The Mathematics Educator
1998, Vol. 3, No. 2, 120-127

The Buzzwords of Mathematical Education

Charles Cooke

Abstract

Three "buzzwords", Constructivism, Multiple Embodiments and Manipulatives, are discussed as examples of recent reinterpretations of ideas which have been in vogue for some time – Learning by Discovery, Varied Examples, and Concrete Apparatus. Old ideas can be reinterpreted in the light of modern developments and can continue to give us useful insights into teaching strategies.

It has become commonplace that ideas which are proposed in mathematical education just as in other fields of human endeavour, containing many partial insights into the problems under discussion, are then grossly inflated, and are seen as being a panacea or cure-all for all our problems. What is particularly depressing is the fact that in many instances these ideas have been in circulation for some time, frequently under different names, and have now been brought forward again as though they were entirely new. As examples we could consider **Constructivism, Multiple Embodiments,** and **Manipulatives**.

Constructivism

The **constructivist** viewpoint can be described as implying that the learner has to construct her own knowledge about the world. At one end of a constructivist spectrum we have what is known frequently as "Radical Constructivism". This must imply that "experience teaches", a view which would not be unfamiliar to many child-centred writers in the early part of this century. The learner reflects on what her experience is, and builds up explanations of that experience. Of course there are many shades of opinion as to what the place of the teacher is in all this; at the extreme end of the spectrum we have the somewhat unrealistic picture of a learner proceeding in a vacuum, and deriving great theories to fit what she has experienced, without the "interference" of a teacher. This is not very far removed from a "Learning by Discovery" view which would expect the learner to make discoveries without the intervention of a teacher. Social constructivism, at the other end of the constructivist spectrum, suggests that not only should a teacher be present, but also should have an important, even a major role to play. But other learners should also be present, and the discoveries made are more as a result of the interactions between the learners, and between the learners and the teacher, and between new experience and old knowledge, than as a result of an individual's efforts. In this respect this approach is more akin to the way in which scientific research proceeds, as a result of team co-operative efforts. Social constructivism then corresponds to a modified "Guided discovery" approach. We could say that any version of the Constructivist spectrum may have useful insights to give us, but no version should be taken as a recipe of universal success. Could we establish a well-known result as a result of adopting a position somewhere along this spectrum? Let us consider the result known as

Euler's Theorem for Polyhedra

F, E, V are the numbers of faces, edges, vertices of a polyhedron. Then **F - E + V = 2**

As a preliminary we could consider what we mean by the term "polyhedron", but the reader is referred to the work of Lakatos (1976) for a discussion of this. At one end of the spectrum we would have a learner provided with a variety of different Polyhedra - cubes, tetrahedra, pyramids and prisms - and asked to reflect on what could be discovered (it could be argued that this is a caricature of what is suggested); at the other end of the spectrum we have the situation where the learner is asked to tabulate the values of F, E and V, and make out a list of values of F - E + V. This is not actually "discovery" at all, as the result is "given away" at the very outset, and the learner may wonder why the particular expression has been chosen. So this also is seen as a caricature, but it has appeared in a text! Instead let us set up a scenario which will operate in a co-operative way. This needs a class of learners; it cannot be undertaken by an individual working alone. The class is divided into groups who are given their instructions independently of other groups. The instructions for the first group would be for each member of the group to draw a network of 5 polygons; the group to look at all the different networks of polygons drawn, and come up with a relationship, if there is any, between the number of edges and the number of faces. They should come up with the suggestion (hypothesis) that the number of edges less the number of vertices is 4. Our second group is given the task of drawing networks with 6 polygons in each; neither group will know that the tasks of the other groups will be significantly but only slightly different. The second group will have as their result, 5. Then the groups can compare their results, and find that they are different, and decide in a social way, what a global conjecture would be. What they should obtain is this

Euler's Theorem for networks of polygons

F, E, V are the numbers of polygons, edges, vertices in a non-overlapping network of polygons. Then **F - E + V = 1**

The equivalence of the two versions of the Theorem would then need to be established. Other examples of how a constructivist approach might work for specific parts of the mathematical syllabus can be found; an approach to Mathematical Induction is possible using game-like tasks such as the Towers of Hanoi.

