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Mathematical Problem Solving
A Teaching Strategy

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Abstract

In this final paper in a series of three we touch on the practical aspects of teaching problem solving. We first discuss the differences between the problem solving approach to teaching mathematics and the traditional one. Next we show how the theoretical aspects of heuristics and metacognition fit with the experimental model of problem solving. This is done by considering specific problems and how they could be approached.

We then discuss five models for the teaching of problem solving as well as three stage lesson format. This leads us on to the concepts of the learning of mathematics FOR, ABOUT and THROUGH problem solving. The next section discusses the concepts of a rich mathematical activity and a quality learning environment.

We conclude with the observation that problem solving is not easy for many teachers. It will take further professional development before it will be a regular part of classroom practice and not just abstract curriculum statements.

Introduction

In Anderson and Holton (1996), there is a report of a questionnaire that was conducted among mathematics teachers to determine what aspects of mathematics are actually being taught using problem solving. While most teachers are making an effort to incorporate problem solving into their pedagogical armoury, it does not seem to be a regular feature of most mathematics classrooms. Here we try to give some practical assistance for teachers who want to commence using problem solving or who wish to take their problem solving teaching further. Much of what we propose here is based upon the theoretical framework provided in Holton and Neyland (1996).

In Holton and Neyland (1996), we defined problem solving and discussed some of its advantages and disadvantages. Before starting to teach problem solving, the teacher should be clear why, apart from the fact that it is in the curriculum, they intend teaching it. It is important too, to note that the problem solving approach to teaching mathematics is different to what has become the traditional approach. However, much of the problem solving approach is not new. Scaffolding, which is not limited to problem solving, has its origins in the Socratic method. Heuristics too have a long pedigree. So how is problem solving different to what has become the standard, chalk and talk, method for teaching mathematics?

The main difference is one of attitude or philosophy on the part of the teacher. The shift is from the so-called "sage on the stage" to the "guide by the side". Philosophically, the teacher needs to change from a giving role to an encouraging role, from "here is how to solve a linear equation" to "how might we solve this linear equation?"

Naturally the students will not be able to invent all the mathematics they need for themselves. There will still be things that they need to be told. They still need to practice both skills and problem solving processes. However, more time needs to be spent by the students exploring mathematics with the teacher as a guide. During this exploration, seeds will be planted and students will develop connections between various parts of their learning which will increase learning, understanding and retention later (see Hiebert and Carpenter, 1992).

So instead of teachers taking students down well trodden paths in a sequential unfolding of mathematical structures, there should be more emphasis on the students themselves structuring mathematical knowledge from specific problem contexts. Certainly the structures that are formed this way have to be justified but not necessarily in the one way shown in the textbook. Where possible, students should be given the opportunity to provide their own justifications. Many of these will be correct and may well be different from the text book proof.

This is not to say that the development of mathematics during the year is anarchic. The teacher should know from the start what material is to be covered. However, there should be some flexibility in the manner and order in which the content arises, and every opportunity to reinforce connections between different parts of the curriculum should be taken. Of course, in high stakes years when there are external examinations, the course may have to be slightly more structured. Nevertheless, it is important even on these occasions for the teacher's questioning to be open rather than closed in order to stimulate the students' thinking and so facilitate their learning.

In the problem solving approach then, there is less emphasis on students applying rules to problems which have been carefully chosen to fit those rules. There is greater emphasis on the creative construction of mathematical structures and solutions in non-routine situations and over a range of contexts.

In summary then, the emphasis on a monological approach to mathematics teaching, learning and reasoning should decrease. This should give way to a dialogical approach.

Incorporating Process and Model

In this section we combine the model of problem solving given in Holton and Neyland (1996) with the heuristics and metacognition discussed in that paper. The aim is to give an overview of the problem solving process. For convenience we repeat the model below in Figure 1.

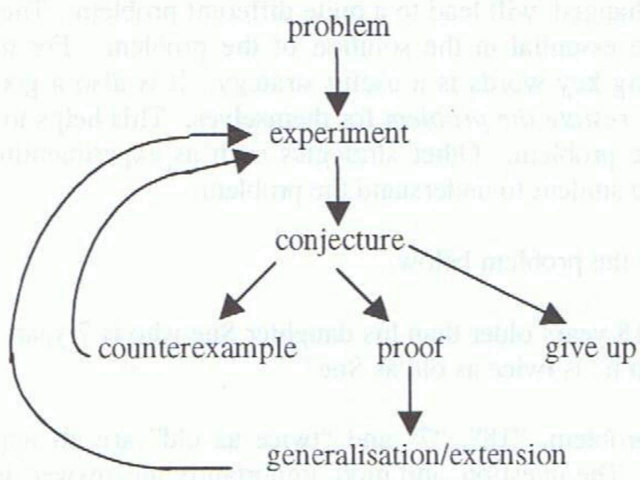

Figure 1

We recall that heuristics are means by which a solution is sought. Most of the heuristics of Begg (1994) are covered in this overview of the problem solving process. We also show that certain heuristics tend naturally to appear at certain stages. In what follows, heuristics are emphasised in italics.

