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# Mathematical Problem Solving: A Theoretical Perspective

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## Abstract

In this paper, we consider some fundamental features of mathematical problem solving and how they relate to its teaching. Specifically, we indicate the importance of heuristics and metacognition in the problem solving process. This leads to the value of the notion of scaffolding as a teaching tool which helps to model the problem solving process for the learner. Practical examples from classroom lessons are used to illustrate the concepts. We finally discuss the use of problem solving not just for its own sake but to introduce new mathematical skills. It is proposed that, although it may not be possible for students to discover every new skill for themselves this way, problem solving is a good technique for planting seeds for new skills and hence enhancing understanding.

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## Introduction

As far as mathematics educators are concerned, problem solving is probably the most important development in recent years. In Anderson and Holton (1997), there is a short history of its introduction to schools in the Western World since 1980 through the publication of An Agenda For Action (NCTM 1980) and that year's NCTM Yearbook (Krulik, 1980).

By problem solving here we really mean two things. First problem solving is solving single closed mathematical problems using strategies and methods which need to be assembled by the problem solver for these particular problems. In solving these problems, students are not following a sample problem that they have just been shown by their teacher. They have to find their own approach, with the teacher's help, of course. This style of problem solving is exemplified in the writing of George Pólya (Pólya, 1945, 1962).

Pólya saw problem solving as a four phase process (Pólya 1973, p. xvif)[1]. The first phase is **understanding the problem**. In this stage the aim is to make the problem your own by becoming familiar with its conditions and by otherwise taking control over it – for example, by forming special cases, drawing diagrams, or describing the problem in different words. If the problem is presented in writing, it is important to gain as much from the wording as you can – what is the unknown, what is the data, and so on.

In the second stage, the problem solver has to **devise a plan**. Here it is necessary to look for connections linking the various components of the problem. By asking such questions as, have I seen anything like this before, can I derive something useful from the data, can I explore a similar problem first, and so on, a method of attack on the problem is produced. The plan needs to be flexible and may have to be altered if no discernible progress is being made.

The third phase of Pólya's process is to **carry out the plan**, being careful to check and justify each step. The final phase involves **looking back**. Can you check the result, can you verify each step, is there a better method of solution, are the kind of questions to be asked. We look at and develop this process further in the next section.

Gradually, the term problem solving has also come to mean an attitude and an approach to teaching mathematics. Here teachers use open situations and questions to explore mathematics and allow students to discover material for themselves or at least use the exploration to plant the seed for further teaching. So, rather than the teacher saying "This is what the graph of $y = 3x + 4$ looks like", she will ask "How can we find the graph of $y = 3x + 4$?". An exploration of the ideas involved will lead to the "discovery" of the linear nature of graphs of the form $y = ax + b$ in general, and to the relevance of a and b.

In this paper we use problem solving to mean both these two things.

The advantages proposed for problem solving in school are that it

(i) bases students' mathematical development on their current knowledge;
(ii) is an interesting and enjoyable way to introduce and learn mathematics and engenders positive attitudes towards mathematics;

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[1] Pólya's "How To Solve It" was first published in 1945. However references here to the book relate to the 1973 edition.

(iii) is a way to learn new mathematics with greater understanding;
(iv) is a useful way to practise mathematical skills learned by other means;
(v) encourages students to learn together in cooperative small groups;
(vi) makes students into junior research mathematicians and helps them see more of the culture of mathematics.

The importance of actively assisting students to move forward from their current state of knowledge has been recognised by Vygotsky and developed by the constructivist movement. Vygotsky (1962) noticed that children who by themselves were able to perform tasks at a particular cognitive level, in cooperation with others and with adults were able to perform at a **higher** level. The region between these two levels, or the potential for a learner's performance, Vygotsky called the **zone of proximal development**. To be able to lead students within and through their zone of proximal development, it is important to know their current cognitive dispositions.

Complementary to this is the constructivists' epistemology. von Glasersfeld (1990), for instance, says that knowledge is something which is constructed by the individual and not something that is outside the individual and simply transported into the individual's mind through teaching. Again with such a philosophy, it is important to understand the learners' current state and to enable them to construct mathematical knowledge through appropriate learning experiences.

Problem solving is an ideal way of causing students to grapple with problems – on their own, in small groups and with a teacher's assistance. In the process they use their current knowledge and develop new knowledge to solve the problem in hand.

Evidence from the classroom suggests that on many occasions problem solving can be more interesting and enjoyable than the traditional approach. There are many recorded instances of teachers who wish to talk to the class as a whole but cannot get their attention because they are engrossed with their problem solving activity. (See Holton, Spicer, Thomas & Young, 1996).