"Multiple embodiments"

This may seem be a new idea, but it has actually been around for some time. The proponents of the Cuisenaire system of rods which was used to teach arithmetic at the early stages, claimed that it not only could be used to meet this aim, but could be used to teach a whole range of mathematical ideas, and that it was the only tool needed. Dienes however suggested that in order to teach a topic we should make use of as many tools as we could find - in other words, many representations. Of course it is an interesting point - if I explain that the idea of a fraction can be thought of in a variety of different ways - such as part of an area (such as a bar of chocolate) or part of a group (such as a class of children) or as an ordered pair (with specific rules of combination) or as an operator (which "does something" to a number), will this muddle the learner? Will having an array of situations where the concept of fraction can have different meanings make it more difficult for the learner? I have heard this idea put forward, but there is little evidence to support this view. The task of the learner in a multiple embodiments approach is to abstract the basic concept from a whole variety of instances. This would seem to be the essential task involved in concept formation, so the idea is at least as old as Piaget! Perhaps what is new is the requirement that for example, algebraic models should be seen in real-world contexts. An example would be to look at a graph and give examples from real life of which the graph would be a representation. Not only are words used as an alternative to a graph, but tables and equations are alternative representations. Given any one of these four ways of describing a real-life situation, we want the learners to develop the skill of giving the other three. A competitive group based mode of approach, where each group gives one of the four representations to describe a real-life situation, and challenges the other groups to produce the other three representations, can be used. The use of graphical calculators to draw a graph given raw data, and to guess at an equation to represent the graph drawn, is encouraged. It is suggested that real understanding lies in the student being able to synthesise the different features which are held to be representative of the concept under consideration; if we consider the students ability to deal with one representation of the concept, we cannot form a true picture of her understanding, and this is not possible even if we consider the students handling of different representations in isolation; it is the interaction of different representations, and the abstraction of the concept from the different representations, which enable us to say that the student has understood the concept. In mathematics we are dealing with abstractions; the concepts may have physical embodiments, but are not objects in themselves. So the concept of a function has different representations - as a graph, or an algebraic expression, as a set of tabulated data, or in verbal form. Each of these is a partial picture of what a function is; it is by moving from one picture to another that the student demonstrates her understanding. In teaching, we generally concentrate on just one of these pictures, sometimes believing that this will enable the learner to develop a facility in dealing with this particular picture of the concept. But this is not the case. If we only taught simple equations of the type where the unknown is on the left hand side of the equation, learners would have little idea how to deal with cases where the unknown is on the right hand side of the equation. If we taught fractions using the "area" picture discussed above, students would require another set of skills in order to deal with other embodiments of the concept of a fraction. Those who would oppose the idea of multiple embodiments, preferring one single embodiment for an idea such as that of a fraction, may be suffering from the fear of introducing "fragile concepts", or rather "fragile understanding". But we proceed from fragile understanding to deeper understanding. It has been suggested that understanding is a function of time; we begin with very little or hesitant awareness of a topic; as our experience develops so too does our comprehension of the topic. The wider the variety of instances which we experience of the topic, so the deeper our knowledge of it grows. This growth is fostered not only by the experience and the reflection on the experience, but also the discussion which takes place with others, who may be at the same or different stages of development of understanding. Our ideas are refined, improved, justified, by this interaction with other learners. One of the learners will be the "teacher", for we could also say that there is always more to be understood about a topic.

The National Council of Teachers of Mathematics (NCTM) in its materials produced to support their desire to have "standards" have considered how geometry could be viewed "from multiple perspectives". Geometry has been viewed from a synthetic perspective; that is, the traditional Euclidean style of proof, which it would be fair to say is based on ideas of congruence; it is concerned with measurement of lengths, areas, volumes; ideas of parallelism and perpendicularity with are derived from measurement of angle; and ideas of similarity, in turn related to ratio, again dependent on measurements of lengths. The traditional synthetic or "direct" proof has still I believe a place in the curriculum; it can take challenging, but accessible forms; for example part of the result known as the "Custard Pie Theorem":

An odd number of persons are standing on a plane surface; the distances between each pair are all different. Each one throws a custard pie at the person who is nearest to them. Prove that nobody is hit by more than 5 pies. Hint: Let X be the position of a person who is hit by two pies thrown by persons at points A and B. What can you say about the size of the angle AXB?

Other "embodiments" of traditional geometrical content are the vector approach and the transformation approach. Examples to illustrate these could be

In a cyclic quadrilateral, lines are drawn which are "medians" at one end, and "altitudes" at the other. That is, in a cyclic quadrilateral ABCD, the line joining the midpoint of AB which is perpendicular to CD is one such line, called a "maltitude". (see fig.1) Prove that these lines are concurrent.

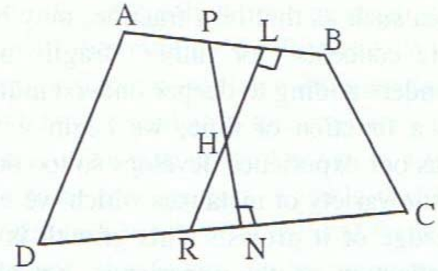

Figure 1

This is amenable to a vector approach, If we take the centre of the circle ABCD as the origin, A,B,C,D have position vectors a,b,c,d so that a.a = b.b = c.c = d.d. If H is the point where two of the maltitudes meet, express facts such as "PH is perpendicular to CD" in vector terms.