I Getting Started

This is the first part of problem solving and includes "problem" and "experiment" from Figure 1. It also covers the first two phases of Pólya's four phase model (Pólya, 1945).

1. Understand the problem.

The problem will either be posed by the students themselves, presented in writing or given verbally. If the problem is not one posed by the students, a first and crucial step is for them to make the problem their own; to become familiar with its conditions, characteristics and variables. If it is in writing the solver has to *read the problem* and read it carefully. Many a solver has begun by solving a problem they thought was there but was actually not.

As the solvers read the problem they should be asking the various questions posed by Pólya. They should also be *looking for key words.* These are words which, if changed, will lead to a quite different problem. They are words or phrases which are essential in the solution of the problem. For novice problem solvers, underlining key words is a useful strategy. It is also a good idea for the problem solver to *restate the problem* for themselves. This helps to emphasise the key aspects of the problem. Other strategies such as experimenting with special cases also help the student to understand the problem.

Consider the problem below.

Peter is 18 years older than his daughter Sue who is 7 years old. How old will Peter be when he is twice as old as Sue?

In this problem, "18", "7" and "twice as old" are all appear to be key words or phrases. The question, and more importantly the answer, is altered if key words are changed. (In actual fact, "7" is an extraneous piece of information here.). On the other hand, "Peter" could be "Peta" and"Sue", "Sam" without changing the answer or the method of solution.

In this particular problem it is obvious that both Peter and Sue age at the same rate, one year at a time. This is nowhere explicitly stated but it is an *hidden assumption*. Many problems have implicit conditions that turn out to be important in finding a solution. So it is vital to make sure than hidden assumptions have been noted.

One of the metacognitive aspects of problem solving is to know when to come back to the original problem and read it again. This may be necessary to see if you are on the right track of a solution or if you have inadvertently started to solve another problem. It may also help you to *try another approach* if you have made no progress with a particular method of attack or *change the point of view*.

Once an answer has been obtained it is important to reread the question to ensure that the solution you have obtained does indeed solve the original problem.

2. Think

Apart from the metacognition already discussed in Holton and Neyland (1996) – which largely relates to monitoring and controlling the problem solving processes – at the start of a problem it is necessary for solvers to go through a mental list of heuristics in an effort to find a few which will get them started.

It is worth asking the following Pólya questions (Pólya, 1973, pxvif).

What is the unknown?
What are the data?
Is drawing a figure useful?
Is there a related problem that has been solved before?
Could it be used here?

In addition it is worth considering

What area of mathematics can be used?
(algebra, geometry, number, etc.)
What approaches might work?

3. Experiment

This is often an early stage in the problem solving process. It has two main functions. The first is to get a feel for the problem. The second is to start to produce some evidence for a conjecture. The experiments may be calculations or measurements or a variety of things depending on the problem in hand.

The experiments should be *systematic*. The aim is to get central, useful information that can be collated in some way, rather than an unconnected jumble of, say, numbers. And the results of experiments should be *recorded* in some logical fashion – in a *list, table* or *diagram*. When appropriate, consider the case n

= 1, then n = 2 and so on. It is worth keeping these experiments until the problem has been completely solved. They may well be useful to provide or inspire a counterexample later.

Another problem that is useful to illustrate some ideas here is the frog problem. Four spotted frogs (S) and four green frogs (G) are sitting on lilypads as in Figure 2. Frogs can move to the next lilypad if it is free or they can jump over another frog if there is an empty lilypad on the other side. What is the smallest number of moves which can interchange the green and the spotted frogs?

S S S S _ G G G G

Figure 2

There is a certain amount of *symmetry* in this problem. Whatever happens next, certainly it doesn't matter whether a spotted or a green frog starts first.

This is a good problem too, to illustrate *solving a simpler problem first*. It's easier to do the problem above once you've tried moving a frog of each type, two frogs of each type, and so on.

4. Panic

This is a common feeling at the start of a new problem. Success and experience will give the solver confidence to continue. However, even expert problem solvers face panic on occasions. Be prepared to think "there is no way I will ever do this problem" and then get on and realise you can.

II Conjecturing

1. Pattern

This can be the most enjoyable and interesting part of the problem solving process. In more difficult problems it is necessary to find a *pattern, rule* or *relation* in order to produce a conjecture.