The argument in favour of achieving mathematical learning with greater understanding, rests partly on "ownership". This is the feeling that "I know this and am familiar with it because I worked it out for myself" (see Ellerton & Clements, 1992). This feeling is assisted by the provision of an environment which encourages learners to actively engage with mathematical concepts. If the problems posed are sufficiently well situated in the learners' zone of proximal development that they can lead to the construction of well embedded mathematical structures, understanding and future recall will be enhanced.

The mathematical curriculum contains concepts and processes which students will not simply construct on their own via a mathematical problem. Some mathematical ideas need to be presented more directly. However, problem solving can sow the seeds for future learning. The case of tree diagrams is discussed at some length in Holton et al (1996). Secondary school students were given problems that led them to see the importance and value of tree diagrams. Though tree diagrams were not specifically invented or appropriated by the students, their problem solving experience had prepared the way for tree diagrams to easily fit into their developing mathematical schemas. This sort of discovery and ground preparation is what Schroeder and Lester (1989) (echoing Hatfield, 1978) refer to as learning mathematics THROUGH problem solving. On this topic, Siemon and Booker (1990) say

> *Problem solving recontextualises mathematics. It provides a rich background against which to interpret meaning, to apply basic ideas, to explore strategies, and to evaluate personal performances. As such, it is much more likely to lead to insights into the processes of mathematical thinking than traditional processes which treat mathematics as a set of isolated procedures to be learnt more or less by rote. In other words, new mathematical ideas can grow out of appropriately posed mathematical problems.*

Problem solving can provide a way of reinforcing mathematical ideas learnt in other contexts. Solving problems is useful here because it provides a novel situation in which to apply skills. It should be noted though that Burkhardt (1988) feels that this works best when the mathematics being used is known very well by the students. This means that they will have probably known the particular mathematical skills involved for some time and had opportunities to practise them in closed settings first.

Problem solving can encourage the use of small collaborative groups because problem solving usually extends students and requires them to think more deeply about their mathematics than in the traditional setting. If the task presented by the teacher is a routine variation of one already done in front of the whole class, or is one that the student finds straightforward, there is little need for the student to seek help from a peer. This kind of task seldom generates discussion between students (see Thomas, 1994). Open tasks do provide the opportunity for students to discuss and to learn from each other.

Finally, in the more traditional mathematics classroom setting, students largely regurgitate the skills provided by the teacher. Consequently they have no feel for the way mathematics is created or for the deeper rules, ideas and general culture of the subject. By putting students into open situations they can begin to understand how mathematics is discovered and learn more of the aesthetic side of the subject. This is partly what Siemon and Booker (1990) are expressing in the earlier quote.

On the other hand, it has been suggested that there are real disadvantages associated with problem solving. These include

(i) the excessive time that problem solving takes, and
(ii) the difficulties involved with the assessment of problem solving.

It is clear that, from some perspectives, a problem solving approach requires a longer time to present to a class than the traditional approach. However, some people feel that time invested in problem solving at the start of a topic is recouped later (see Anderson & Holton, 1997). Others feel that this is time well spent as it pays off in greater understanding. And still others point out that if students cannot use the mathematics they are learning, the whole exercise is futile. Nevertheless, at this stage it does seem difficult to advocate using problem solving throughout the year and across all topics, especially in the later years of school where high stakes examinations have to be undertaken. It also seems reasonable to vary pedagogical approaches because it is evident that students have a range of learning preferences. Possibly not all students gain from the open approach. For instance, teachers in one study (Holton et al, 1996) felt that weaker mathematics students should spend less time in problem solving than the more able students. This area would benefit from further research.

The assessment of problem solving is an area of concern for teachers. Although some progress has been made in this area (see Clarke, 1992 and Webb, 1993) and alternative modes have been developed, there is more to be done. While fixed time examinations continue to be a major form of summative assessment, there will be difficulties for problem solving because it is not easily assessed in this way. Alternative forms of summative assessment are being attempted. One such is in the Victorian Certificate of Education in Victoria, Australia (see McCrae, 1995).

Unfortunately simply giving teachers good problems to pass on to their students does not produce good problem solving. Neyland (1994), for example, after having observed a wide variation in the way good problem starters were used in the classroom, notes

> *Some well designed [problems] do not result in the active learning event intended. There appears to be two main reasons for this. Firstly the way the activity was originally presented to the class diminished its potential. And, secondly, the interactive component of the teaching-learning process was not adequately prepared and unanticipated problem arose. (p. 11)*

Similarly Lovitt (1995) says

> *All the early problem solving efforts were mostly devoted to the creation of suitable problems in the belief that teachers could present these in classrooms and generate effective learning with the same maths they used for expository teaching. It has taken some time to recognise that this is not the case.*

In the balance of this paper we will discuss some aspects that we see as crucial for problem solving other than the availability of "suitable problems". These are heuristics, metacognition and scaffolding. We comment on each in some detail and indicate their importance for both teacher and learner.

