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Mathematical Mantras

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Abstract

The failure to distinguish between mathematical facts that result either from definitions (e.g., $x^0 = 1$) or from logical inference (e.g., $\sqrt{a}.\sqrt{b} = \sqrt{ab}$), and those that require a formal axiomatic proof (e.g., $1 + 1 = 2$) prevents many teachers from offering a simple proof to these so-called "obvious" results. We will look at some of them - how they can easily be derived intuitively without resorting to any formal proof.

Introduction

It is said that if we don't know why something is true, or how we know something is true, then we probably don't understand it. If we are honest with ourselves, we are often guilty of reciting basic mathematical facts without actually knowing how these truths came about.

Unlike *axioms* (undefined concepts like "point" and "line"), many teachers go through decades of teaching without ever asking how certain seemingly intuitive, mathematical results could be derived. For example, our rationalisation about $\sqrt{a}.\sqrt{b} = \sqrt{ab}$ may satisfy our intuition but not our sense of proof. We become like mathematical monks reciting those *mathematical mantras* to hundreds of students every year by furthering the mathematical gospel of ignorance.

I have listed down ten common mathematical mantras, and we will see that we no longer need to put our blind faith in them, because they can all be easily demonstrated to elementary school students.

Mathematical mantras

1. Never divide by zero.
2. Negative times negative equals positive.
3. To divide a fraction, invert and multiply.
4. To multiply and divide decimals.
5. $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.
6. Any number times zero is zero.
7. If $a \times b = 0$, then either $a = 0$ or $b = 0$ or both are zero.
8. $a - (b - c) = a - b + c$.
9. $x^0 = 1$.
10. $x^{\frac{1}{2}} = \sqrt{x}$.

Never divide by zero

Why can't we divide by zero? The answer involves the notion of consistency.

Division by zero leads to either *no* number or to *any* number.

To divide a by b means we need to find a number x such that $bx = a$, whence $x = \dfrac{a}{b}$.

If $b = 0$, there are two different cases to discuss (i.e., $a \neq 0$ and $a = 0$).

Case 1: a ≠ 0

Since $a \neq 0$ and $b = 0$, $x = \dfrac{a}{b} = \dfrac{a}{0}$, or $0 \,.\, x = a$.

What number x, multiplied by 0, will yield a, where a is any fixed number ($\neq 0$)?

Since any number multiplied by 0 is 0, there is no such number x.

Case 2: a = 0

Since $a = 0$ and $b = 0$, $x = \dfrac{a}{b} = \dfrac{0}{0}$, or $0 \,.\, x = 0$

Because any number multiplied by zero is zero, x can take any number.

Therefore, division by zero leads either to *no* number or to *any* number.

Negative times negative equals positive

(a) Using an inductive method

Consider the following pattern with negative two being multiplied by different numbers in decreasing sequential order.

$(-2) \times (+3) = -6$

$(-2) \times (+2) = -4$

$(-2) \times (+1) = -2$

$(-2) \times 0 = 0$

As the second factor decreases by 1, the product increases by 2.

Extrapolating the above pattern, we have the following:

(-2) x (-1) = +2

(-2) x (-2) = +4

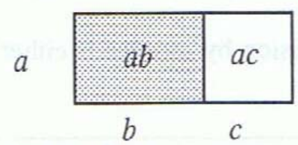
(-2) x (-3) = +6

(-) x (-) = (+)

Therefore, the product of two negative numbers is a positive number.

(b) Using the Distributive Law [a x (b + c) = (a x b) + (a x c)]

Consider the product (-2) x (-3).



(-2)(-3) = (-2)(-3) + (0)(3)
= (-2)(-3) + [-2 + 2](3) [-2 + 2 = 0]
= (-2)(-3) + (-2)(3) + (2)(3) [Distributive law]
= (-2)[-3 + 3] + (2)(3) [Distributive law]
= (-2)(0) + (2)(3) [Zero times any number is 0]
= (2)(3)
= 6

Any numbers or letters may be used in place of 2's and 3's to show that

$$(-a)(-b) = ab.$$

To divide a fraction, invert and multiply

We all know that when a number is multiplied by 1, the product is equal to the original number, e.g., $\frac{2}{3} \times 1 = \frac{2}{3}$.

We also know that there are different names for 1, such as $\frac{5}{5}$, $\frac{7}{7}$, and $\frac{\frac{2}{3}}{\frac{2}{3}}$.

Consider the following division: $\frac{2}{5} \div \frac{3}{7}$.

Clearly, $\frac{2}{5} \div \frac{3}{7}$ can be rewritten as $\frac{\frac{2}{5}}{\frac{3}{7}}$.