As we have already said, some problems give up the correct conjecture immediately (that for the Four Colour Theorem was discovered by a teenage in one evening). Other problems take more time. In harder problems it may be necessary to discard a number of conjectures before obtaining the right one. Some people find some problems easier to make conjectures about than others. And conjectures require practice. So it is necessary to start with simple problems and work up.

2. Sense

Does the conjecture make sense? Is it *consistent* with previous knowledge and the data that has been assembled in the experimental phase. Does the conjecture imply anything that *feels wrong*? Intuition and common sense can and should be used here. A conjecture which implies that the height of a mountain is 5cm must surely be wrong. Any conjecture should be *justifiable*. Even though it cannot be proved at this stage, there should be good reasons for choosing one conjecture over another.

III Proof/Counterexample

The more difficult problems will require the solver to trade off *conjecture* against *proofs* against counterexamples. The process is a dynamic one which is only completed when the final step in the solution is written down. Frequently, trying to prove a conjecture gives an idea for a counterexample and looking for a counterexample can give an idea for a proof. Whether starting down the proof or counterexample trail the solver may see that the conjecture needs adjusting and the process starts again.

It is difficult to see whether to first try to justify a conjecture or show it is wrong. The decision as to which to try first depends on the solver and the problem. If testing a few cases will cover all possible counterexamples, then the solver should go down that path. This will either produce a counterexample or strongly confirm the conjecture. In the latter case, of course, a proof will still be required.

Some problems look like others and this may suggest a way to proceed to a proof. A slight change of a known proof might work. Or the problem may be one which suggests a well known proof technique such as *proof by contradiction* or *mathematical induction*.

1. Extreme Cases

In trying to find counterexample it is often valuable to try *extreme examples* or *extreme cases*. These examples are somehow at the edge of the spectrum of values that are being used. For instance, what happens if a number is very small or what happens as n approaches infinity? What happens if the triangle is isoceles? Extreme cases are usually easier to handle than more general situations so it is worth testing a few before trying a proof. Even if they do not provide a counterexample they may well give the solver some useful information which can be used later. Here is a problem which can be quickly solved by looking at an extreme case: Do all 1 litre milk containers have the same surface area? The answer, no, can be easily obtained by imaging a 1 litre box-shaped milk container with a base the size of a tennis court and a height a fraction of a millimetre. Clearly this 1 litre container has a surface area much larger than the one in the fridge.

2. Special Cases

A conjecture can also be tested against *special cases* rather than extreme cases. Special cases are typical cases such as the ones during the experimental phase. Where possible, cases would be tested because they are straight forward or easy to test. But sometimes more complicated cases are forced on the solver.

3. Simpler Problems

Sometimes problems are far too difficult to solve first time. They may involve far too many variables to be able to comprehend. In that case reduce the problem to a *simpler problem* in some way. For instance, in the frog problem using fewer frogs is a good way to start. Similarly a gambling problem involving five dice might be reduced at first to one involving only two. Insight gained from the two dice case may well lead to the solution of the original problem. Here the aim is to keep as many of the essentials of the problem as possible while reducing the problem to a manageable size. Using simpler problems is also something that is useful at the experimental stage, as we saw earlier. Sometimes problems can be simplified in more than one way and students should learn to choose the best from the range of available simplified situations. For example, the problem how many squares are on a chess board, can be simplified in two quite different ways: (i) how many squares are there on a 1 x 1 board, on a 2 x 2 board, and so on; (ii) how many 1 x 1 squares are there on a full chess board, how many 2 x 2 squares, and so on. In this case both these simplifications lead to a solution, but this is often not the case and students need to learn to select the most useful simpler problems.

4. Exhaust All Possibilities

One of the very simple methods of proof is to *exhaust all possibilities*. If there are sufficiently few cases to handle (under fifty, say), then by looking at each one in turn, the solver can learn about the entire problem. This way any conjecture about the situation is immediately verified or disproved. Simple combinatorial problems such as the behaviour of two dice are often open to this approach. For instance, how many different ways are there for two dice to give a sum of 4 (see Anderson and Holton, 1996)?

When a solution has been obtained by older children by the exhaustive approach, it is worth asking the Pólya phase four question "Can you derive the result differently?" They should be looking for more sophisticated approaches. There are times, however, when no other method is available. Certainly this will be the case with younger children. And finding a justification, any justification, is better than no justification at all.

5. Guess And Check

One of the simplest ways of tackling a problem is via *guess and check*. This is sometimes also called *trial and error*. In the problem of Peter and Sue, an answer can be found by guessing and checking. It is therefore a good strategy for simple problems. There are drawbacks, however. First, if the problem involved has a large number of cases, for instance a five dice gambling game, then it may not be easy to guess the correct answer in a reasonable time. Second, guessing and checking will give an answer. However, the method cannot tell you whether the answer is unique. It may well be that the problem has a number of answers. Only an exhaustive search will be able to determine all answers in such a problem. Finally, the guess and check method in complicated situations may only reveal the conjecture required; it may not justify that conjecture.