**Heuristics**

Heuristics is an old area of philosophical discussion that was brought to the mathematical problem solving arena by Polya (1945). Its aim is to study the methods and rules of discovery and invention. According to Polya

> *Heuristic reasoning is reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem. (p. 113)*

> *Heuristic, as an adjective, means 'seeing to discover'. (p. 113)*

The point about heuristics is that they are not necessarily the means of directly obtaining a solution to a particular problem. Rather they are general strategies that will give the problem solver a means of searching for a method of solution. In difficult problems, certain heuristics will only help to begin the problem; other heuristics will aid the problem solver in other parts of the problem solving process. We take up this side of heuristics towards the end of the chapter. First we discuss some heuristics and their use in problem solving.

Begg (1994, p. 188f) has produced the following list of heuristics. These are strategies that we might expect to see students use in the mathematics classroom. It is not an exhaustive list.

- guess and check
- make a list
- draw a picture, table or graph
- find a pattern, a relationship, and/or a rule
- make a model
- solve a simpler problem first
- work backwards
- eliminate possibilities
- try extreme cases
- write a number sentence
- act out a problem
- restate a problem
- check for hidden assumptions
- change the point of view
- recognise when procedures are appropriate (and justify this)

Some of these can be organised together under more general headings. For instance, "make a list" and "draw a table" are specific examples of "being systematic". Mathematics in the New Zealand Curriculum also groups various heuristics together (Ministry of Education, 1992, p. 25).

One heuristic not on Begg's list is to ask if the current problem is like any other problem you have seen. It seems that inexperienced problem solvers see every problem as a new example unrelated to other problems they have seen. As students gain experience they begin to see that the wording of a problem masks a solution technique and many problems have a similar strategy. To aid students to gain this understanding, Norman in Holton et al (1996) did more than simply put heuristics on a board in his classroom so that they were readily available to students. In addition, under each heuristic students wrote the name of problems for which each heuristic was useful. This enabled them to see that some problems are similar and thus amenable to the same solution technique.

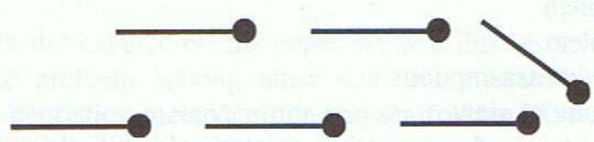
While it is important that students of mathematical problem solving learn a range of strategies for solving non-routine problems, it is also important that they learn to test these strategies in the context of the particular problem they are working on.

Students of all ages and all levels of experience with problem solving use a range of strategies for mathematical reasoning. In many instances they use a strategy which appears to them to be reasonable and they derive a solution which is logically consistent with this chosen strategy. When asked to justify the solution, they do so by showing how it follows logically from the chosen strategy. Sometimes, though, their chosen strategy is incorrect.

Let's look at three examples. Firstly, one from the junior/middle primary school: the matchstick triangle problem.

<u>Example 1</u>

Triangles are made with matches as follows:



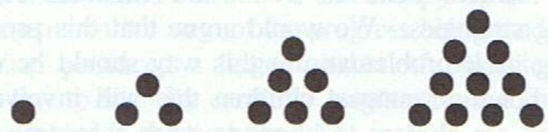
How many matches are needed to make 100 triangles?

It is not uncommon for children to use the following sort of strategy: by making 10 triangles it can be established that 21 matches are needed; 100 triangles, being 10 times this number of triangles, will require 10 times as many matches; so the answer is 10 x 21 matches, i.e., 210.

Now clearly the answer 210 is incorrect. But it is logically consistent with the chosen strategy. And the strategy is reasonable. It does work for many other problems. In this situation it is really the strategy rather than the answer which is incorrect.

<u>Example 2</u>

The next example is from the upper primary or junior secondary school: the triangular number problem.



1 3 6 10

1, 3, 6 and 10 are the first 4 triangular numbers. What is the $100^{th}$?

In this problem it is not uncommon for students to use another plausible but incorrect strategy and to derive a solution logically consistent with it. Some argue that, because a triangle is half a square the solution can be found by forming a square with 100 dots along the base, calculating the total number of dots (100 x 100), and then halving the result to give a solution of 5000. Once again the strategy is reasonable and works elsewhere, and the answer is consistent with it, but in this case the answer is incorrect.

<u>Example 3</u>

As a final example, we would like to draw on the research on heuristics in mathematics carried out by Tversky and Kahneman (1982). These researchers investigated the heuristics people use when reasoning in situations involving uncertainty and have identified a number of useful reasoning strategies which are often used wrongly, even by people working at a tertiary level. One of these heuristics is what they call **representativeness**. For example, many people incorrectly reason that in a family of 6 children the sequence BGBGBG (boy, girl, boy, girl, boy, girl) is more likely to occur than either BBBBBG or BBBGGG. Tversky and Kahneman argue that these people use **representativeness** as the reasoning strategy for making this judgement. BBBBBG does not appear to be as representative of the near 50-50 distribution of boys and girls in the population as BGBGBG, and BBBGGG does not appear to be as representative of the random way in which children of particular genders are born. Once again the reasoned judgement follows logically from the choice of strategy, and the choice of strategy is plausible.