One convenient way of multiplying $\frac{\frac{2}{5}}{\frac{3}{7}}$ by 1 is to find a fraction which, when multiplied to both the numerator and denominator of $\frac{\frac{2}{5}}{\frac{3}{7}}$ produces a fraction whose denominator is 1.

$$\frac{2}{5} \div \frac{3}{7} = \frac{\frac{2}{5}}{\frac{3}{7}} = \left(\frac{\frac{2}{5}}{\frac{3}{7}}\right) \times \left(\frac{\frac{7}{3}}{\frac{7}{3}}\right) = \frac{\frac{2}{5} \times \frac{7}{3}}{\frac{3}{7} \times \frac{7}{3}} = \frac{\frac{2}{5} \times \frac{7}{3}}{1} = \frac{2}{5} \times \frac{7}{3}$$

Therefore, $\frac{2}{5} \div \frac{3}{7} = \frac{2}{5} \times \frac{7}{3}$

In general, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

Alternatively

From the definition of division, $\frac{2}{5} \div \frac{3}{7} = x$ if and only if $\frac{2}{5} = \frac{3}{7}x$.

So we need to find a number, x, such that $\frac{3}{7}$ times the number is $\frac{2}{5}$.

Now $\frac{2}{5}$ x 1 is $\frac{2}{5}$, so if we could find a number that, when multiplied by $\frac{3}{7}$, would yield 1, the answer would follow immediately.

Now $\frac{3}{7} \cdot \frac{7}{3} = 1$.

So $\left(\frac{3}{7} \cdot \frac{7}{3}\right) \cdot \frac{2}{5} = \frac{2}{5}$ or $\frac{3}{7} \cdot \left(\frac{7}{3} \cdot \frac{2}{5}\right) = \frac{2}{5}$. $\quad [(ab)c = a(bc)]$

But since $\frac{3}{7}x = \frac{2}{5}$, where $x = \frac{2}{5} \div \frac{3}{7}$

Therefore $\frac{2}{5} \div \frac{3}{7} = \frac{2}{5} \times \frac{7}{3}$.

To multiply and divide decimals

Let us multiply two decimal fractions, say 1.4 and 0.36.

Rewriting the decimals as common fractions:

$$1.4 \text{ x } 0.36 = (14 \times \frac{1}{10}) \text{ x } (36 \times \frac{1}{100})$$
$$= (14 \text{ x } 36) \text{ x } (\frac{1}{10} \times \frac{1}{100})$$

$(\frac{1}{10} \times \frac{1}{100})$ determines the number of decimal places the product will have if 1.4 and 0.36 are converted to whole numbers first before multiplication.

$$\text{Thus } 1.4 \times 0.36 = 504 \times \frac{1}{1000} = 0.504$$

The number of zeros in the denominator corresponds to the number of decimal places.

To divide decimals

Example

Divide $\frac{4.055}{0.25}$.

$$\frac{4.055}{0.25} = \frac{405.5}{25} = 16.22 \text{ (by division)}$$

The division is carried out by converting the decimals to fractions, and performing the division according to the rules of fractions.

$$4.055 = \frac{4055}{1000} \text{ and } 0.25 = \frac{25}{100}$$

$$\begin{aligned}4.055 \div 0.25 &= \frac{4055}{1000} \div \frac{25}{100}\\ &= \frac{4055}{1000} \times \frac{100}{25}\\ &= \frac{4055}{1000} \times 4\\ &= \frac{16220}{1000}\\ &= 16.22\end{aligned}$$

$\sqrt{a}.\sqrt{b} = \sqrt{ab}$

Let $p = \sqrt{a}$ and $q = \sqrt{b}$

Then $p^2 = a$ and $q^2 = b$

Now $(pq)^2 = p^2q^2$

$= ab$

$pq = \sqrt{ab}$

But $p = \sqrt{a}$ and $q = \sqrt{b}$,

therefore $\sqrt{ab} = \sqrt{a}.\sqrt{b}$

Any number times zero is zero

Let us prove that $a \times 0 = 0$ for every integer a.

Since by definition of 0, $a + 0 = a$, we have $a \times (a + 0) = a \times a$ (1)

Moreover, by distributive law, $a \times (a + 0) = (a \times a) + (a \times 0)$ (2)

From (1) and (2), we have $(a \times a) + (a \times 0) = a \times a$

Hence $a \times 0 = 0$ because 0 is that number which when added to any number b gives b. (In this case, $b = a \times a$)

If a x b = 0, then either a = 0 or b = 0 or both are zero

If $a = 0$, the theorem is proved.

If $a \neq 0$, there must then exist an inverse, $\frac{1}{a}$.