The efficiency of guess and check can be increased with metacognition by using *guess and improve*. Here subsequent guesses are improved using the data of past guesses. Again in simple problems where a unique answer is almost certain, this is a useful strategy. In Peter and Sue's problem, we could first guess that Peter was 30, in which case Sue would be 12. The check $2 \times 12 \neq 30$, shows that we have guessed incorrectly. We could then make other possible guesses for Peter's age and then do the corresponding check.

To make sure that we don't keep making the same guesses, the results could be recorded in a table such as that in Figure 3.

Peter's Age	Sue's Age	2 times Sue's Age	Correct
25	7	2 x 7 = 14	X
30	12	2 x 12 = 24	X
40	22	2 x 22 = 44	X
.	.	.	.
.	.	.	.

Figure 3

Another advantage of the table is that it enables us to move our guesses in the right direction; it enables us to see how best to use guess and improve.

With the guess of 30, 2 x 12 = 24 which is **less** than 30. With the guess of 40, 2 x 22 = 44 which is **more** than 30. The next guess should then be directed **between** 30 and 40. Guessing and thinking allows us to hone in on the correct answer.

We actually came across a nice solution of this problem where the student had *made a model*. On the edges of two pieces of paper he wrote the numbers 1 to 30. He then put the edges of the paper side by side and moved one of them until corresponding numbers were 18 higher on one edge (see Figure 4). He then looked along the edges to try to find where one number was twice the other. (Actually he didn't put down enough numbers, but what he had was enough to enable him to see the right answer.)

| 1 2 3 4 5 6 7 8 9 10 11 12 13 . . .

. . . 14 15 16 17 18 19 20 21 22 23 24 25 26 27 |

Figure 4

8. Work Backwards

Work backwards is often useful in problems which involve a sequences of moves, events or operations leading to a known end point. By reversing the process the starting configuration can be obtained; this is a bit like running a video in reverse. It is useful for investigating two-person games, for instance. The game of Nim is well worth analysing in this way in order to find a winning strategy. Start from the last move and work back to the starting position. Work backwards is the basis of many commonly used methods for solving equations. This strategy is also useful when dealing with proofs of trigonometric identities.

9. All Information

If the solver is unable to make any headway it is often worth rereading the question in an attempt to ensure that *all of the information* given in the problem has been used. Sometimes there is an implicit detail that is not mentioned specifically but which is vital to the solution. Knowing general facts such as the sum of the dots on opposite sides of a dice is 7, may well be required to solve a problem. (See hidden information in the experimental stage.)

10. Check

Having obtained a complete solution the solver should go through the justification and check every step. Even if the proof passes the check test, sometimes the solver still has a nagging doubt. In all cases, it is a good idea for the solver to get someone else to check their solutions. (It has been said that solutions should be checked first by you, then by a friend, and then by an enemy.) At this stage too, sometimes a solver may see a quicker more efficient way to solve the problem. The new method needs to be written down and checked too.

IV Generalisation/Extension

1. Generalisation

A *generalisation* is a problem which contains the original problem as a subset of special case. Generalisations can often be found by increasing the number of digits (in some number problems), stepping up the dimension (in a geometry problem) or by increasing the number of variables in some way. For instance, you may want to do the frog problem with an arbitrary number of frogs (n, say) on either side. This is a generalisation of the four frog case.

The reason for considering generalisations is that they give a result which is true for a much wider class of objects. One difficulty with generalisations is that they may require a quite different justification from the justification used in the original problem.

2. Extensions

An *extension* of a problem is one which is related to the original problem but not by way of a generalisation. These can be found by changing one of the conditions of the original problem in some way. It may be that a problem can be extended by changing addition to multiplication. As with generalisations, the proof of an extension may not be linked in any way to the proof of the original problem.

V Give Up

1. Why?

Some problems are too difficult to be solved right now. So, at times, students will have to abandon some problems. If they are forced to abandon **all** problems they tackle, then the problems are too difficult.

2. When?

This will depend on the teacher and the student. It becomes a matter of priorities. However, any reasonable problem will take more than 10 minutes and less than 2 hours (unless it is an extended investigation). In general, we suspect that students can go further than we usually expect. This is especially true of the better students. And it should be remembered that expert problem solvers like to sleep on a problem if no solution is at first forthcoming.

3. Help

In cases where the problem is too hard for both the teacher and the student, outside help should be summoned. This may be from another teacher, a book or a local university.

Implementing Problem Solving

If you are a teacher who has been convinced philosophically of the importance of problem solving in mathematics, or if you see it in the curriculum and feel a responsibility to teach problem solving, the next question to be resolved is how? The first thing to point out is that there is no unique way to teach problem solving. Teachers and schools will need to develop their own styles. Each teacher will need to develop an individual style which will no doubt develop and be a function of the school, the students and many other things.

