be matched with the corresponding outcomes and probabilities of the contextual problem. For example, if a problem situation involved the random drawing of "one of six videos from a basket," students could model this probability situation using a spinner with six equal sectors where each sector represented one of the videos to be drawn from the basket.

Two probability generators are said to be equivalent if they produce the same probability distribution. More precisely, two generators are equivalent if their sample spaces correspond and corresponding sample points have the same probabilities.

Conceptual Framework

In this study we explored students' thinking as they attempted to establish models for both one-dimensional and two-dimensional probability situations. For example, modeling the "the selection of one snack item" from a vending machine whose button labels have been erased illustrates a one-dimensional situation; whereas, modeling "the selection of two snack items," one from each of two such unlabeled vending machines, represents a two-dimensional situation.

The major goal of the study was to investigate elementary, middle, high school, and college students' thinking when they attempted to model a probability situation embodied in a real world context. Although research indicates that even young children possess varying levels of understanding of sample space and theoretical probability (Jones et al., 1997; Watson, Collis, & Moritz, 1997), we found few studies that looked at students' probabilistic thinking when they try to model contexts that incorporate probability (Shaughnessy, 1992). It is one thing to identify the outcomes and determine the event probabilities associated with a probability generator; it is another to identify or construct a probability generator that will faithfully represent the probability distribution embodied in a contextual task.

This goal was addressed by observing and probing students' thinking as they selected and justified their choice of probability generator(s) for six tasks located in three contextual situations. As a consequence of using this set of common tasks, the researchers were also able to observe cross-sectional changes in students' thinking with age and mathematical experience.

Even though there is a substantial body of research that has looked at students' thinking in probability situations that involve experiments (Konold, Pollatsek, Well, Lohmeirer, & Lipson, 1993; Shaughnessy, 1992; Jones et al., 1997; Kahneman & Tversky, 1972), we found little research that investigated students' capabilities in modeling contextual problems. Some exploratory studies by Watson (1980) and Everton (1984) suggest that these processes are enjoyable but challenging for high school students.

The second research goal is an extension of the first and seeks to examine the ease with which students can identify different probability generators to model the same contextual problem. This also begs the question as to whether students can recognize when and why two probability generators are equivalent. While research is sparse on both of these questions, a recent study (Jones, Langrall, Thornton, & Mogill, 1999) reported that elementary students who exhibited strong numerical reasoning in probability were able to recognize when two probability generators were equivalent. However, this study did not address the broader question dealing with students' ability to identify and justify multiple generators to model a contextual problem.

Methodology

Subjects

Seven Midwestern USA students ranging in school level from second grade through second year university were the sample for this study. The sample was purposefully chosen in the sense that the students were seen by their teachers to be at least average achievers in mathematics and fluent communicators in terms of sharing their thinking. They also covered the wide range of age and mathematical maturity desirable for this study. The students, represented by pseudonyms, were: Adam, a second grader at a small parochial school; Beth, a fourth grader at a public elementary school; Cathy, a sixth grader in a public middle school; Dunicia, a tenth grader of Indian heritage at a university laboratory school; Edward, an African American student in grade twelve at the same school as Dunicia; Faith, a university sophomore majoring in business at university; and Ginny, a student who had completed high school two years previously and dropped out of university before she had completed one semester. Adam, Beth, and Cathy had no prior experiences in studying probability; Dunicia and Edward had studied some introductory and largely theoretical probability in their mathematics courses. Ginny and Faith had undertaken some formal study of probability in a finite mathematics course.

Procedure

All 7 students responded to the assessment protocol in an interview situation conducted by the first author. Interviews lasted approximately 30 - 45 minutes and were audiotaped (some were videotaped) in a quiet space that facilitated high quality transmission. An arrangement whereby the student sat behind a table opposite the interviewer enabled probability generators and other materials associated with the assessment tasks to be clearly seen and operated by the students. Students were also provided with pencil and paper and were encouraged to write or draw diagrams to explain or clarify their thinking.