In each of the 3 examples cited above, people used reasonable but incorrect strategies to deduce logically consistent answers. Clearly it is not sufficient to ask learners in these situations to merely justify their answer. From their point of view their answers are perfectly logical. They need to be asked to justify, and hence test, their **strategies**.

Accordingly, learners need to be shown methods for testing their mathematical reasoning strategies. We would argue that this process of making mathematical reasoning itself problematic in this way should be occurring at all levels of the school. For the youngest children this will involve helping them become aware that there are **choices** to be made when reasoning mathematically (there are different ways of looking for patterns, there are different ways of adding numbers, and so on). Once they are aware that there are choices they can be encouraged to start to think about whether some choices are better than others in particular circumstances. Older students can be encouraged to make proof strategies problematic (e.g. by asking whether it is proof by contradiction or mathematical induction, a valid proof strategy and why).

**Metacognition**

Basically **metacognition** is "thinking about your thinking". According to Schoenfeld (1987) metacognition encompasses the three areas of (i) self-regulation, or monitoring and control; (ii) knowledge of our own thought processes; and (iii) beliefs and intuition. All of these areas have an impact on the way we tackle problems and, in fact, on almost every aspect of our lives.

Self regulation is particularly relevant in problem solving. However, its importance is generally not realised until a problem is tackled which is of some difficulty to the solver. Schoenfeld (1992, p. 335) suggests that "it's not just what you know, it's how, when and whether you use it" that is important.

Schoenfeld (1992) reports on an experiment in which he compared the control behaviour of a novice problem solver with that of a research mathematician. The novice read the problem very quickly and then embarked upon a method of solution. The novice stuck to this single approach for the full 20 minutes allocated, despite clear evidence that they were making no progress. At the end of the session they were asked how that approach might solve the problem and were unable to give a reason.

Schoenfeld feels that this behaviour is typical of inexperienced problem solvers. They make a quick decision on how a problem is to be tackled and keep to that approach despite their lack of success.

On the other hand, 'expert' problem solvers are much more aware of their progress and continually monitor how things are going. In Schoenfeld's experiment, the mathematician spent more than half of his allotted 20 minutes making sense of the problem. Time was spent purposefully exploring the problem before he found what turned out to be a correct approach. The expert also made a number of comments on the state of the progress being made. During the problem, the mathematician produced a large number of possible solution methods but selected only a few to try. By carefully monitoring the chosen method of attack, the problem was solved in the time available.

In Schoenfeld's experience, students can learn the control strategies of the expert. However, these skills need to be taught explicitly. Schoenfeld suggests that this can be done by 'coaching' with active interventions as students work on problems. We will return to this ideas of 'coaching' later when we discuss scaffolding.

To help students learn to master this control phase, Schoenfeld (1992, p. 356) suggests using the following general questions.

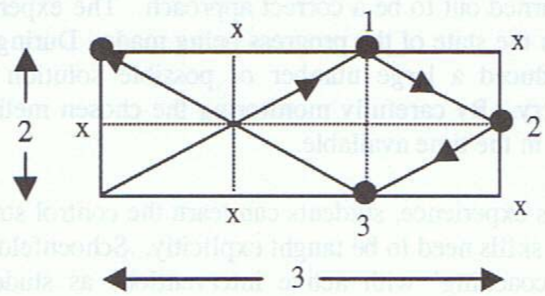
*What (exactly) are you doing? (Can you describe it precisely?)*
*Why are you doing it? (How does it fit into the solution?)*
*How does it help you? (What will you do with it when you have it?)*

Students also need to think "Have I been using the strategy for too long? Is it time to try another approach?"

The second aspect of metacognition, knowledge of our own thought processes, has not been pursued to any great extent in the mathematical education literature. It is clear though that the third aspect, beliefs, can have an effect on problem solving. For instance, if a student believes that all mathematical problems can be solved in 10 minutes, they will stop working on problems after a little while and so may not come to grips with problem solving at all. It is clearly up to the teacher to engender positive beliefs and attitudes in students.

How can teachers go about teaching mathematics students to analyse a problem, to test the success of a chosen strategy and if necessary to select an alternative? In our experience it is important to encourage students to spend time identifying as many potential solution strategies as possible before one is selected for prolonged investigation. Once a particular strategy is selected the strategy itself needs to be treated as problematic and evidence needs to be sought, on the one hand, to justify its further use, and on the other, to refute its selection. Here is an example.