An easy way to start is by using a series of one-off problems that may not necessarily be related to the content of the remainder of the lesson. This allows the teacher to gain confidence and gather together a string of useful problems. However, the danger is that students will not realise the relevance and importance of problem solving that stands alone, not integrated with the rest of the curriculum.

Sigurdson, Olson and Mason (1994) suggest five models of problem solving teaching that incorporate problem solving into lessons in a greater way than the one-off problems discussed in the last paragraph. These models are

1. Pólya's Problem Solving Strategies;
2. Strategy Problems as the Basis of a Lesson;
3. The Investigation Model;
4. A Teaching Approach to Problem Solving;
5. A Problem-Process Approach.

In the first model, students work on problems with a view to learning general problem solving strategies. The problem solving tends to be taught as a separate subject within mathematics and is not related to other curriculum material.

The second model uses problems as a basis for the lesson. For instance, the age problem from the last section involving Peter and Sue, may well be used to introduce simultaneous linear equations.

The Investigation Model uses problem which are more open-ended than the age problem or the frog problem. They often entail students working in groups for relatively long periods of time. Again, they do not usually link directly with the mathematical curriculum. This model appears to be very much like the first but with the problems being more extensive.

In the fourth model, teachers approach all of their teaching from a problem solving perspective. Here the teacher will interact with the students and engage in "probing discussions" with them. There would also be a concentration on developing problem solving skills. Students would be encouraged to try to solve even text book exercises in a number of ways. Sigurdson et al note that while this approach is to be recommended, it is difficult to implement. This is because it means that teachers need to rethink their whole approach to teaching. This may not be easy.

The final model on the list suggests the use of about 10 minutes a day on problem solving. The problems used in this model are not as extensive as either the Pólya or Investigations model but they are related to the content of the rest of the mathematics lesson. Rather than working in groups, the problems are solved in a whole class setting with teacher-student interaction. Problem solving strategies are emphasised by the teacher in this model.

Our research suggests that problem solving lessons work well around a three stage format which can be incorporated into all but the Problem-Process Model. The first stage is a whole class format. Here the problem is discussed by teacher and student, the students may suggest possible heuristics.

In the second stage, students work in small groups. The aim of this stage is to enable students to become involved with the problem and attempt to solve it for themselves. During this period, the teacher is able to go from group to group to provide scaffolding. However, our research suggests that most scaffolding during group work is undertaken by the students themselves as they give peer tuition to their group members.

The third stage is a reporting back stage which gives the students an opportunity to say what they have done and how they did it. Here the spotlight is again on the students who are reporting on their work. Different methods of solution are to be encouraged during this stage. It is important that student thinking and learning takes place in this final stage. One reason for the reporting is to improve students' ability to communicate but even more important is the chance for students to see alternative approaches to a problem. These alternative approaches should be useful on future occasions. In this stage too there is an opportunity for students to see link between different parts of mathematics which is again another aid towards understanding.

We have observed these three stages in a variety of problem solving classes. They seem to work equally well with weak and strong students and with young and old students. A general rule of thumb is that the three stage cycle should be repeated with young students and weak students. For these children the three stages should take no more than, say, 15 minutes. Very good students may only go around the cycle once in a one hour lesson.

In Holton and Neyland (1996) we quoted a paragraph from Siemon and Booker (1990) in which they say that problem solving encompasses the nature of mathematics and suggest that problem solving is more likely to lead to mathematical insights than the traditional teaching approach. This is based on the work of Hatfield (1978) who proposed the learning of mathematics, FOR, ABOUT and THROUGH problem solving.

By FOR problem solving, he meant that mathematical skills need to be learned so that they can be used in problem solving. But students also need to learn ABOUT problem solving. It is not enough to solve problems. Students need to learn the strategies of good problem solvers. Hence they need to be aware of a range of heuristics and be using metacognitive processing. Hembree (1992) and other authors provide evidence that students who have been made aware of problem solving strategies become better problem solvers. That is also our own research experience.

The final approach is THROUGH problem solving. Consider the following problem, which probably fits into the Investigation Model of problem solving teaching. The problem can be found in Lovitt and Clarke (1988).

The students are given cards in sets of three which contain respectively, the heads, bodies and rear portions of several animals. The cards can be put together to form Silly Beasts by taking a head, a body and a rear at random.

How many Silly Beasts can be made? How many Silly Beasts are (i) no parts zebra; (ii) one part zebra; (iii) two parts zebra; (iv) three parts zebra?