Instrumentation

The research questions and the conceptual framework guided the design of the assessment protocol which comprised a warm-up-activity and six tasks, located in three contexts (see Table 1). These contexts were chosen so that they would be relevant to elementary through university students. An additional task was used with the fourth grader and two high school students, but was omitted in the remaining interviews to keep the interview time within 30 minutes. This additional task will not be discussed further.

In the warm-up task, each student was asked to use the probability generating devices which included: a spinner with six congruent sectors; a spinner with two congruent sectors; a standard die; a bag containing bears in six different colors; and a two-sided chip, red on one side and yellow on the other. After being asked to carry out a single trial with a device, students were asked to list the possible outcomes. All students except Adam, the second-grader, were able to list the possible outcomes for each generator.

Table 1. Summary of Probability Tasks

Task	Type of Problem
1a. A group of six friends are gathered to spend an evening together. They decide to watch a video. Each person wants a different movie. The videos they want are: the Disney movie <u>Aladdin</u>; the Disney movie <u>Beauty and the Beast</u>; the Disney movie <u>The Lion King</u>; the animal adventure <u>Babe</u>; the animal adventure <u>Incredible Journey</u>; and the space movie <u>Apollo 13</u>. Suppose you wanted to give everyone an equal chance of getting his/her movie. How could you do that using one of these? (Point to spinners, dice, cup of colored bears.)	1-dimensional; event of one outcome
1b. How could you use this (pick up first probability generating device used) to find the chance of getting a Disney movie?	1-dimensional; event of multiple outcomes

Table 1 (…cont'd)

Task	Type of Problem
2a. It's dark. Terry wants to get dressed, but can't see what color the clothes are. In the top dresser drawer are six shirts, each a different color: red, blue, white, green, yellow, and tan. In the bottom drawer, there are two pairs of pants: one pair is gray and the other pair is blue. How could you use one or more of these to model what Terry did?	2-dimensional; 2 events of one outcome each
2b. Suppose Terry has in the top drawer three red shirts, two blue shirts, and one green shirt, and the bottom drawer still has the same pants: one pair gray, one pair blue. How could you use these (point to tools) to find the chance that Terry got a blue shirt and gray pants?	2-dimensional; 1 event of multiple outcomes
3a. You go to a vending machine to get a snack. Through the window, you can see six choices: apples, oranges, bananas, pretzels, crackers, and your favorite kind of candy bar. The buttons are all worn off so you can't tell which button goes with which snack. You really want pretzels or crackers. What is the chance that you get what you want? Could you use one of these (point to tools) to model your problem?	1-dimensional; event of multiple outcomes

Table 1 (...cont'd)

Task	Type of Problem
3b. A few days later, you and a friend go to a different place and both want snacks. You find two vending machines side by side, but the words are all rubbed off the buttons again and you can't tell which button gives which snack. The snacks in the machines are identical: apples, oranges, bananas, pretzels, crackers, and candy. The machines are not necessarily the same, so you can't count on the third button on both machines giving the same snack. You and your friend decide to each press a button on one machine at the same time. What is the chance that you both get fruit? How could you use these (point to tools) to model this problem?	2-dimensional; 2 events of multiple outcomes

The six tasks are presented in Table 1. Tasks 1a and 1b are set in the context of a party where each one of six friends at the party wanted to see a different movie. After the context for the tasks was read and the students were shown the list of movies, they were asked in Task 1a how they could use one or more of the probability generators to give everyone an equal chance of getting his/her movie. In Task 1b, they were asked how they would use the probability generators they had selected in Task 1a to find the chance of getting a "Disney Movie" (three of the six movies were Disney movies). In Task 2a, Terry randomly selected a shirt and a pair of pants from six different-colored shirts and two different-colored pairs of pants, respectively. Students were asked how they could use one or more of the probability generators to model what Terry did. In Task 2b, with a different composition of shirts (see Table 1), they were asked how they would use the probability generator(s) to find the chance that Terry chose a blue shirt and gray pants.

