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Exploring Understanding In Mathematics - A Review of *Understanding in Mathematics, by Anna Sierpinska*

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For many years, the complexity of understanding has perplexed and fascinated many educators, psychologists, and philosophers, who pursue to catch its meaning and inferences in various settings and to invent certain instructional environments for enhancing educational effectiveness. Specifically, in the current reform movement of school mathematics, teaching and learning mathematics with understanding has been widely accepted and greatly emphasized (e.g., Davis, 1992; Hiebert & Carpenter, 1992). Educational researchers and educators have been struggling to find efficient ways to instruct students for achieving a better understanding of mathematics.

As understanding has been explored in various contexts with different perspectives, studies on understanding in mathematics (e.g., Hiebert & Carpenter, 1992; Michener, 1978; Putnam, Lampert, & Peterson, 1990) have had a different focus and sphere from studies on understanding in mathematical problem solving (e.g., Cummins, Kintsch, Reusser, & Weimer, 1988; Kintsch & Greeno, 1985; Mannes & Kintsch, 1991; Mayer, Lewis, & Hegarty, 1992). Understanding as one step in the problem-solving process has piqued the interest of many cognitive psychologists, educational researchers, and mathematics educators. Studies on this topic have generated many interesting cognitive theories, models, and instructional strategies (e.g., Cummins et al., 1988; Newell & Simon, 1972; Polya, 1957). However, studies on understanding in mathematics have tended to relate to the theory of learning (e.g., Hiebert & Carpenter, 1992) and to adopt theoretical perspectives from other disciplines like anthropology and sociology (e.g., Hatano, 1988; Lave, 1988; Saxe, 1991). There are relatively few attempts to summarize discussion of understanding in mathematics. To promote our understanding and discussion of understanding in mathematics, this book, *Understanding in Mathematics*, provides us a synthesized viewpoint on understanding in mathematics from strands of mathematics, philosophy, the psychology of mathematics education, logic, and linguistics.

The aim of this book is "to contribute to a better understanding of how real people understand mathematics in real life, "(P. XV). Stemmed from her concerns about the practical problems in mathematics education such as: "how to teach so that students understand? Why, in spite of all my efforts of good explanation they do not understand and make all these nonsensical errors? What exactly don't they understand? What do they understand and how?" (P. XI), the author felt the necessity to synthesize and develop various ideas on understanding. Along this direction, the author drew together a wide range of literature contributed by a variety of scholars such as, Ajdukiewicz, Bachelard, Hall, Piaget, and Vygotsky. Different perspectives on understanding in the literature from time to time drove the author to struggle for the meaning of some common concepts, such as understanding and meaning, and the conceptualization of understanding process and good understanding, based on the notion of the act of understanding. Various examples, mainly drawn from the historical development of mathematical knowledge as well as college students' learning of mathematics, were adopted to support the arguments raised by the author. A total of five chapters were organized in the book to present its contents.

Chapter One: "Understanding and Meaning"

The author perceived that there is a paradox between understanding and meaning in existing literature, where, for example, Dewey defined understanding by meaning and Ajdukiewicz explained meaning with understanding. The first chapter discussed the relationships between the notions of understanding and meaning, which aimed to clarify the notion of understanding. Related to the word of understanding, the author discussed various ways that the word of understanding is used in ordinary language and academic communications, including the object of understanding, the content and ways of understanding, and the basis for understanding. By adopting C. S. Peirce's theory, the author discussed the meaning of a *sign* in general. When a sign is to be interpreted for its meaning, the levels of interpretations determine the levels of meaning of the sign in ones' discussion. The levels of meaning of a sign in discussion include: (1) sign itself, (2) immediate interpretant of the sign (the meaning of the sign), (3) dynamic interpretant of the sign (individual acts of interpretation), (4) final interpretant (the convergent of individual interpretations). Specifically restricted in the language that is also a type of sign, the author presented more detailed discussion about meaning in our language as located in a specific social and practical environment. For example, the meaning of mathematical terms used in classroom is not only relied on their definitions or rules but also determined by the persons who are using them and how the terms are being used. With these discussions, the notions of understanding and meaning were connected under the assumption of the same object (i.e., a sign) for understanding and meaning: What a sign represents forms the basis for our understanding of the sign. This viewpoint tried to avoid the paradox between understanding and meaning, but did not aim to solve the paradox. Because the meaning (with some publicity as compared to understanding) of a sign is always established from individual mind (i.e., understanding), we do not know how we can decide whether understanding or meaning should come into being first. Besides these discussions, the first chapter also introduced several other related issues that served as a rich background for the follow-up chapters.

