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Occam's Razor And Understanding In Mathematics

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Abstract

The philosophical principle known as Occam's Razor suggests that we should not make unnecessary assumptions; a simple explanation is to be preferred to a complex one. A number of different analyses of types of understanding have been made, starting from Recorde (1551) to the modern work of such writers as Skemp, Mellin-Olsen and others. Over-elaboration of models may not be of much help to the practising teacher. A simpler division of experience into natural, conflicting and alien experiences, described by Duffin and Simpson (1994, 1995) which has much in common with the work of Piaget and Davis, may be a more fruitful approach.

Introduction

In 1551, Robert Recorde made the distinction between different types of understanding of mathematics, predating our present interest in this matter by nearly 450 years! He distinguished between those who "study principally for learning" and those who wish to acquire knowledge for some purpose or other, but have "no time to work for exacter knowledge". In 1976, Mellin-Olsen made roughly the same distinction – between those who wish to learn because they attached **personal significance** to what they have learned, and those who see knowledge as **instrumental** in attaining other goals, which may have very little to do with mathematics itself.

Instrumental and Relational Understanding

Mellin-Olsen, and later Skemp, went on to distinguish two types of learning or understanding. **Instrumental** understanding is displayed when a student gives evidence that he knows how to apply a principle or procedure, without necessarily appreciating its relation to a mathematical structure, or the reason why a procedure works. This could also be called **"procedural knowledge"**. **Relational** understanding, which could be said to lead to

"conceptual knowledge" is shown by such responses as point to a student's acquisition of knowledge about a structural relationship, and reasons why procedures work. Skemp pointed out that instrumentalism was often described as "rules without reasons", and was frequently regarded as not evidence of understanding at all. For others the possession of a rule was regarded as evidence of understanding; you would not want to say to a student who you suspect has the rules but not the reasons, "you may think you understand, but you don't" – to which the student may very reasonably reply "Of course I do – look at all these right answers". We end up devaluing the student's achievement. We might say that both types of understanding are needed, possibly in different situations. It would be a salutary experience for all of us to make a list of occasions when we offer instrumental explanations to our students, perhaps because of time or other constraints. But they may want to know why, for example, the rule for multiplying fractions works.

Intuitive and Formal Understanding

Having given a division of understanding into two types, it was not long before other writers seized on this and produced models of their own, with more than two types. Byers and Herscovics (1997) accepted the two types already considered, but introduced two more. The first of these was called **Intuitive** understanding, where all or most of the "formal proofs" were omitted and replaced by "heuristic arguments or appeals to physical reality". In demonstrating that she was at this level of understanding, a student would be expected to rely on imagery, using guessing, not give a well-ordered sequence of steps in proof, show little awareness of the process of solution of a problem, and have difficulties in reporting her work. We believe that flashes of sudden insight often precede proof, and proof often means confirming what we know already. It may be that the use of a Dynamic Geometry program such as the Geometer's Sketchpad leads to this sort of temporal distinction as to how we work. The use of this type of software might lead students to be convinced about the truth of the concurrence theorems for a triangle, and how these might be extended to the case of quadrilaterals, and then see the need for a formal written proof of the type we have become accustomed to in older textbooks, and which have all but disappeared in modern texts. What exactly do I understand if I drag a triangle around the screen and always the medians are concurrent, so that I am convinced that it will always be so? Is this inferior knowledge to that shown by writing out a formal proof? This leads us to consider the fourth type of understanding proposed by these writers, which is called **Formal** understanding, where mathematics is seen as a cultural product which is inseparable from symbolism and notation. This is shown by the ability to connect symbols and notation with relevant ideas and connect them into chains of logical reasoning. But an undue emphasis on this type of formal reasoning may preclude the development of relational understanding. It may smack of rote learning of, for example, the way in which to write out a proof. This **Tetrahedral** model of understanding does have the merit that it shows how the four types are related to each other; few learners will have only one type; at least, they will operate in different ways at different times and with different types of material. It is suggested that the types of understanding can interact – either reinforcing or hindering each other.

"Formal" is sometimes replaced by "Conceptual" but we do not "teach" for conceptual understanding; rather we avoid giving rules, but give experience from which the learners may abstract the rules. So we test for understanding of concepts by seeing if the learner can use the concept in different situations. The concept of a Factor is tested by asking such questions as "what are the factors of 15" or "tell me some numbers which have 8 as a factor". The greater the range of situations in which the learner can use the concept, the greater we will say what his conceptual understanding is. If also he can consider prime numbers and prime factors, then we will feel that the learner has an even deeper understanding of what a factor is. As evidence that a student understands X, we want to know that the student can apply X in situations different from that in which it was learned or previously encountered. Both relational and instrumental understanding seem now to be coming together, when we come to the crux of the matter – how to test for understanding.