In Tasks 3a and 3b, the context involved vending machines for which the six snack choices were random because the names on the buttons had worn off. In Task 3a, the students were asked how they would find the probability of a compound event (pretzels or crackers) using one or more of the probability generators to model the one-dimensional situation. In Task 3b they were asked how they would solve a problem in the same context, but involving a two-dimensional situation.

Each of the six tasks was designed to address both research questions. Tasks 1a, 1b, and 3a focused on one-dimensional situations, while Tasks 2a, 2b, and 3b focused on two-dimensional situations. In carrying out the assessment interviews involving these tasks, the first researcher maintained strict adherence to the protocol. However, general probes like "tell me more," "could you say that in another way?" "show me," were used to encourage students to clarify and elaborate on their responses.

Data Sources and Analysis

Data on students' probabilistic thinking were collected from three sources: (a) students' taped responses to the three tasks in the assessment protocol, (b) written responses or artifacts produced by students in relation to the three tasks, and (c) researcher field notes taken during the interview.

Following the interviews, the first researcher generated transcripts for each of the seven interviews, wrote summaries of her field notes, and examined student artifacts. A double-coding procedure described by Miles and Huberman (1994) was used to code the transcripts based on the six assessment tasks. Using this procedure, both researchers independently coded the responses on each task using their own multiple codes, such as preferred probability generator, solution strategy, number of probability generating devices used, and misconceptions. For example, in the solution strategy code, sub-codes were used to characterize the strategy (subjective, qualitative, quantitative) and its validity (invalid, valid). Following independent coding by the two researchers, consensus was reached on which codes and sub-codes were to be used. Subsequently, both researchers used the agreed-upon coding system to review their coding of the data on the 7 students' responses to the assessment tasks.

During the coding process, both researchers used a grounded-theory approach (Bogdan & Biklen, 1998) to discern key thinking patterns exhibited by the 7 students in modeling contextual probability tasks. These patterns were correlated, synthesized and then used to describe and interpret students' probabilistic thinking with respect to the two research questions.

Results

Table 2 categorizes students' responses to the 3 one-dimensional and 3 two-dimensional tasks in the assessment protocol. As shown in this table, students' response strategies were classified into 4 groups by mathematical validity and reasoning process. The students' reasoning was coded as mathematically <u>valid</u> when they identified one or more probability generators that appropriately modeled the contextual problem and were also able to explain why their model worked. Otherwise, it was coded as <u>invalid</u>. A number of codes were generated by the researchers to describe the reasoning processes used by the students. These codes--idiosyncratic, one-to-one and other correspondences, procedural probability knowledge, and conceptual probability knowledge--will be explained and illustrated below.

A student's reasoning was coded as <u>idiosyncratic</u> if the student used subjective reasoning or deterministic thinking to exercise irrelevant or unwarranted control over a probabilistic situation. For example, when asked if he could use one of these (probability generators) to model the first vending machine (Task 3a), Adam replied "... I'd just try and find the janitor" [who would open the machine for him].

Table 2. Student Strategy and Validity Summary

Students (Grouped by Strategies)	Strategies - Validity • Reasoning Category	
Name (grade/age)	One-Dimensional Tasks 1a, 1b, 3a	Two-Dimensional Tasks 2a, 2b, 3b
Adam* (grade 2)	- Invalid • Idiosyncratic	- Invalid • Idiosyncratic
Beth (grade 4) Cathy (grade 6) Dunicia (grade 10)	- Valid • 1.1 and other correspondences	- General invalid • Idiosyncratic
Faith (university sophomore)	- Valid • 1.1 and other correspondences • Procedural probability knowledge	- Valid modeling with prodding/Generally invalid reasoning • 1.1 and other correspondences • Procedural probability knowledge

Table 2 (...cont'd)

Edward (grade 12) Ginny (20 year-old)	- Valid • 1.1 and other correspondences • Procedural probability knowledge • Conceptual probability knowledge	- Generally valid • 1.1 and other correspondences • Procedural probability knowledge • Conceptual probability knowledge

* Adam was not asked to solve Task 3b.