Chapter Two: "Components and Conditions of an Act of Understanding"

The second chapter focused on the notion of an act of understanding, which served as the basis for the whole conceptualization in this book. The notion of an act of understanding was described through its components. They include (1) the understanding subject: the person who understands, (2) the object of understanding: the object that a person intends to understand, (3) the basis of understanding: the artifact(s) that a person's thought is based on in an act of understanding, (4) the operation of the mind: mental action that links the object of understanding with its basis. Further discussion of these components of the act of understanding presented various perspectives and the author's own thought on each of these components. For example, in discussing the basis of understanding, the author presented existing perspectives on "What is the basis of understanding?" such as, representations, mental models, or apperception. In discussing mental operations involved in understanding, the author identified four basic operations that include identification, discrimination, generalization, synthesis, and discussed their relationships. With related to an act of understanding as happened in mathematics classrooms, the author discussed internal and external conditions of an act of understanding such as, students' attention and intention in their acts of understanding and the language communication in a certain classroom social setting. For a better clarification about an act of understanding, the author also discussed the differences between understanding and knowing, understanding and invention (or discovery), an act of understanding and an activity of reasoning, human understanding and current approach of computer simulations in studying human understanding.

Chapter Three: "Processes of Understanding"

The process of understanding as "lattices of acts of understanding linked by reasonings" has been proposed in chapter 3. Because the acts of understanding do not have the characteristics of inference or derivation, the acts of understanding were differentiated from various reasonings. Moreover, the author treated reasoning as a linkage among the acts of understanding. Various simple reasonings that use only one process of inference or deduction were classified, based on Ajdukiewicz's work, into two categories: spontaneous reasonings and problem-directed reasonings. In the category of problem-directed reasonings, there were proving, verifying, and explaining. Because different reasonings, examples, figures of speech (e.g., "metaphor" and "metonymy"), activities, previous knowledge and experience have different functions in the process of understanding, the author discussed their roles in the process of understanding separately. In each discussion, the author presented, organized and compared existing variety of related perspectives, issues, and examples drawn from the historic development of mathematics and students' learning of mathematics.

Chapter Four: "Good Understanding"

As another main concern of this book, "What is good understanding?", has been discussed in chapter 4. The importance of study on good understanding is located in the purposes of our education. However, the difficulty of study on good understanding is inherited in the subjectivity and relativity of our assessments. Existing research on understanding has generated several different approaches for studying understanding. As adopted by the author in this book, the historico-empirical approach was used to discuss the meaning of good understanding in mathematics. Based on some identified commonalties (i.e., certain mechanisms of the development of understanding, bifurcation of ways that words change their meaning) between individual students' understanding and the historical development of understanding in mathematics, the author argued that good understanding is achieved through significant acts of overcoming certain obstacles. To support her argument, the author presented some philosophical discussions on epistemological obstacles and one example about the development of a mathematical theorem (i.e., Bolzano theorem) in the follow-up sections in chapter 4.

Chapter Five: "Developmental and Cultural Constraints of Understanding"

In chapter 5, the author discussed the developmental stages of understanding subject and the influences on understanding from the surrounding culture as two constraints for the development of understanding and its evaluation.

Based on Vygotski's theory of the development of concepts, the author discussed (1) the process of developing generalization in children between the ages 2 and 7 or 8 that grounded on concrete objects and their relationships, (2) the process of developing mental operations of identification and discrimination in elementary-school children that grounded on objects' features and their relationships, and (3) the process of developing the operation of synthesis in adolescents that grounded on abstract thoughts. Because epistemological obstacles are assumed to locate in the sphere of conceptual thinking, they will not become obstacles to children before they reach the stage of conceptual thinking. However, "the obstacles grow on the soil of complexive, childish thinking -- they have genetic roots. But the fertilizers (the challenges that make them grow) come from the surrounding culture, from the implicit and explicit ways in which the child is socialized and brought up at home, in the society, in the school institution." (P. 159)