The Billiard Table Problem




If a ball is fired at 45 degrees from one corner of a n x m billiard table, where n and m are whole numbers, how many bounces will occur before it lands in a pocket? In the example shown in the diagram, a 2 x 3 table results in 3 bounces.

We have seen many students investigate this problem and most approach it much as Schoenfeld's novice does, by selecting a single strategy: make a table of values of m, n and b (the number of bounces) and look for a pattern. Usually these students use a small number of other strategies on route, and many successfully arrive at a conjecture for the relationship between n, m and b. However, few students seem to consider any other possible strategy and even fewer offer a proof (or for that matter any kind of justification, other than by testing a number of cases) for their chosen conjecture.

We tried teaching a new group of students some strategies for analysing this problem before they adopted a particular strategy. The students came up with a range of possible solution strategies.

- Make a table and look for a pattern.
- Look for symmetrical patterns formed by the path of the ball.
- Look for classes of common shapes of ball pattern, e.g., a 2 x 3 and 4 x 6 table will produce the same pattern.
- Look at the classes of table with sides of the same ratio.
- Don't look for the number of bounces, look for the number of "misses" (shown by crosses on the diagram above).
- Divide the table into square proportions of length the size of the shorter side and examine what happens with the remainder of the table.

- Solve a harder problem first, e.g., if the angle was 50 degrees. (Yes, this is sometimes a useful strategy and in this case often leads to a quick result and proof!)
- Bounces off a cushion are a bit like reflections in mirror. Can this similarity be exploited?
- Divide the problem into a series of smaller problems and solve each special case one by one. For example, square tables, 1 x n, 2 x n tables, and so on.
- Work backwards from the pocket the ball falls into.

This group of students approached the problem in a style more similar to Schoenfeld's expert, and were able to justify and prove their result.

**A Model for Solving Problems**

In this section we give a model which describes the problem solving process and we show how this incorporates the heuristics and metacognition that we have discussed earlier. This model is additional to, and complementary to, Pólya's Four Phase Model which we presented earlier.

Before considering the model, it is worth noting that there are three words associated with problem solving that we use here in a special way. By the **answer** to a problem we mean the final outcome. This may be a number, a formula, a set of conditions (Mary is the farmer, John is the candlestick maker, etc.), or something else. Whatever its form, the answer settles the question raised in the problem. Getting to the answer requires a **method**. This method will have implicit and explicit parts. Heuristics and metacognition will be used implicitly, as will regular mathematical skills. Explicitly the final method, if written out in full, will show the logical steps, with appropriate equations and diagrams, which justify the answer. The sum total of method plus answer we will refer to as the **solution**. In obtaining the solution to a problem though, it is important to note that heuristics, metacognition and cognition are all used. We illustrate this in Figure 1.

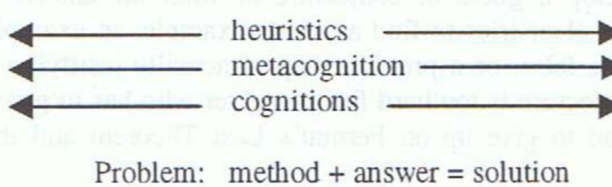



Figure 1

It is important to note here that the processes by which solvers move from problem to solution are almost always nonlinear. Except in very simple problems, an heuristic will not produce an idea which produces a method and then an answer. Almost always there is a shunting backwards and forwards between these various quantities. It is not uncommon for us to "know" the answer before we have the method to justify the answer. Fermat "knew" the result of his famous Last Theorem over three hundred years before it was finally proved. The fact that only four colours were needed for the Four Colour Theorem was "known" a hundred years before it was proved.

These facts that we "know" without justification are called **conjectures**. In essence they are just mathematical guesses. Solutions to difficult problems go through the process shown in Figure 2.

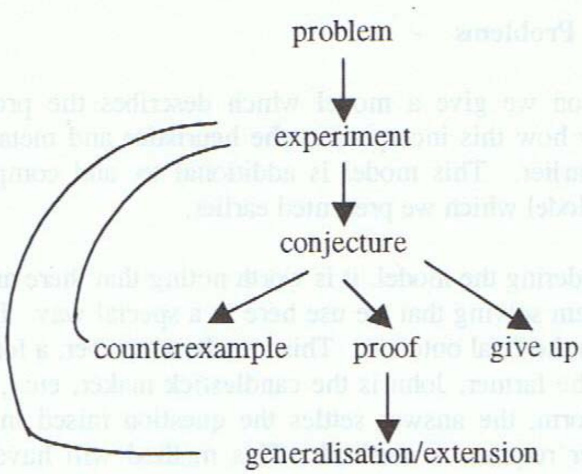



Figure 2

The model in Figure 2 gives some idea of the framework in which problems are solved. Given the problem, a certain amount of experimentation takes place. During this experimental period, the solver begins to understand the problem and develop a guess or conjecture of what the answer might be. By various means the solver tries to find a counterexample, an example which shows the conjecture to be false, or a proof, a way of actually justifying the conjecture. Sometimes the conjecture is too hard for the solver who has to give up. Countless mathematicians had to give up on Fermat's Last Theorem and the Four Colour Theroem.