Since $a \times b = 0$ and any number multiplied by zero is 0, we have

$$\frac{1}{a} \text{ x } (a \text{ x } b) = \frac{1}{a} \text{ x } 0 = 0$$

Also $\frac{1}{a}$ x (a x b) = ($\frac{1}{a}$ x a) x b = 0 $\qquad$ [$x(yz) = (xy)z$]

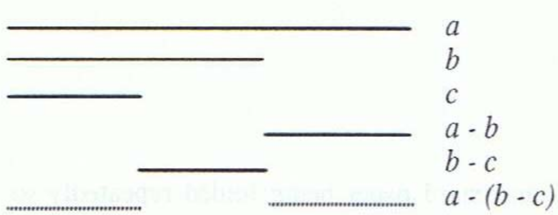
Now since $\frac{1}{a}$ x a = 1, 1 x b = 0

But since 1 x b = b, therefore b = 0

Hence the theorem is proved.

a - (b - c) = a - b + c

"Seeing is believing"



Anything raised to the zero power equals 1

It is not possible to *prove* that $x^0 = 1$ because the notation x^0 makes no sense without an agreement, which mathematicians called a "definition".

To maintain consistency in the laws of exponents, the definition $x^0 = 1$ has to be the only possible one.

Given that $x^m \div x^n = \frac{x^m}{x^n} = x^{m-n}$

Let $m = 3$ and $n = 3$, say

Then $\frac{x^3}{x^3} = x^{3-3} = x^0$

From first principles, $\dfrac{x^3}{x^3} = \dfrac{x \times x \times x}{x \times x \times x} = 1$

This implies that a reasonable meaning for x^0 is 1.

Let us test it using other examples.

$$x^2 \text{ x } x^0 = x^{2+0} = x^2 \qquad [x^a \,.\, x^b = x^{a+b}]$$

This is consistent with x^2 x $1 = x^2$ [Any number times 1 equals that number.]

$$x^2 \div x^0 = x^{2-0} = x^2$$

which is consistent with $x^2 \div 1 = x^2$

So the meaning that $x^0 = 1$ is settled.

Alternatively

A visual proof

Consider a large rectangular sheet of uncreased paper being folded repeatedly so that each fold divides the surface into two equal rectangles.

When the sheet is opened out again after any folding operation, it is found to be divided by the creases into an equal number of parts equal to some power of 2.

Thus 1 fold gives 2^1 part
2 folds give 2^2 parts and so on.
Hence n folds give 2^n parts.

For the original unfolded sheet n was equal to zero, and that it contained 1 part,

so $2^0 = 1$

***x raised to the power of half is the square root of x*, i.e. ($x^{\frac{1}{2}} = \sqrt{x}$)**

To give $x^{\frac{1}{2}}$ any meaning, we make use of the following formula:

$$(x^m)^n = x^{mn}$$

If we let $m = \frac{1}{2}$ and $n = 2$: $(x^{\frac{1}{2}})^2 = x^{\frac{1}{2}\times 2} = x^1 = x$

The above statement implies that $x^{\frac{1}{2}}$ is that number whose square is equal to x.

By definition, the square root of x times the square root of x is x, i.e. ($\sqrt{x} \times \sqrt{x} = x$).

So $x^{\frac{1}{2}} = \sqrt{x}$

Using a similar argument, we can generalise the index notation to fractional indices of the form $\frac{1}{3}, \frac{1}{4}, \frac{2}{5}, \ldots$ by letting $a^{\frac{1}{n}}$ mean the nth root of a, that is, the number x such that $x^n = a$.

Conclusion

In the light of the above, we need to be sensitive to the so-called obvious mathematical results. It is our responsibility to point out to the students that some results are merely definitional, while others demand a rigorous proof which is beyond their grasp at that point of time. But for those results that can be derived using basic facts, we owe our students a proof. Often, only a partially intuitive discussion will suffice, especially when the audience we are addressing to, is not a mathematically sophisticated one.

We need to note that in the process of demonstrating mathematical justification *vis-à-vis* rationalisation, intuition is often used to create the postulates.

For example, in proving (-2)(-3) = +6, we assume that the additive inverse of an integer is unique, and that the distributive law applies to both positive and negative integers. We only hope that students will not be under the impression that to prove a relation, all we need is to come up with enough assumptions.

Let us stop acting like mathematical monks, but take up our missionary role to convince the disciples under our charge that nothing should be taken on trust and used on the sole ground that the result works.

Listed below are other common mathematical mantras often chanted by teachers. Can you give a simple argument why they work?

More mathematical mantras

1. Zero *is* an even number.

2. $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$.

3. Area of a circle is πr^2.

4. If $ax^2+bx+c=0,\ a\neq 0$, then $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

5. If α and β are the roots of the equation $ax^2+bx+c=0$, then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

6. *Pythagoras Theorem*: $a^2+b^2=c^2$.

7. $\sin(A+B)=\sin A\cos B+\cos A\sin B$.

8. $x^2-y^2=(x-y)(x+y)$.

9. In a Δ, $\alpha+\beta+\gamma=180°$.

10. $x^{-1}=\frac{1}{x}$

References

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Skemp, R. R. (1986) *The psychology of learning mathematics*. Penguin Books.