Procedural or Propositional Learning

Procedural learning can be changed into what is sometimes called **propositional learning**; rather than putting the emphasis on learning "how to" we put the emphasis on learning "that"; if a student says that she learned "that $x^2 - 2x$ can be factored" then many questions can be asked which will build up a relationship between knowing that, knowing how, and knowing why. These will include such questions as – how is it different from expressions which are not factorizable; what are the procedures for carrying this out, and are there more than one way to do it; what is the connection between this procedure and the underlying mathematical structure, and perhaps most importantly of all, why should we be concerned with factorizing at all? Miller, Malone and Kandl quoted by Kieren (1992) suggest this "3 dimensional space" in which varieties of understanding reside. It consists of "knowing that" which will range from knowledge of discrete bits of information to knowledge of an integrated body of information; "knowing how", which ranges from knowing how to perform simple procedures to knowing how to perform complicated procedures; "knowing why" which ranges from an intuitive knowing to a rigorous knowing. This is useful for teachers because they can place a student's present understanding in relation to the goal position.

This can be linked to the view suggested by Tall (1978) that we should see understanding as a function of time. If we are teaching or learning instrumentally, we are within the comfort of a closed system. The teacher can always fall back on the answers at the back of the textbook, or in the "teachers guide". But when we try to make a coherent mental picture of a body of knowledge, rather than of single disconnected pieces of information, as we move from a comfortable situation to a search for a coherent pattern, we are going towards relational understanding.

Occam's Razor

As more writers discussed the question of levels of understanding, the number of suggested types of understanding increased. Even Skemp himself increased his original 2 categories to 4, and among the adjectives used to describe understanding are **symbolic; formal; logical**. Kieren and Pirie (1992) suggested 7 styles, but these are essentially stages on a growth in understanding, and describe what the learner can do; these are primitive knowing, image making, image having, property noticing, formalizing, observing, structuring and inventing. Nested rings are used to illustrate the model; more sophisticated levels have access to less sophisticated levels. Again, the use of this model may be to place the individual learner at some point on the pathway, in time, from a very simple knowing to a complex knowing. Otherwise we would want to adopt the philosophical principle known as Occam's razor, which suggests that it is preferable to adopt a simple explanation rather than a complicated one. To think of understanding as something which takes place in time, with the learner proceeding along a path which goes from superficial understanding of isolated knowledge to a deeper understanding of mathematical structures seems to be an acceptable approach. Unfortunately attempts to formalize what a school curriculum in Mathematics should be seem to lead inevitably to the compartmentalization of knowledge, and an emphasis on testing which is associated with this approach prevents the attainment of a global view as to what the subject is about, and what a school curriculum in Mathematics could be.