In discussing the cultural roots of epistemological obstacles, the author adopted Hall's theory of culture. Three fundamental levels in culture (i.e., the "formal", the "informal", and the "technical") were introduced as the basis for further discussion. In mathematical culture, specifically, the formal level is rooted in people's beliefs, values, and world views; the informal level is embedded in people's schemes of action and thought; the technical level is located in the mathematical theories that are rationally justified, logically coherent, and explicit. These three levels interplay in the historic development of mathematical knowledge, which has been illustrated with several examples presented in chapter 5. The interplay among these levels also illustrates that the path to good understanding is propelled through struggling, overcoming obstacles, and occurring changes in the frame of a person's mind.

A Further Look: Let Us Continue the Conversation and Study on "Understanding"

The book is an important endeavor to help us to reach a better understanding about this topic in current reform of school mathematics. As demonstrated in the book, the ideas using the concepts of *an act of understanding* and *(epistemological) obstacles* as the bases for discussing understanding in mathematics are interesting. With several examples of students' understanding in mathematics presented in the book, we can see how these ideas may be used in understanding students' understanding and in developing our pedagogical means. For example, a more detailed example of an epistemological analysis about the concept of the limit of a convergent numerical sequence illustrates the potential utility of epistemological analysis for informing instruction (Sierpinska, 1990). Besides this proposed way of looking at students' understanding, another appealing feature of this book is its inclusion of social and cultural influences on understanding. This inclusion provides us a broader view on the development of students' understanding in mathematics than many previous studies that solely focused on cognition.

However, there are also some questions that emerged from my reading of the book. I have pulled out two concerns below with different scopes for further considerations.

The scope versus focus of this book The good aspect of this book is that it presents us a wide range of theoretical perspectives and approaches on various aspects of understanding. However, this way of discussion, at the same time, may also be a deficit, since there are no focused issues being theorized in a coherent structure for the book. It seems that the approach used in discussion in this book is neither top-down nor bottom-up. There is no systematic theoretical structure for the conceptualization in the book. It also lacks empirical studies to serve as a ground for its conceptualization. Consequently, the conceptualization is not well founded. For example, it is unclear why the notion of "an act of understanding" was used as the basis for the conceptualization in this book. Why can the process of understanding be viewed as the lattices of the acts of understanding linked by reasonings? Cited perspectives and issues did not answer these questions but rather might distract from main arguments that the author might want to convey. If restricting the scope of this book, the author might have been able to develop her own theory in a rigorous way.

The notion of epistemological obstacle and the historico-empirical approach The book has shown that the notion of epistemological obstacle is useful in studying understanding, although researchers have not reached an agreement about its definition. However, it is not clear from this book how well the notion of epistemological obstacle can be used in relating the discussion of individual students' understanding and the discussion of the development of mathematical content knowledge. Based on Bachelard's concept as adopted by the author, epistemological obstacles are "ways of understanding based on some unconscious, culturally acquired schemes of thought and unquestioned beliefs about the nature of mathematics and fundamental categories such as number, space, cause, chance, infinity, ... inadequate with respect to the present day theory." (P. XI) With this concept, the author adopted a historico-empirical approach in discussing epistemological obstacles in chapter 4, and presented us several examples of epistemological obstacles in the historical development of mathematics. However, because the epistemological obstacles in individual students' understanding have also been discussed in the book, it is unclear how to make a consistent mapping between these epistemological obstacles with the historico-empirical approach. In fact, in order to improve students' understanding in school mathematics, a cognitive account of the development of students' understanding may deserve more attention than the historical account of the development of mathematical knowledge.

The topic of "understanding in mathematics" discussed in this book is not new but very difficult. As research efforts have generated various theoretical perspectives on "understanding", the author's efforts in pulling them together should be applauded and appreciated. The wide range of existing theoretical perspectives and approaches on understanding have been well discussed and integrated in this book with a specific conceptualization about understanding. In particular, many relevant studies done by scholars in European countries were introduced in this book. It is certainly helpful for us to stop for a while to try to understand and communicate with each other about the problems related to students' understanding as the author anticipated. This book itself has set a starting point for us to further conversation and study on understanding in general and students' mathematical understanding in specific.

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