It a counterexample is found, more experimentation is required to obtain a new conjecture. It a proof is found, then the problem might be generalized or extended in some way and the cycle begins again.

**Scaffolding**

Earlier we mentioned the idea of "coaching" with respect to metacognition. The questioning process which is implicit here is an important part of what is now called **scaffolding**. The concept developed from Vygotsky when he noted that children could go further when they worked in conjunction with an adult than they could on their own. The term first appeared in Wood, Bruner and Ross (1976).

Greenfield (1984) described scaffolding in more detail as follows.

> *The scaffold is a metaphor to describe the ideal role of the teacher. The scaffold, as it is known in building construction, has five characteristics: it provides a support; it functions as a tool; it extends the range of the worker; it allows the worker to accomplish a task not otherwise possible; and it is used selectively to aid the worker where needed. To illustrate this last point, a scaffold would not be used, for example, when a carpenter is working five feet from the ground.*
>
> *These characteristics also define the interactional scaffold provided by the teacher in a learning situation. That is, the teacher's selective intervention provides a supportive tool for the learner, which extends his or her skills, thereby allowing the learner to successfully accomplish a task not otherwise possible. Put another way, the teacher structures an interaction by building on what he or she knows the learner can do. Scaffolding thus closes the gap between task requirements and the skill level of the learner.*

Cambourne (1988) sees the most common sequential interactions in scaffolding as (i) focusing on a gap which the learner needs to bridge; (ii) extending the student by raising the skill level to bridge the gap; (iii) refocusing by encouraging clarification and redirecting by offering new information if there is a mismatch between the learner's interest and the message or in the teacher's expectations of the capacity of the learner.

Bickmore-Brand and Gawned (1990) provide implications of scaffolding for the classroom, the teacher's role, the learner's role and curriculum activities/materials. As far as the classroom goes, it is important to provide an interactive environment for mathematics. Learners might, for example, be expected to work together in groups of two to four on specific tasks.

There are wider implications for the teacher's role as now this is one of a roving consultant rather than a director of learning. The teacher needs to be a major participant in the mathematical activities, scaffolding the student and the activity so that the student is extended in both mathematics (skills **and** processes) and in developing the language to handle mathematics. So it is necessary for the teacher to encourage and model purposeful, exploratory, supportive, reflective classroom talk. This involves such questions as "Why do you think ...?", "How might ...?", "What would happen if ...?", "Do you remember ...?", "I don't believe ..."

Students need to be aware that the teacher values interacting with them. If a student is stumbling verbally, "encouragers" may be used to help them extend the length of their turn. The teacher should also consider leaving a pause after speaking to encourage a response.

Learners should feel free to ask "curiosity" questions and to seek opinions from a wide variety of sources, not just the teacher. They should have the opportunity to explore ideas and practise their knowledge of mathematics and the new language of mathematics that they are acquiring.

An effort should be made to relate mathematical activities to the learner's background knowledge. Tasks that are meaningful, and where mathematics is used as a means to an end, are worthwhile here. Activities could also be experiential and interactive, and related to the learner's interests. It is also important to provide open-ended activities where the resolution of the task can be obtained by a number of methods.

The point of scaffolding is to help students overcome a barrier in their progress to successfully solving a problem. Naturally the teacher could give the student the piece of information required but what advantage would that be to the student? How would the student cope with the next barrier?

Through scaffolding, the teacher is modelling problem solving behaviour and is showing the problem solver the kind of question that it is useful to ask when a barrier is reached. The aim is for the student to internalise the scaffolding so that these questions can subsequently be asked by the student. Thus we might think of metacognition as being **self-scaffolding** or of scaffolding as being **public metacognition**. Whichever of these terms you prefer, the goal of scaffolding is to help make the student an independent problem solver. Consequently, students need to know why it is that teachers are asking these apparently obtuse questions, rather than supplying direct answers. This was clearly in Pólya's mind when he wrote

> *There are two aims which the teacher may have in view when addressing to his students a question or a suggestion of the list. First, to help the student to solve the problem at hand. Second, to develop the student's ability so that he may solve further problems by himself. (Pólya, 1973, p. 3)*

In problem solving, students seem to encounter difficulties in three areas. The obvious areas are strategic and mathematical but there is another area. This is in the affective domain, where students are experiencing some difficulty for instance, because of lack of motivation, lack of belief in their own ability or because they cannot cope with the frustration caused by lack of progress.