Different Experiences

Recent work which has been helpful includes the work of Duffin and Simpson (1994, 1995). They attempt to deal with understanding by trying to see what is going on in the learner's mind, and see what the implications are for the teacher in the classroom as she tries to see if the learners are developing a deeper understanding of the content being studied. They consider how learners deal with new experience, how they fit these experiences into their existing understanding. These experiences are classified as being either **natural, conflicting or alien**. A natural experience is one which fits in easily with a learner's existing understanding, and will reinforce his current thinking. For some learners the idea that $2^0 = 1$ is a natural experience; for others this idea is difficult to comprehend, and may be a conflicting experience; one which does not fit in with the learners' existing mental structures concerning the laws of indices. Their existing understanding of 2^n as meaning "2 multiplied by itself n times" or more precisely "the product of n factors, each of which is 2", does not help, and the learner may feel that 2^0 does not have any meaning. However experience with division may help as $2^3/2^3 = 2^{3-3} = 2^0$, but also = 1, as anything divided by itself equals 1. A conflicting experience may affect existing understanding unfavourably and detrimentally, but it may also lead to a positive approach, where the existing structure is seen as coping with a limited class of examples, and has to be replaced by an improved structure which will cater for the new experience. This has similarities with the ideas of Davis (1967) on "Monster-barring and Monster-adjustment" and indeed with the Piagetian ideas of Accommodation and Assimilation. An alien experience is one where the learner finds no point of contact with anything she has already learned. The most obvious way of dealing with this is to ignore it and to move on to something else. Characteristics of understanding can fall into two groups; the first of these with "being able to do" as a major descriptor; the second deals with abilities, such as "being able to follow logical reasoning" with the result that the learner will be able to retrieve "organised, relevant knowledge". Duffin and Simpson found the first group to be alien to their ideas, although it has much in common with the idea of instrumental understanding; the second category was found to have some points of contact with the ways in which they were looking at understanding. For them, understanding occurs when we "feel comfortable; feel the thing belongs to us; could explain it in our own words; are able to recognise it in different contexts". The value of this view for the teacher in the classroom is that the emphasis is placed on how the learner feels about what she is studying, and how she can communicate her understanding. This has considerable relevance when we come to consider how we can assess a learner's understanding. The authors suggest that "on its own, the ability to do or to reproduce a procedure is not enough to ensure that understanding is present because it may merely imply a good memory". Stronger evidence for understanding comes from other abilities – to recognise things in different contexts, and to explain things in our own words. In explaining something, we are using connections between what has to be explained and the simpler language in which we want to express it. So deeper explanations come about because we have been able to make deeper connections – that is, with a greater body of knowledge which is at our disposal. This is of value not only in assessing students, but for the teacher herself in the process of teaching, we find that students will say that they do not understand, and the teacher is asked for a better or clearer explanation. Here the teacher has got to be able to make deeper connections; this time not with structures known to her, but to structures which are already in the repertoire of the students. As we will note, assessing doing or reproducing procedures is a comparatively simple task, but assessing structural and communication skills is a complex one. We are considering how to identify not internal but external indicators: "If my student understands, then I will see……" Duffin and Simpson (1995) found that teachers tend to use external indicators to see if their pupils can do….that is, can perform a task, to see if they can reconstruct a concept or process; but also can put something into their own words, to explain it to the teacher or to their classmates. But the problems still remain; the pupil may be reproducing or recalling rather than reconstructing and may be giving somebody else's explanation and not their own.

Conclusion

We need guidance as to how to match up the main two types with regard to these three factors – the teacher's style; the student's preferred learning strategy; and the learning materials – textbooks, software, manipulatives which are being used. It is not that we are expecting to be able to produce an exact match between the types across these factors – that is, a teacher teaching instrumentally a class of learners who wish to learn instrumentally using a textbook or a traditional type, presenting proofs and giving practice exercises. Rather we require some flexibility with regard to all aspects of the procedure. When should I teach relationally, and to which types of pupils? What parts of the curriculum can be effectively dealt with in an instrumental manner? How should the methods of assessment used match up with the strategies adopted? If we are teaching a curriculum where the subject matter has been broken up into small segments, we find that the method of assessment will differ from the style of teaching exemplified in the Shell Centre's "Teaching Strategic Skills". To introduce this style of teaching and assessing into an overcrowded curriculum implies that something has got to be deleted from the existing syllabus. This may mean the loss of some cherished pieces of Mathematics. Of course it is not a permanent loss; it will appear somewhere, at a later stage, but perhaps not at all. It was decided that memorizing the formula for the general solution of a quadratic equation was a bit of mathematics which could be dispensed with, for a pupil at age 16. Some might wish to defend the continued inclusion of this formula in the curriculum, but it surely is a more productive approach to consider factorization methods, than "completing the square", and finally derive the formal, but once this has been done it is consigned to a table of formulas for use. If a student can solve a quadratic equation using factorization, we would say they are operating relationally. If they use the formula, they may be operating instrumentally. I say "they may be", and this brings me on to the difficult task of assessment. Assessing procedural knowledge is easy; assessing conceptual knowledge is difficult.

Stewart and Cohen (1996) suggest that although we may not feel that we understand what is going on, the system – that is, you together with the rules you are putting into operation, does. To say that you "internalize" the rules, perhaps by memorizing them, assumes that your brain has a virtually unlimited capacity. "The only rule that you could internalize would be some kind of meta-rule; a rule about rules". If we apply this to the task of devising a Curriculum in Mathematics (or indeed, any other subject) we see that the emphasis should be on an overview of the subject with ideas such as problem-solving, and a cultural and historical approach having a preeminent position, rather than the requirement to reproduce discrete pieces of information. This is a task which should concern us as we approach the 21^{st} century.

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