In the problem the students will certainly need to be systematic but in classes observed by Holton et al (1996) it was hoped that students would discover tree diagrams as a method of solution. Clearly the problem can be solved by tree diagrams but there are other ways of being systematic which won't directly lead to the new skill of tree diagrams. None of the students were able to "invent" the diagrams for themselves. This may have been because insufficient numbers of animal cards were given to the students. The number was usually three and this is small enough for students to produce a correct systematic list (Holton et al 1996, Chapter 6).

The exercise was not a failure, though. Once the students were shown tree diagrams and how they could be used to solve the Silly Beasts problem, they were able to use tree diagrams in other situations. Hence the problem had prepared the ground for the learning of a mathematical skill.

At this point it is important to point out that whether students discover new skills directly from a problem or indirectly as in the Silly Beast example, they are likely to remember them better than if the skills are learnt "by rote", that is, by the more traditional approach. This is because the ground has been prepared, the new concept fits into their experience and the students see a reason for it. However, it does not mean that the students will necessarily remember the concept for ever without further practise. Problems involving tree diagrams will need to be used in the class from time to time so that the concept is not forgotten.

But can **any** mathematical skill be introduced this way? What about mathematical induction? A problem solving approach here would be to present it as a tentative proof strategy. The problem wold be "Is this a valid proof strategy?" Ideally this would be asked after the students have been working on a problem involving the use of mathematical inductive reasoning strategies using number patterns. For example, they could have been exploring triangular numbers and developed the conjecture that the *nth* triangular number is $\frac{1}{2}n(n+1)$. This leads to the question, can this be proved in all cases? So the teacher puts forward a proof strategy as a conjecture (the proof strategy is mathematical induction). The question for the class is, "Is it a valid proof strategy and why?" The proof strategy is treated as a conjecture to be refuted or justified. As a conjecture, various strategies could be used to attempt to refute or justify it. For example, does it make sense in finite cases? In cases where n is not a natural number? Can we think of analogous situations such as a string of islands coming off the mainland – if I can get from the mainland to the first island and I can get from any island to the next does this mean I can get to every island? So we are suggesting that mathematical induction can be taught using a problem solving approach provided (i) it comes at the end of a suitable patterning problem and (ii) mathematical induction, as a proof strategy, is itself treated as a conjecture to be refuted or justified using problem solving strategies.

If approached the right way then, any mathematical skill can be taught using a problem solving perspective. Hence it can be taught THROUGH problem solving. As we have remarked earlier, there may be good reasons for using other approaches on occasions. However, the importance and value of using problem solving on a regular basis in the mathematics classroom cannot be over stated.

OPE-N Plan

In Holton et al (1996) an OPE-N plan for the teaching of problem solving was proposed. This suggests a method of incorporating problem solving into a more traditional mathematics programme. It is a half way house between the traditional programme and the full problem solving model of Sigurdson et al (1994). At the same time, it tries to incorporate; the learning of mathematics THROUGH problem solving.

The initials O, P, E and N represent four stages as follows

O Observing problem solving;
P Practising problem solving;
E Employing problem solving;
N Normal mathematics teaching.

Certainly students still need to learn the normal mathematical skills. No problem solving can be effected without some mathematical knowledge. In the first instance, this can be continued in the normal way, alongside the problem solving development. Students do not become problem solvers automatically though they need to observe problem solving in action. This can be done formally by the teacher doing problems in a whole class setting and pointing out the strategies being used (see the Pólya model of Sigurdson et al, 1994). On the other hand, it can be done more informally, with the teacher pointing out what has been happening in group sessions. So students observe both their teacher and their peers in problem solving situations.

Naturally students then need to practise problem solving. This is both a consolidation phase, where they absorb some of the skills they have observed, and a learning phase where they may discover new heuristics.

In the employment stage, students use problem solving to learn new mathematics. Here the teacher, and then the students, should tackle curriculum material from a problem solving standpoint.

It is worth noting that there is not necessarily a linear time relation between the O, P and E stages. Often the observation stage comes after practising some problem solving. During the reporting back stage students will no doubt observe different methods of solving problems that they can use on a future occasion. On the other hand, we have seen 5 year old children employing problem solving to lay the foundation for the four mathematical processes before they have practised much problem solving at all.

The point of mentioning these three stages is not to ensure that they are presented in order but to ensure that the teacher realises that all three aspects need to be engaged at some time.

For primary teachers, the OPE-N style of teaching should not cause any difficulties. It reflects the style of teaching that they use in other areas. Language, for instance, taught in the style of Cambourne (1988), is very similar to the way we see mathematical problem solving being taught.

Finally it is worth pointing out that some teachers prefer to retain the "normal" style of teaching alongside the problem solving approach because it provides variety to their teaching. There is also a belief that it is more efficient in terms of time. As was noted in Anderson and Holton (1996), there is evidence that the so called "loss" of time in the initial stages is really an investment, and that it leads to efficiencies later. Lau (1996) in his working teaching 16 year old students their first calculus course, has found that a problem solving approach can fit in the same period as the traditional approach. In addition, the problem solving approach appears to lead to better understanding.