For one-to-one correspondence, the student represented the contextual situation with a probability generator in such a way that each simple outcome of the contextual probability tasks was matched with a corresponding outcome of the probability generator and their respective probabilities were also matched. Other correspondences were sometimes used. For example, in Task 1b, Beth modeled the event of selecting a Disney movie by putting six different-colored bears in a bag. She said, "Blue, purple, and red could be [a Disney movie]... if you drew one of those you could watch any Disney movie you wanted to." She used a three-to-three correspondence in the sense that 3 different-colored bears represented the 3 Disney movies, although she did not identify individual matchings. Edward solved the same task (1b) using a two-portioned spinner and a three-to-one correspondence, "...you could say if it lands on the red side of the spinner, you watch a Disney movie. If it lands on the other side, you watch a movie that's not Disney." In essence, Edward matched the three Disney movies with the red side of the spinner.

Procedural probability was the code used when a student applied a probabilistic formula without explanation or indication of conceptual understanding. For Task 3a, a vending machine task, Edward figured the chance of getting pretzels or crackers as "two over six...[because] there's six total different items, so two divided by six equals like .33333, one-third."

A response was coded as conceptual probability if the student explained his or her thinking using probability concepts that went beyond formula recitation. In deciding how to model the selection of shirts and pants in Task 2a Ginny reasoned, "Well, she could pick any of these six shirts with the gray [pants] and that would be six different outfits and any of the six different shirts with the blue and that's another six outfits so in total that would be twelve outfits. So she picked one outfit, green with blue, and that's one outfit out of twelve that she could have picked."

Students' probabilistic reasoning

Our first research question asked what probability thinking was used by students when they attempted to identify probability generators to model a contextual task. In accord with Table 2, we have reported these results of students' probabilistic thinking under two categories: one-dimensional tasks, and two-dimensional tasks.

One-dimensional tasks. When modeling contextual tasks, Adam, a second grader, exhibited idiosyncratic or deterministic thinking rather probabilistic thinking. For example, when modeling the selection of a video (Task 1a, Table 1), Adam responded that the children should vote to determine what movie would win. He said, " [I would] just pick two movies and see how many people want that movie [sic]." In other tasks, Adam consistently used a voting-elimination process by pairing off elements in the sample space and reducing the number of choices. In essence, Adam did not reason probabilistically when asked to model contextual tasks.

Beth, a fourth grader, used one-to-one correspondence to solve several tasks. For example, she used six different-colored bears, the six portions on a spinner, or the six numbers on a die to represent the six videos in Task 1a. In Task 1b, choosing a Disney movie, Beth used three-to-three correspondence (three colored bears to represent three Disney movies) and three-to-one correspondence involving a spinner. In the case of the three-to-one correspondence, she chose a two-portioned spinner and said, "The pink represents the Disneys and these [blue] represent the other movies, so you have the same chance - the same amount."

Cathy, a sixth grader, also used one-to-one correspondence in Task 1a. After validly modeling this task using bears, the six-portioned spinner, and a die, Cathy attempted to use the two-portioned spinner. She said, "Well, this one [the two-portioned spinner], I don't think you could use [it] ... unless you paired them off two by two. And say, like Aladdin and Beauty and the Beast, you spun it and Beauty and the Beast won and Beauty and the Beast and Lion King would have to verse off and Lion King won and you could keep on going until one person wins." Cathy didn't seem to realize that her elimination process would not produce equally likely probabilities for the six movies. Although her use of correspondences was strong, it was not as consistently valid as that of Beth.

Dunicia, a tenth grader, used one-to-one correspondence and three-to-one correspondence in modeling the selection of a Disney video (Task 1b). She said, "This side will be Disney [blue]; this side won't be Disney [pink]... you could spin it and whichever side it goes on [determines if a Disney movie is watched]." She also used many-to-many correspondence when modeling with the die, "One through three could be the Disneys and four through six could be the non-Disneys." Dunicia described the videos as "the Disneys" and the "non-Disneys", an indication that she understood the concept of an event and its complement.

