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On Teaching Strategies In Mathematics

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Abstract

In this paper we focus on the importance of strategies in the mathematics curriculum and in teacher training.

We discuss examples of control strategies of different types, namely:

- select the best technique: decision-making in a restricted and straightforward domain, like solving indefinite integrals or deciding on the convergence of series;
- step by step control: a strategy for graphing functions;
- general control on the solution process: a strategy for heuristic problem solving.

The examples are taken either in first year university education or in higher secondary school. The development of similar strategies for lower levels of mathematics education should take an important place in the training of future mathematics teachers.

Introduction

Curriculum standards for school mathematics as well as professional standards for teaching mathematics have been discussed at large by the American National Council of Teachers of Mathematics (1989, 1991). The two volumes establish a broad framework to guide a reform in school mathematics. It is not my intention to talk about these standards, perhaps other people will. Nevertheless I would like to recall the central theme in these curriculum standards, namely the development of mathematical literacy. This term is defined as an individual's ability to explore, to conjecture and to reason logically; to use a variety of mathematical methods efficiently to solve problems; to communicate about and through mathematics; to connect ideas within mathematics and between mathematics and other disciplines.

In order to attain these goals it must be recognized that mathematics is more than a collection of concepts and skills to be mastered. Its teaching must also include methods of investigation and reasoning, strategies that help students in solving problems, means of communication and notions of context.

Curricula should reflect these goals, and emphasize major shifts in the actual mathematics teaching (NCTM, 1991):

1. away from merely memorizing, toward mathematical reasoning;
2. away form mechanistic answer-finding, toward conjecturing, investigation and problem solving;
3. away from isolated concepts and procedures, toward strategies and connections;
4. away from the teacher as sole authority for right answers, toward logic, mathematical evidence and control strategies as verification.

Realizing these shifts in class is only possible if we train our mathematics teachers in this direction, but therefore we really have to improve the teaching of mathematics at all levels. The above described shifts should not only happen in primary and secondary school mathematics, but also in mathematics teaching at tertiary level and in teacher training. In fact, beginning teachers are very much influenced by what they have experienced themselves as students. They often try to teach in the way they have been taught, imitating those they considered to be great teachers.

Since changes have to be realized throughout the complete education period, training good teachers is really a long term project. In order to succeed in the realization of meaningful changes, we must keep in mind that the quantity of mathematics taught is not essential. It is its quality, in terms of the goals formulated above, which is of utmost importance.

In this paper I want to focus on the teaching of strategies. In view of the goals stated above strategies are important, and teaching them can be done within the actual framework, taking up only limited extra time.

A reason why a systematic approach to problems is not taught in general at the university is that most faculty assume that students are able to, and will develop their own strategies, as they succeeded in doing themselves. But not all our students are future university professors and many of them are just unable to develop strategies without help. The same is of course true for secondary school. A second reason why strategies are lacking in secondary school teaching is that textbooks do not explain strategies. And many teachers just cover in a linear way the material offered by their textbook. An important point in teacher training is therefore to make future teachers aware of the fact that they should not explain or teach mathematics the way it is written down.

In the next paragraphs I will discuss examples of control strategies of different types, in order to attain different goals:

1. select the best technique: decision-making in a restricted and straightforward domain, like solving indefinite integrals, deciding on the convergence of series;
2. step by step control: a strategy for graphing functions;
3. general control on the solution process: a strategy for heuristic problem solving.

The examples are taken from either first year university education or higher secondary school. Similar strategies can be developed for lower levels of mathematics education.

A Strategy for Solving Integrals

Indefinite integration should cause students little difficulty because the prerequisite algebraic and differentiation skills are mechanical, and the techniques of integration such as substitution, integration by parts and partial fractions are straightforward algorithms. And students do well as long as they know which technique they are supposed to use, i.e. at the end of a particular class or a particular section in a textbook.

But my own experience and more elaborate studies (Schoenfeld, 1978, 1985) show that problems arise as soon as students have to select the integration technique themselves (at the end of a chapter or at an exam): in general students are very inefficient in their selection, using time-consuming techniques without even checking whether simpler techniques are available. This contradicts the general assumption that, with much practice and the help of comments of teachers, students are able to develop efficient selection mechanisms themselves.