Strategic barriers arise when a student lacks the ability to completely formulate the right approach or strategy for solving the problem, part of a problem or a related sub-problem. Mathematical barriers arise when a student's lack of mathematical knowledge prevents progress towards a solution.

The three types of barrier are not mutually exclusive. This is evident in a situation where a student is unable to formulate a strategy because he or she does not have mastery of a particular piece of mathematical content. If at the same time the student has a difficulty in the metacognitive area relating to beliefs and intuition, then the student's impasse is the simultaneous result of all three types of barriers.

The questioning that teachers may use to help students overcome these barriers, also appears to fall into three categories. The first we mention is **cognitive scaffolding**. This scaffolding is required when students are having difficulty determining a method of attack or an appropriate piece of knowledge, skill or procedure. Generally here the students will be aware of their need for support. First the teacher needs to determine the precise nature of the difficulty. This is often facilitated by a quiet period of observation or some discrete questioning. If the student's problem is caused by the lack of a suitable heuristic, then the teacher needs to determine where the student's difficulty is in the problem solving process. This means that the teacher will need to be aware of Pólya's Four Phase model.

If the student's difficulty is related more to a mathematical skill, the student will need to be reminded of the piece of mathematics. In cases where the student has never met this skill, then it will need to be taught before the problem can be solved, unless there is another way of tackling the problem.

**Metacognitive scaffolding** is help given in response to a metacognitive need. This type of scaffolding differs somewhat from cognitive scaffolding in that the student may not be aware of the need for such assistance. In fact, a desirable feature of such scaffolding is that the teacher tests and questions students with regard to the overall management of the task requirements whether they "need" it or not. The point of this is to improve students' awareness of how well they are managing and monitoring their progress, whether they feel "in control" or not. Hence, students should expect this sort of interaction at any time, not just when the teacher (or student) thinks things are not going too well. This is crucial. If the teacher only requests justification when things are not going well, the students come to see these requests as a cue to search for an error rather than a prompt towards forming good habits.

There are other key features of the help offered by teachers' scaffolding which are not explicitly expressed as part of cognitive or metacognitive scaffolding. We refer to this as **other scaffolding**. These include the transfer of responsibility for the task labour from the teacher to the student (which may include asking the student to give justifications for assertions made); the scale of the assistance given, whether it be for a small part of the journey from starting the problem to finishing it (**micro-scaffolding**), or whether it be more concerned with a significant portion of the journey from start to finish (**macro-scaffolding**); and the clarification of the ideas raised, and refocussing on new targets to be reached.

**Scaffolding Examples**

In this section we discuss some specific scaffolding examples from the transcript of a lesson which was part of the research reported in Holton et al (1996). The class is a Form 3 class (the students are approximately 13 years old).

The problem under discussion was to construct two dice so that each of the sums 1 to 12 was equally likely when the dice were rolled. The teacher decided to check first that the students could calculate the probabilities of the various sums obtained from normal dice.

In starting, the teacher says:

> *Here's the problem, but before we do it, we're going to have to just check a little bit of background and relate it to some of the work from earlier on. So I'll just read it through for a start, then we'll think for a minute ... I want you to think, firstly, if there's another problem or another activity that we've already done that was a bit like this, OK.*

Getting the students to think of a previously solved problem that might be useful is a problem solving heuristic that we have mentioned before. This provides an example of both cognitive and metacognitive scaffolding to overcome a strategic barrier.

Then the teacher went on to talk about the fact that the problem was going to be tricky and that they would possibly feel frustrated by it. This provides an example of an affective domain barrier being overcome by metacognitive and other scaffolding.

> *T: (to class) ... Right ... I want to do an activity today which isn't going to be easy, all right? It's going to involve you in doing a bit of problem solving ... it's going to involve you in getting annoyed and thinking, "... blow, I think I just want to stop this problem." So I felt like that ... I felt like that at one stage when I first worked this one ... just did this one recently, I thought, "Oh, Gill, blow." ... I was a bit angry, so you might feel like that for a minute and you might think, "Oh, this is too hard, and I'm not going to do it." But I want you to try and get over that and think a little bit more, and try maybe to ask the people in your group when you feel like that, and see if they feel like that, and then you think, "OK, let's start again and think about it another way." I want you to just try and just persevere for a little bit ... it's not easy.*

In the transcript below the teacher is working with a group of four students. They are trying to work out the changes of getting a total of 4 with two normal dice.