Faith, a university sophomore, used one-to-one and other correspondences and procedural probability knowledge in solving these tasks. She used correspondences much like the students described previously, but also demonstrated some probabilistic reasoning. For example, she said "The chance of getting a Disney movie is one out of two or three out of six," and later, "there's three Disney movies, six total." Notwithstanding these exemplars of her probability knowledge, Faith relied almost totally on the use of correspondences to establish probability models and didn't explicitly make links between the correspondences and the procedural knowledge.

Edward, a high school senior, used one-to-one and many-to-many correspondences in a manner similar to the others. When he attempted to use 6 two-sided disks (red on one side and yellow on the other) in Task 1a, Edward initially used an invalid representation: "You could have each kid throw one up in the air all at the same time...if they all land red, that's Aladdin, and if one of them lands...yellow and the others are red, then it's Beauty and the Beast...two of them Lion King, and so on. Recognizing his error he observed, "Everybody wouldn't [have the same chance] because there's no way they are all going to land red." Although Edward recognized his error he was not able to produce a valid model with the 6 disks. Like Cathy, it seems that he was more comfortable with models that readily fostered correspondences.

However, Edward did demonstrate greater ability than Faith and the other school students in making connections with probability knowledge. For example in Task 3a, Edward noted he had a thirty-three percent chance of getting crackers or pretzels from a vending machine and selected his model to represent this ratio. He explained, "two over six... I wanted either pretzels or crackers and there's six total different items, so two divided by six equals like .33333, one-third."

When Ginny, the twenty-year-old who was not in college, was asked about her use of colored bears to model Task 1a, she reasoned, "there's six different colors of bears... six friends which meant six videos." She explained that this model gave every one an equal chance, "Because ... there's a different colored bear for each video which represents each person. So you can just pick at random and no one person knows any better than the other." Showing conceptual understanding of the probabilistic aspects of this task, she indicated that no one controls the result and even linked this to the term "at random." Similarly in her solution to Task 3a on the vending machine, Ginny responded, "you could push a total of six buttons or you could get six different things, but you only want two of them (pretzels or crackers) so it'd be two-sixths, which is a one-third chance." In the subsequent modeling process she used a six-portioned spinner, and built on her probabilistic reasoning. Like Edward, Ginny's thinking was enhanced by strong conceptual links in her probability knowledge.

Two-dimensional tasks

As was the case in one-dimensional tasks, Adam (grade 2) showed little or no probabilistic thinking in two-dimensional modeling (Tasks 2a and 2b). His reasoning was typically idiosyncratic. Given the consistency of his reasoning, the interviewer did not ask Adam to solve Task 3b (with two vending machines).

In contrast to their thinking on the one-dimensional tasks, Beth (grade 4), Cathy (grade 6), and Dunicia (grade 10) generally solved the two-dimensional tasks in an idiosyncratic manner. For example, in Task 2a (selecting a shirt and pants), Beth's initial response was typically idiosyncratic, "if you pack your own clothes, then you would know what was where." When she did come up with a model, she used the red side of the chip to represent the first three colors of shirts and the yellow side the other three. She paired off the colors of shirts and used a "play-off" strategy until a final shirt color was determined. Then she used a similar strategy for the pants. Her thinking is probabilistic but it is invalid because it doesn't produce an assignment of equal probabilities. Moreover, Beth's strategy reduces the task to 2 one-dimensional problems. Such a reduction strategy, also used by Cathy and Dunicia, models the selection of shirts and the selection of pants, but it does not produce a model which pairs shirts and pants as required by the problem context.

Faith, the university student, used appropriate modeling, but generally invalid reasoning in solving two-dimensional tasks. For Task 2a, she said, "you could have one chip which represents the pants--the blue and the gray. Six bears represent the different shirts." Subsequently she constructed three additional valid models for this task by using the two-color chip with a die, the two-portioned spinner with the six bears, and the two-portioned spinner with the die. Faith realized that the task was two-dimensional, but struggled to rationalize the probabilities implicit in her models. She said, "If you have blue pants, you have one out of two probability and for getting a red shirt you have a one out of six chance and so getting the two together you have a one out of two chance. That doesn't seem right."