In order to overcome this problem, Alan Schoenfeld wrote a module (1977) which provides the students directly with a general procedure for approaching and solving integrals. The procedure is based on the observation of "experts" working on integrals. Their strategy selection shows two patterns:

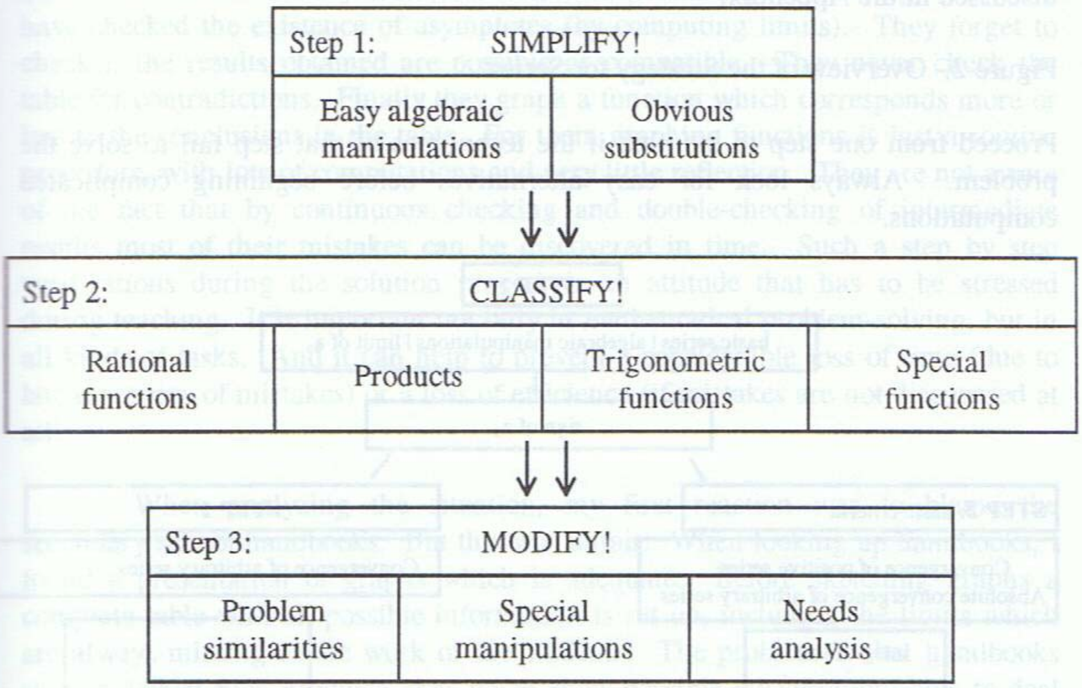
1. a domain-specific one: the form of the integrand, which often points to an appropriate technique of integration;
2. an efficiency component: never do anything difficult or time-consuming until you have checked simpler and faster alternatives.

The strategy is written as a set of instructional materials and outlined in two tables (an overview and a more detailed table). The strategy consists of three steps as shown in Figure 1, taken from Schoenfeld (1985, p. 103).

Alan Schoenfeld gave me permission to translate his module into Dutch and to expand and adapt it to the needs of my students (Grandsard, 1986). Since Belgian students study integration techniques in secondary school, the translated module is used, with success, as a "self-study, review and improvement package" for first year university students.

Figure 1. A Broad Overview of the Integration Strategy

Proceed from one step to the next when the techniques of that step fail to solve the problem. Always look for easy alternatives before beginning complicated calculations. If you succeed in transforming the problem to something easier, begin again at Step 1.