Rich Learning Activities

In the survey reported in Anderson and Holton (1996) almost three-quarters of the teachers reponding said that pre-prepared resources and problems that would fit in with content strands were their most urgent needs with regard to problem solving. But what sorts of problems are teachers looking for? There is a remarkable degree of consensus about this. The list of criteria for these problems that New Zealand teachers repeatedly come up with, is very similar to the list produced by United Kingdom teachers in the book "Better Mathematics" (1987). New Zealand teachers concur with their United Kingdom counterparts and judge that the following list describes the sorts of problems they are looking for. (For more on this topic see Neyland, 1994).

It must be accessible to everyone at the start.
It needs to allow further challenges and be extendible.
It should invite children to make decisions.
It should involve children in speculating, hypothesis making and testing, proving or explaining, reflecting, interpreting.

It should not restrict pupils from searching in other directions.
It should promote discussion and communication.
It should encourage originality and invention.
It should encourage "what if" and "what if not" questions.
It should have an element of surprise.
It should be enjoyable.

However many of the teachers we asked to suggest such a list of criteria did not know what problem solving in mathematics was, and so did not produce their list from any prior philosophical commitment to problem solving. These teachers were asked instead to list the ideal criteria for a general mathematical classroom activity which is rich in learning potential. How do we account for this similarity? The explanation, we believe, lies in the fact that the processes of learning and those of problem solving are very similar. Teachers, because of their experience with teaching and learning, recognise the sorts of mathematical activities which students learn from and find interesting. It is no surprise, either, that the same sorts of approaches are used by mathematicians when they create new mathematical ideas. Through problem solving the mathematician is producing new knowledge in the public arena, the student of mathematics is producing new knowledge in the personal arena. So the process of learning and problem solving can be thought of as similar, and this fact places problem solving in a central position in the mathematics classroom.

Stanic and Kilpatrick (1989) point out the similarities between Pólya's idea about the processes of mathematical discovery and invention and Dewey's ideas about reflective thinking. In fact, Stanic and Kilpatrick argue that what we call problem solving, Dewey called reflective thinking. Pólya, a mathematician and teacher, sought to explain how mathematics is created; his aim was to identify the thinking and reasoning processes used. He wanted learners to experience these and gain some insight into them. He wanted to discourage those forms of mathematics teaching which placed undue emphasis on the learning of mechanical procedures. Instead, he wanted to promote the more ambitious, but ultimately more successful, aim of teaching insight mathematics.

> *If the teaching of mathematics gives only a one-sided, stunted idea of the mathematician's thinking, if it totally suppresses those "informal" activities of guessing and extracting mathematical concepts from the visible world around us, it neglects what may be the most interesting part of the general student, the most instructive for the future user of mathematics and the most*

inspiring for the future mathematician. (Pólya, 1966, pp. 124-125, cited in Stanic & Kilpartick, 1989.)

Dewey, perhaps this century's foremost philosopher of education, argues that reflective thinking (problem solving), is not just one of the ways human beings deal with the world, it is the very essence of human thought. Dewey, like Pólya, believed that problem solving strategies need to be taught, and that this aim should not be separated from the process of organising mathematical subject matter. In other words, the same processes which lead to reflective thinking also lead to the structuring of mathematical knowledge.

For Dewey, the learner's experience is the essential starting point for learning. Problems arise naturally within experience, and teaching and learning involve the reconstruction of this experience, leading to the structuring of subject matter. This reconstruction of experience requires, in Dewey's view, reflective thinking (or problem solving). He went further, arguing that instruction in mathematics that "does not fit into any problem already stirring in the student's own experience", or that is not presented to the student in such a way that it will "arouse a problem" is "worse than useless for intellectual purposes". If teaching doesn't stimulate any process of reflecting its results are likely to remain in the mind" as so much lumber and debris" and become a barrier to effective thinking when a problem is encountered (Dewey, 1910, p. 119, cited in Stanic & Kilpatrick, 1989).

Pólya and Dewey both consider problem solving to be an art form that needs to be taught and learned, but not one that can be reduced to cast iron rules or algorithm procedures. Skills is involved in problem solving but problem solving is not just a skill.

It is clear, then, that for Pólya and Dewey, problem solving is central to all learning in mathematics, and for all learners of mathematics. For them learning mathematics is learning, FOR, ABOUT and THROUGH problem solving.