Although all her models implied that there were 12 pairings, she wasn't able to build on this to produce a valid probability statement.

Both Edward (high school senior) and Ginny (post-secondary but not in college) were successful in modeling Tasks 2a and 2b, dealing with shirts and pants. They used a variety of models in combination and talked about the problem as two-dimensional, always linking a shirt with a pair of pants. Ginny drew her own diagram using lines to connect pairs of shirts and pants. She also indicated there were twelve different outfits in Task 2a. She then used her diagram as a referent to select generating devices and explain her reasoning.

For Task 3b (two friends at two vending machines trying to get fruits on both, with fruits as three of six options), both Edward and Ginny initially modeled a <u>single</u> vending machine with two buttons for each option. After probing they realized this was not a valid model. For example, Ginny had used one cup containing 12 bears, 2 from each of 6 colors. When the interviewer asked, "How does this model the vending machines?" Ginny responded, "Well, there's twelve different choices." Then she paused briefly and revised her model, "We should do this separate. We should do it like this and then we should have another cup for these." Ginny divided the bears into 2 groups, with 1 of each color in each of the two cups. Then she said, "So I have three in six chance over here and she has three in six chance because we don't put them all together...."

As is evident in the excerpt above, Ginny and Edward appeared to visualize the original task and then set up two-dimensional models: six bears in each of two different cups, six numbers on each of two dice, or two spins of a six-portioned spinner. While this problem challenged them, both were able to work their way through to valid models using correspondence and connections to their conceptual and procedural knowledge.

Equivalence of generators and multiple models

Our second research question focused on students' probabilistic thinking when they were asked to identify more than one probability generator to model a contextual problem and, ipso facto, whether they were able to recognize when two probability generators were equivalent. Adam (grade 2) attempted to use several different probability generators after much probing. However, because he did not construct appropriate or consistent models of the same contextual problem, there was no reason for him to focus on the equivalence of probability generators. Note the disparity of his models for Task 1a: a two-portioned spinner using a play-off strategy; one die for each video, the video with the highest scoring die to be watched; and all the colored bears in spite of different numbers of each color. Not surprisingly, Adam did not consider equivalence or recognize any relationship between his chosen probability generators.

All of the students except Adam identified more than one valid probability generator for some tasks. This multiple modeling seemed to be based on an understanding of equivalent probability generators. It may be that the frequent use of correspondences fostered recognition of equivalent models. Beth (grade 4) was able to select different probability generators when asked to do so and her explanations of how she used different generators showed parallel reasoning. For example, in Task 1a (video selection), she said, "There's six bears, six different colors and you could put them in a jar [sic] and shake the bag up and you could reach in and pick one. And you supposedly pick orange for the person who wanted The Lion King ..." When asked to use another tool, she identified the six-portioned spinner, explaining, "It's got six different little utility stickers so you could spin the spinner and...if you got yours [sticker/sector] you would pick your movie." What is somewhat unclear is whether Beth recognized the need for equal probabilities in establishing correspondences.

Cathy (the sixth grader) moved fairly quickly from one model to another in solving Task 1a. She chose a six-portioned spinner and explained the correspondences. At the end of her description, she said, "And the same thing with this, you could just roll the dice..." and she then pointed to bears, describing a valid model when asked to explain. As noted earlier (p. 12), after using several equivalent probability generators appropriately, Cathy sometimes used the remaining devices invalidly. Once again, it is not clear whether Cathy took cognizance of sample space probabilities when establishing correspondences.

Dunicia (grade 10), Edward (grade 12), Faith (university sophomore), and Ginny (post-secondary but not in college) generally used probability generators appropriately and moved quickly among models, indicating representations that were equivalent. They demonstrated understanding of equivalent models by the ease with which they moved among them and in their explanations which referred to both sample points and corresponding probabilities.