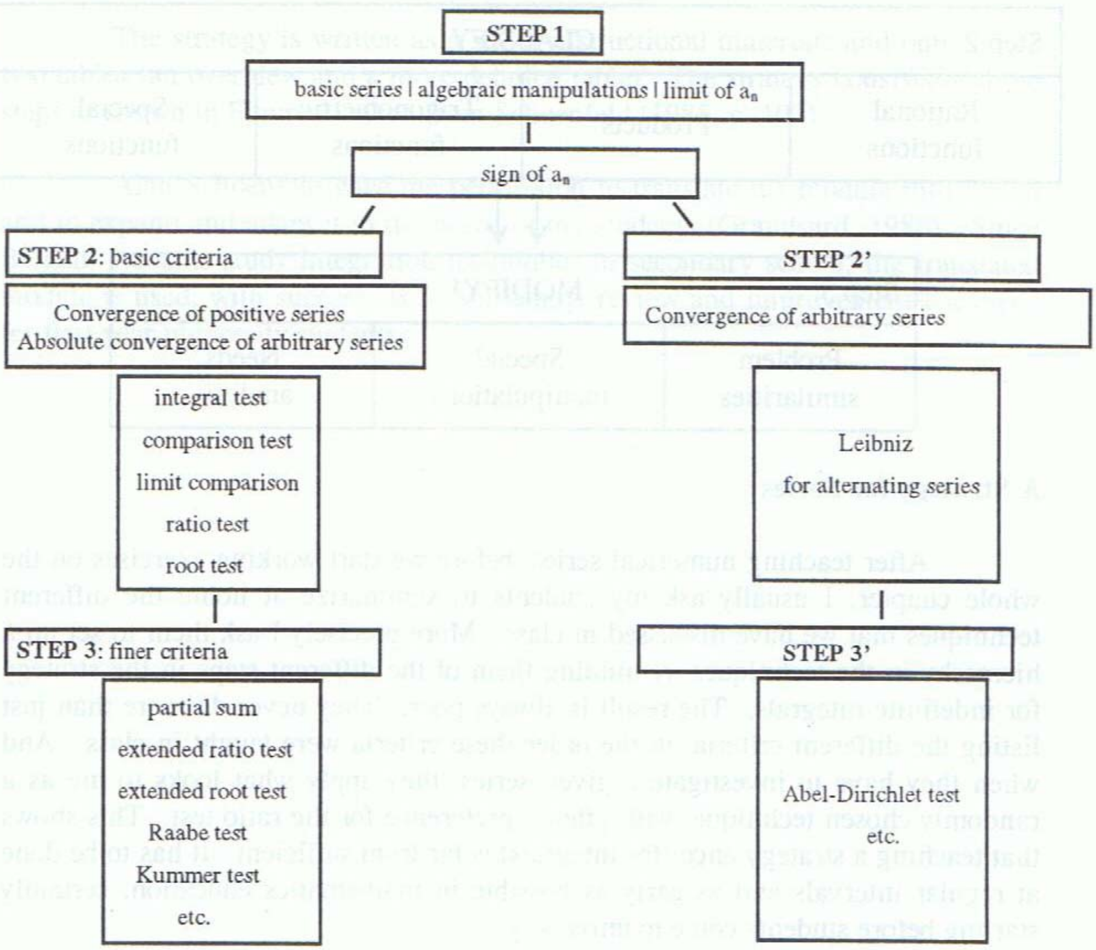
A Strategy for Series

After teaching numerical series, before we start working exercises on the whole chapter, I usually ask my students to summarize at home the different techniques that we have discussed in class. More precisely I ask them to set up a hierarchy in the techniques, reminding them of the different steps in the strategy for indefinite integrals. The result is always poor. They never do more than just listing the different criteria, in the order these criteria were taught in class. And when they have to investigate a given series, they apply what looks to me as a randomly chosen technique, with often a preference for the ratio test. This shows that teaching a strategy once (for integrals) is far from sufficient. It has to be done at regular intervals and as early as possible in mathematics education, certainly starting before students come to university.

Series are of course less straightforward than indefinite integrals, conceptually (limit of the sequence of partial sums) as well as with respect to the skills needed to apply the different criteria (limits, estimates). In fact deciding on the convergence or divergence of a given series can require great ingenuity. Nevertheless for exercises at student level one can easily set up a strategy comparable to the one for integrals, as shown in Figure 2. This strategy is briefly discussed in the Appendix.

Figure 2. Overview of the Strategy for Series

Proceed from one step to the next if the techniques of that step fail to solve the problem. Always look for easy alternatives before beginning complicated computations.



A Control Strategy for Graphing Functions

In my country, calculus is on the secondary school program, but it is reviewed in many first year mathematics courses at the university. The behaviour of most of our entering university students when graphing functions is the following. They compute f' and f", and then set up a table with f', f" and conclusions about f. They omit to indicate limits in the table, although they may have checked the existence of asymptotes (by computing limits). They forget to check if the results obtained are possible or compatible. They never check the table for contradictions. Finally they graph a function which corresponds more or less to the conclusions in the table. For them graphing functions is just a routine procedure, with lots of computations and very little reflection. They are not aware of the fact that by continuous checking and double-checking of intermediate results most of their mistakes can be discovered in time. Such a step by step verifications during the solution process is an attitude that has to be stressed during teaching. It is important not only in mathematical problem-solving, but in all kinds of tasks. And it can help to prevent a considerable loss of time (due to late discovery of mistakes) or a loss of efficiency (if mistakes are not discovered at all).