> *A: Well, you can have ...*
> *T: (To C) So what have you got?*
> *C: 1 and 3*
> *T: 1 and 3 ... so what else could you have?*
> *B: Um ...*

*C: ... and 3 and 1.*
*T: That's two ways. What else could you have?*
*C: 3 and 1 ...*
*B: 2 and 2, and then another 2 and 2.*
*T: Is there another 2 and 2? Tell me about that.*
*C: ... 3 and 1 and 1 and 3 ...*
*B: Yes, because they're opposite other ... dice.*
*C: They're different dice.*
*T: Oh, wait a minute ... OK ... so you say ... (to C) show me 2 and 2 on here.*
*C: (with dice) Here.*
*T: Right, that's 2 and 2 ... right, now, so that ... that lands like that.*
*Now, show me the different thing.*
*C: (demonstrating) No, it isn't ...*
*B: No ... oh ... (laughs)*
*C: It is!*
*B: (laughs)*
*T: Now, if they land like this ... is that different to if they land like this?*
*C&B: Yeah ...*
*C: They're different dices!*
*B: Yeah, but they're the same ...*
*A: Like you could ...*
*C: ... so they can't change*
*T: Right, so how many ways are there of getting 4?*
*B: Three.*
*C: Oh, three!*
*T: You're right, then. (Teacher leaves group)*

In this transcript, the teacher continues to ask open questions. With statements like "Tell me about that" and "show me", the teacher is putting the onus on the students and requiring them to justify their assertions. By suitably manipulating the conversation, the students are led to see that there is only one way of obtaining 4 using 2s but two ways with 1 and 3. This section of the lesson gives examples of strategic and mathematical barriers which have been overcome with cognitive scaffolding.

The next transcript shows where an opportunity for the use of metacognitive scaffolding to overcome a mathematical barrier was missed.

> *T: OK, so we've done some dice problems with chance, and you can think about the one you did the other day. Can anyone even think back to last week when you did anything like this? ... Any total from 1 to 12? Does that not ring a bell with anybody? What did you play last week?*
>
> *G & H: Pass the Pigs.*
>
> *T: Right, anything else? Anything else that had anything about totalling them up?*

In her response to "Pass the Pigs" the teacher has shown that this was not the game she was thinking about. She could have scaffolded around this by asking what the important features of "Pass the Pigs" were and how they linked to the current problem.

So far we have assumed that everything goes smoothly in the scaffolding process. However, there are at least two difficulties noticed by Holton et al (1996). The first problem was what should a teacher do if the students do not "hear" the scaffolding.

In a Form 2 class, we observed a group of mathematically able students who ran into a problem. The teacher arrived and gave what appeared to be good scaffolding but this was ignored by the students. If they had listened to what he said they would have found a way around their difficulty. However, they seemed to be so engrossed in the problem that they did not appear to really be listening. They were not being rude; they were concentrating more on their own thoughts than on what the teacher was saying. The teacher apoeared to realise that there was an impasse and moved on to another group. Later the students solved the problem for themselves.

Such instances do occur from time to time. For some reason the students are unable to use the teacher's scaffolding. Possibly this is because the teacher is working outside the students' zone of proximal development. It may also be the case that the students are concentrating so much on their own train of thought (which may or may not be correct) that they do not really hear what the teacher is saying. The best strategy on such occasions appears to be to leave the group and return later when they are more receptive.

The other scaffolding problem is whether or not students should be told an answer. Must the students produce all answers for themselves? There are going to be occasions when the teacher will have to give an answer. This may be because of time pressure or because of the frustration level of the students. There are also going to be times when the teacher is unable to think of an appropriate open question. The aim, though in general, is to use questions that will provide a maximum number of opportunities for the students to supply the answer for themselves.

**Discussion**

We have outlined some of the theoretical streams which feed contemporary discussions of problem solving and we have illustrated some of these with classroom examples. Three things will, we hope, be evident by now. Firstly, problem solving is not a single, precisely defined method in mathematics teaching. It is much better thought of as a braid of several fibres each contributing to the overall colour, texture and strength of what we are thinking as the problem solving approach in mathematics.

Secondly, the braid is still being woven. There is a great deal we do not yet know about the teaching and learning of problem solving and we are really still in the early days of the problem solving curriculum. But we do know that, even within the realm of current knowledge about problem solving, there is plenty of room for individual variation by teachers.

Thirdly, there is a widespread commitment among teachers, curriculum developers, and education researchers, to the problem solving approach to mathematics teaching. This is because problem solving approaches avoid many of the difficulties associated with earlier teaching methods and because problem solving approaches have embedded within them many of the educational characteristics believed to be important for a style of mathematics teaching will be appropriate for the early years of the third millennium.

We wish to make it clear that we have only outlined some of the available theory which supports the problem solving mathematics curriculum. In particular, there is one major current of related theory we have not outlined above for reasons of space. This current, itself drawing on a range of traditions, integrates problem solving pedagogy (Freire, 1972), critical mathematics teaching (Frankenstein, 1983), connected mathematical teaching (Belenky et al, 1986), and the "public educator" approach to mathematics teaching (Ernest, 1991). The inclusion of this major current in the above discussion would not have changed the conclusions drawn above. It would, however, have strengthened the theoretical basis we outlined for problem solving and broadened the range of its applications within mathematical education.

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