Edward illustrated the fluency with which the more mathematically mature students constructed multiple probability models. For Task 1b (Disney video), he said, "You could say if it lands on the red side of the spinner [two-portioned], you watch a Disney movie. If it lands on the other side, you watch a movie that's not Disney. And you could also... say if it lands on a[n] odd number --Disney movie, even number non-Disney movie... And you could flip one of these [two-sided disks, red and yellow]-- red Disney. You could grab two bears and assign Disney to one color." In contrast with the less mathematically mature students who hesitated between models and often needed further probing, Edward's thinking was very fluent. He moved from one generator to another, often volunteering several models without being asked. This fluency was also characteristic of Faith and Ginny.

Discussion

Recent reforms in mathematics curricula have produced a renewed emphasis on the learning of probability and, in particular, on learning of probability modeling (AEC, 1991, 1994; NCTM, 1989). This study addressed the need for further research on probability modeling by examining students' reasoning when they were asked to identify one or more probability generators that model a series of contextual tasks. The 7 students whose probabilistic thinking was sampled ranged from lower elementary through post-secondary.

All students in the sample, except Adam (grade 2), demonstrated some valid probability strategies when asked to identify probability generators that modeled contextual situations. Adam consistently used subjective judgments and deterministic reasoning in a manner that is consistent with earlier research on young children's probability thinking, albeit research that did not involve modeling contextual tasks (Acredelo, O'Connor, Banks, and Horobin, 1989; Fischbein et al., 1991, Jones et al., 1997). The remaining 6 students, at least in one-dimensional contexts, showed that they were able to use 1-1 and other correspondences to establish valid probability models, in some cases prior to any instruction in probability. The thinking demonstrated by Beth (grade 4) and Cathy (grade 6) on most one-dimensional tasks supports this conclusion because neither of them had studied probability in school. Moreover, because all of these six students used 1-1 and other correspondences even when they had access to additional conceptual and procedural probability knowledge, there is evidence that the concept of "correspondence" is a powerful and natural one in tasks involving probability modeling and experimental probability.

Even though we recognize the presence and power of "correspondence" in students' probabilistic thinking in modeling processes, we do not claim that all six students identified in the previous paragraph had a complete understanding of the meaning of correspondence in probability modeling. In fact, our data suggests that only Edward and Ginny were able to consistently make the kind of connections that linked the sample points and their probabilities in the contextual problem and with those of the probability generator. There are instances when the other four students made complete connections (sample points and their probabilities) as with Beth (grade 4) in Task 1b, "The pink represents the Disneys and these [blue] represent the other movies, so you have the same chance--the same amount." However, these are rare; even Faith (a university sophomore) seldom verbalized connections between her procedural probability knowledge and her approach to modeling. In spite of these limitations, the fact that the elementary school students in this study except Adam brought prior knowledge of correspondence has implications for classroom instruction in probability modeling and simulation. It suggests that teachers can use correspondence as a powerful concept on which to build.

The results of this study vis-à-vis two-dimensional probability modeling reveal how cognitively challenging this process is, even for older students. Only Edward (grade 12) and Ginny (post-secondary but not in college) demonstrated valid probability strategies and reasoning in the two-dimensional modeling tasks in this study. Although previous research has already documented the difficulty of two-dimensional theoretical probability tasks for school students and adults (Fischbein et al., 1991; Fischbein & Schnarch, 1997; Green, 1991; Lecoutre & Durand, 1988; Lecoutre, 1992), this study has added to the earlier finding by revealing similar cognitive complexity for two-dimensional probability modeling. For example, in Task 2a, the two-dimensional modeling task involving shirts and pants, Beth initially regressed to subjective reasoning. Later in the same task she appropriately translated the two-dimensional task into 2 one-dimensional models but treated each separately and hence was not able to model the two-dimensional context. Interestingly, Cathy and Dunicia modeled Task 2a in the same manner, ignoring the need to connect the 2 one-dimensional models.