When analyzing the situation, my first reaction was to blame the secondary school handbooks. But this was unfair. When looking up handbooks, I found a presentation of graphs which is adequate. Before sketching graphs a complete table with all possible information is set up, including the limits which are always missing in the work of the students. The problem is that handbooks show polished final products, they never show possible mistakes and how to deal with them, and they rarely spell out a complete strategy. This is the task of the teacher: one does not explain mathematics in the polished form it is written down. But it seems that our teacher training is falling short of these objectives, and that teachers stick too much to their written sources.

At the Sixth South East Asian Conference on Mathematics Education in Surabaya, I presented a paper on teaching a control strategy for graphing functions (Grandsard, 1993). This strategy can easily be incorporated in a calculus course (higher secondary level or first year university). It gives us an almost perfect example of the way in which mathematics must be taught: as something that fits together meaningfully. We should never teach to look at f', f" and limits separately, but at the whole set of information about f that can be derived from them.

The strategy for sketching graphs looks similar to what is done in most calculus books. The differences are that limits are computed before the derivatives, and that at each step of the procedure the students are asked to mark the results obtained in a big table, showing x, f, f' and f'' and to check the feasibility of these results. Moreover at each step in the procedure there are hints about graphing, suggesting that the graph should be built up progressively, using the results obtained step by step. This results in working on three sheets of paper simultaneously: one for the computations, one for the table and one for the graph. For a description of the strategy we refer to Grandsard (1993, 1995).

My grading of graphs reflects the emphasis I put on reflection and control during the whole procedure. Computational errors are sanctioned much less than lack of control. A graph that does not correspond at all to the results obtained in the table is not worth much.

A Control Strategy for Heuristic Problem-solving

Mathematical problem-solving via heuristic techniques, as described by Polya (1971), is a much more difficult task than the previous ones we discussed (integration, series, graphing functions). Not only is the number of heuristics the student has to manage larger by an order of magnitude than the number of techniques in the previous examples, but the heuristics are far more complex to implement. In fact they are not algorithmic, and even if they seem appropriate, they give no guarantee of success.

Alan Schoenfeld (1980, 1985) describes a managerial strategy for general problem-solving via heuristic techniques. This strategy was the foundation for his courses in problem-solving, as well as for ours (Grandsard, 1989).

The strategy is again based on detailed observations of good problem solvers working unfamiliar problems. It shows the same two patterns as the strategies for solving integrals and series:

1. it points to particular domain-specific techniques wherever possible;
2. for efficiency, it insists on trying easier techniques before more difficult ones.

The strategy is given in flowchart form. It is however not to be considered as a "program" that the students have to implement mechanically, but as a guide to use in case the student does not know what to do next. It suggests which heuristics may be appropriate and in what order. If the student reduces the original problem to a more manageable one, then the process begins again with the new problem.

The strategy shows five major stages in the problem-solving process: analysis, design, exploration, implementation and verification, as shown in Figure 3 (Schoenfeld, 1985, p. 110).

End of those stages is explained in detail and examples of appropriate heuristics for each stages are given.