However, as we quoted in Holton and Neyland (1996), Lovitt (1995) says that problem solving requires more than rich mathematical activities. In addition, teachers need to be independent problem solvers, need to know how to use heuristics, metacognition and scaffolding and need to provide a quality learning environment. This environment is typified by Lovitt and Clark (1988) as follows

starts from 'where the pupils are at'
recognises that pupils learn at different rates and in different ways
allows pupils time to reflect on their own thinking and learning

involves pupils physically in the learning process
encourages pupils to expand their mode of communicating mathematics
responds to the interests, concerns and personal world of the pupil
conveys the wholeness of mathematics, rather than presenting it as a disjointed collection of topics
recognises the importance of risk-taking for effective learning
encourages pupils to learn together in cooperative small groups
involves the power of visual imagery
recognises the power of story-shells
is non-threatening and encourages participation of all pupils
encourages a wide variety of strategies in problem solving and investigation
recognises the key role parents play in the pupil's development
uses the full range of available and appropriate technology
recognises the special needs of particular pupils
uses a range of assessment procedures which reflect the approaches to teaching and learning mentioned above.

Discussion

Of the models of teaching proposed by Sigurdson et al (1994), the one that comes closes to the ideal is the Teaching Approach to Problem Solving. However, a number of authors Burkhardt (1988), Holton, Spicer and Thomas (1995), and Prawat (1991), feel that it is a major undertaking for teachers to begin this approach as it involves a rethinking of their whole approach to teaching mathematics. The difficulties for teachers arise for many reasons. Some of these relate to potential loss of control of the situation in going from a closed to an open environment where it is never clear in which direction the students will head next. Some relate to the wider range of teaching skills required to handle the richer environment, the many heuristics, the metacognition and the scaffolding. Other difficulties are due to social phenomena including student and parent expectations. But it is our experience that most of these difficulties are surmountable by many teachers, even though at times the teachers may be close to giving up.

In our current research, one teacher was in despair after what she thought was a bad lesson. The next problem solving lesson she taught was extremely successful, with the students seeing links that she herself had not previously seen. From that moment she has been sold on problem solving.

Another teacher had worked solidly doing a competent job for over two terms with a weak streamed class. In a regular third term exam (not based on problem solving) her students were 10% above similar students in another class. Her students had gained considerably on word problems. The practice they had had in problem solving situations had given them confidence to successfully tackle the word problems in the exam.

Of course, not all teachers feel comfortable with the problem solving approach, even after some practise. It is not clear yet why some teachers take to this method and others find it difficult. It is clear though, that teachers will decide whether or not problem solving takes place in mathematics classrooms. It will not happen simply because it is in the curriculum. Consequently problem solving still needs to be the subject of professional development. As most teachers were not brought up on a diet of problem solving, it is not an approach that comes naturally to them.

For teachers who wishes to try problem solving, but do not have a great deal of confidence, we suggest that they start with the Pólya Problem Solving Strategy model, move to the Problem-Process model and the Investigation Model before trying Strategy Problems as the Basis to a lesson and finally the Teaching Approach to Problem Solving. Along the way, it is valuable to watch other teachers teaching problem solving and to discuss with them the benefits and difficulties that they experience. But at all stages it is important to point out to students what heuristics they are using and how metacognition can be used. Hembree (1992) and Holton et al (1996) emphasise the value to students for the deevlopment of their problem solving potential of knowing what strategies they have used. Included in "strategies" here are both heuristics **and** metacognition.

In the last section we mentioned the need for rich mathematical activities. There is also a need for year long curriculum plans which show teachers how to incorporate problem solving into their overall programme and thereby enhance their whole mathematics teaching.

Finally, what are the key aspects for teaching problem solving? What are the prerequisites for teaching problem solving? First teachers need to have solved mathematical problems for themselves. They need to experience the pain and the joy and appreciate the processes, heuristics, metacognition required to produce a successful solution. Certainly in the early days they need to have completely solved all the problems they plan to give to their students. This way they will know that the problems they are proposing are not too easy or too hard, and do give practise in whatever heuristic or skill they intend.

The second need of teachers is that they think like a student, are able to put themselves in the position of a student and are able to see the student's point of view. This is crucial if teachers are to be able to give good scaffolding and lead students over whatever is their current difficulty. Working through problems before class helps here too, because the teacher will no doubt have experienced some of the difficulties that the students will experience.

The third teacher need is flexibility and openness. If rich mathematical activities are given to students, then the students will undoubtedly produce solution methods which had not occurred to the teacher. In scaffolding, the teacher needs to be flexible and help students through to a solution **using their method** if at all possible. It is relatively easy to insist that they back off and solve it the way the teacher has in mind. However, this route stifles creativity and it stifles learning on the part of both the student **and** the teacher. Furthermore, it's against the whole spirit of problem solving.

And finally a teacher needs patience. Our research suggests that students meeting problem solving for the first time take some six months before showing appreciable gains. The teacher therefore must not expect short term success.

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