A priori, it seems reasonable to conjecture that two-dimensional probability modeling tasks would be less complex than two-dimensional probability theoretical tasks. Such a position is arguable on the grounds that a two-dimensional modeling task doesn't require a complete listing or counting of all the outcome pairs; only recognition that an outcome will be an ordered pair of the Cartesian product of 2 one-dimensional sample spaces. Notwithstanding this argument, our data offers little support for the conjecture that two-dimensional modeling tasks are less complex than two-dimensional theoretical tasks. For example, in the two-dimensional shirts-pants task, Beth, Cathy and Dunicia all started appropriately by recognizing the need for 2 one-dimensional models (one for shirts and one for pants), but none of them realized that the modeling required a paired outcome--one from the shirts' probability generator and one from the pants' probability generator. For these three students it seems that inability to model is closely associated with a lack of basic theoretical probability knowledge. By way of contrast, Edward and Ginny were successful in modeling two-dimensional contexts and their success seemed to be linked to their ability to make conceptual and procedural connections with theoretical probability. Even though we did not pursue theoretical probability solutions, the reasoning of both of these students suggests that they could have solved the two-dimensional tasks using their theoretical probability knowledge. Only Faith's actions produced somewhat contradictory evidence that could be construed as suggesting that two-dimensional modeling tasks are less complex than two-dimensional theoretical tasks. She was able to model two-dimensional contexts even though she was consistently unable to make confirmatory conceptual connections with the theoretical probability aspects of the context. Given the need for students to forge strong links between probability modeling knowledge and theoretical probability knowledge, an implication for instruction is to adopt an approach that enables students to build explicit connections between the concepts of experimental and theoretical probability.

With the exception of Adam, all students showed some facility in identifying multiple probability generators to model the contextual tasks, albeit largely restricted to one-dimensional tasks. Notwithstanding these limitations, even the younger students, especially Beth (grade 4), demonstrated appropriate strategies in identifying multiple models for contextual tasks. Once again the key strategy was "correspondence" and this worked effectively for Beth and Cathy (grade 6), providing the sample points and their probabilities were explicit. Cathy, in particular, tended to overgeneralize, forcing correspondences where they did not readily exist or reverting to subjective reasoning. What distinguished the thinking of older students like Dunicia, Edward, Faith and Ginny from younger students like Beth and Cathy was the former's ability to make justifiable and explicit connections between context and probability generator in terms of <u>both</u> sample points and their probabilities. In essence, the students with greater mathematical maturity had acquired a stronger conception of when two probability generators were equivalent. Given the importance of the concept of "equivalent probability generators" in the modeling process, curriculum designers would be well advised to incorporate this concept into early probability experiences undertaken by elementary children.

While the size of the sample demands that caution be exercised in making conclusions about the development of probability reasoning, the results of this study suggest that there is a positive relationship between mathematical maturity (age) and students' ability to engage in probability modeling. However, in accord with the research of Fischbein and Schnarch (1997), there are inconsistencies in this pattern. Faith, the university sophomore with arguably the most experience in the study of probability, consistently demonstrated weaker probability reasoning than Edward (grade 12) and Ginny (post-secondary but not in college). Moreover, Beth (grade 4) often demonstrated more complete strategies and conceptions than Cathy (grade 6).

Future research needs to examine in greater depth the conceptions and misconceptions in probability modeling that elementary and middle school students bring to the classroom. In addition, research needs to trace students' learning in probability modeling during instruction--especially the way that students develop concepts like correspondence. Modeling is, of course, only part of the process of determining an experimental probability through simulation; the process includes other components such as: defining and carrying out a trial, making observations and collecting data on repeated trials, and using this data to find the experimental probability (Gnanadesikan, Scheaffer, & Swift, 1987). Students' thinking in these other components needs to be investigated and, more particularly, future research needs to examine the intuitions and conceptual knowledge students use in the entire process of simulation.

This study has sought to open up research in the field of probability modeling. Its results reveal that even younger students bring powerful intuitions like correspondence to the modeling process. The research also reveals that even mathematically mature students don't necessarily make connections between the modeling process and their conceptual and procedural knowledge of theoretical aspects of probability. Clearly, more research on students' thinking in modeling and experimental probability is needed if the expectations of international reforms are to be met in relation to the learning of key processes in mathematical probability (AEC, 1991, 1994; NCTM, 1998).

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