Again consider this problem :

A triangle PQR is inscribed in a triangle ABC; that is, point P is somewhere on AB, point Q is on BC and point R is on CA. We want the perimeter of the triangle PQR to be a minimum. (see fig.2). Prove that CP is perpendicular to AB, with similar results for AQ and BR (The triangle PQR is sometimes known as the "Pedal" triangle of the triangle ABC).

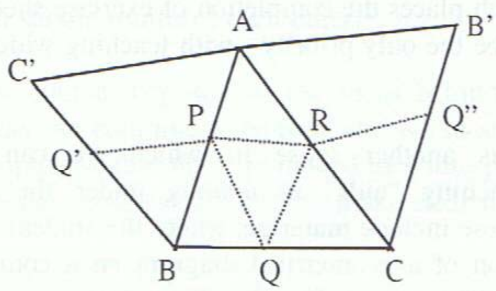

Figure 2

This is amenable to a transformation approach. Hint : Reflect the triangle ABC in AB so that the image of Q is Q'; reflect the triangle ABC in AC so that the image of Q is Q''. The perimeter of triangle PQR is the zigzag line Q'PRQ''. When will this be a minimum?

In both cases, we are dealing with the building blocks of geometry – points, lines, triangles, concurrence and so on.

Manipulatives

This merely means something which can be manipulated, in a physical sense. Traditionally we spoke of concrete apparatus. This meant any materials whether commercially produced or home - made. They ranged from the detailed over - structured materials such as Cuisenaire or Colour - Factor rods, to simple materials to help in the construction of Polyhedra and other geometrical structures. It is interesting to note that now little is heard of the complicated expensive type of materials; a lot is heard of the "do it yourself" variety. It is not merely financial considerations which have caused this. Ohanian (1992) asks in what way all the K-3 projects in Mathematics resemble each other. The answer is manipulatives; "those brightly coloured geometric pattern blocks, interlocking Unifix cubes, ancient Chinese Tangram puzzles, two colour counters, geoboards and Cuisenaire rods" which you find in every primary classroom. So it is ironic that "manipulatives" are now put forward as though they were new, when for example, Unifix cubes were introduced over 40 years ago. She makes the telling point that the hard bit is not putting the manipulatives in the classroom, it is replacing a style of teaching which places the completion of exercise sheets at the top of a list of priorities - or maybe the only priority - with teaching which sees manipulatives as tools for learning.

But there is another sense in which we can view some recent developments in teaching "aids" as coming under the general heading of "manipulatives". These include materials where the student is able to control the size, shape or location of a geometrical diagram on a computer screen. Using software such as the "Geometer's Sketchpad" students can manipulate a triangle with its associated medians, altitudes, angle bisectors or side bisectors, and develop results which appear to deserve the title of "theorems". Whether this approach can be accepted as a valid method of proof is open to discussion, but it surely is true that by manipulating geometric entities in this way, the student develops a "feel" for what is going on, equally as valuable as "playing" with concrete apparatus. Galindo (1998) refers to two methods of justifying in traditional geometry - the empirical method, which depends on a number, possibly a large number, of examples, and the deductive method, which is based on logical reasoning. It is believed that a student's understanding of geometry will be deeper when they can make connections between computer - generated constructions, paper and pencil drawings, the "ideal" geometric objects in the world of proof, and physical objects in the real world. He makes it clear that there is a distinction to be made between a "construction" and a "drawing". If we have constructed a triangle, for example, on the computer screen, then relationships associated with the triangle which are invariant will be unchanged when we drag a vertex to create a "new" example; we will be able to establish what these invariant properties are. This is why the "dynamic geometry" programs are different from making drawings, on the screen or on paper. So if a student is asked to write an explanation of how she made a construction, we will be able to see what sort of connections the learner is making, and what sort of understanding is present. As an example of the type of assessment task which can be set, we have what has been called "black boxes" which the student has to "deconstruct". A diagram is given, such as Bhaskara's diagram for the theorem of Pythagoras, without any indication as to how it has been constructed. The student has the task of finding out how it was constructed; there are a number of ways of going about this; possibly by selecting a point in the diagram and seeing what happens when it is dragged. Alternatively the student could attempt to repeat the construction - but then a comparison between the given and the new needs to be made to see if indeed they are constructed in the same way. This may need to involve "dragging" techniques. Galindo suggests that the empirical (including computer constructions) and the deductive justification should reinforce each other.

I am not, of course, opposed to the ideas behind these "buzzwords". These three "old" ideas can continue to be used and we should be clear about two aspects; we should emphasise that they are indeed new interpretations of tried and trusted ideas from the past; and we should also make clear the limitations of each approach.

References

Enrique Galindo (1988). Assessing Justification and proof in Geometry Classes taught using Dynamic software in *Mathematics Teacher*, 91(1), 76 – 82.

Imre Lakatos, (1976). Proofs and Refutations, Cambridge, Cambridge University Press.

Susan Ohanian (1993). Garbage Pizza, Patchwork Quilts and Math Magic, New York, W H Freeman.