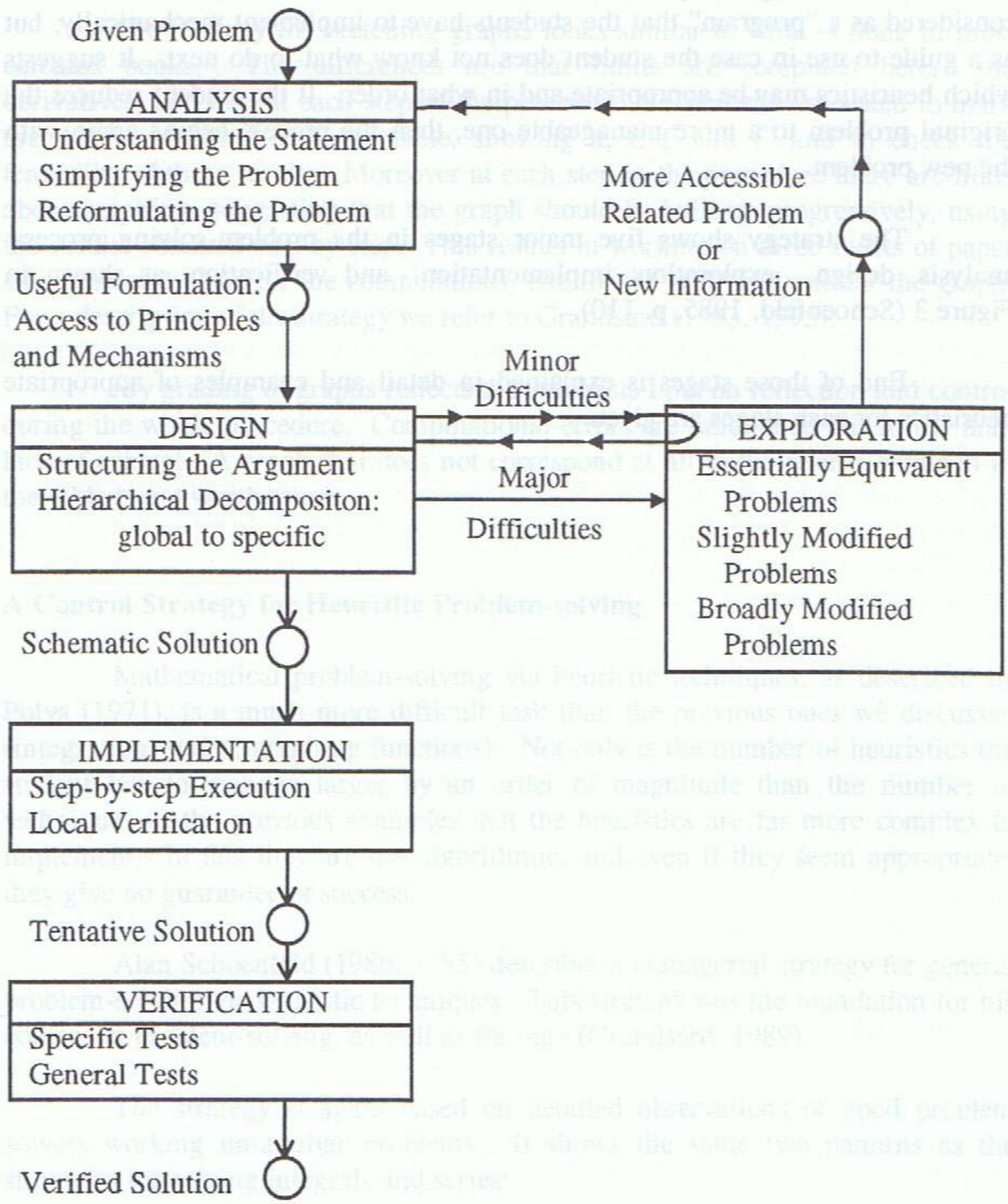

Given Problem
ANALYSIS
Understanding the Statement
Simplifying the Problem
Reformulating the Problem
Useful Formulation:
Access to Principles
and Mechanisms
More Accessible
Related Problem
or
New Information
Minor
Difficulties
DESIGN
Structuring the Argument
Hierarchical Decompositon:
global to specific
Major
Difficulties
EXPLORATION
Essentially Equivalent
Problems
Slightly Modified
Problems
Broadly Modified
Problems
Schematic Solution
IMPLEMENTATION
Step-by-step Execution
Local Verification
Tentative Solution
VERIFICATION
Specific Tests
General Tests
Verified Solution

Figure 3

Conclusion

In this paper, we considered four examples of control strategies. The first two strategies help to select the best technique in two straightforward domains (integration and convergence of series). The third one is a strategy for continuous control during the process of graphing a function. The last one is a control strategy in the complex domain of problem-solving by heuristic strategies.

Most students are unable to develop even the simplest of these strategies on their own. On the other hand, teaching this kind of techniques helps the students in improving their performance in mathematics. It is therefore clear that strategies must be included in mathematics teaching, from an early stage on. And one can hope that teaching many strategies starting with very simple ones, will help students to develop strategies of their own.

In order to realize this I therefore suggest that

1. the discussion of strategies should take an important place in teacher preparation;
2. future teachers should do projects on developing strategies at different levels;
3. strategies should be discussed at professional meetings of teachers of mathematics, with examples of strategies at different levels;
4. strategies should be discussed in mathematics courses at university level and in mathematics courses for future teachers;
5. strategies should be part of the secondary school curriculum.

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Appendix

Discussion of the Strategy for Series

Step 1 consists of **recognizing series** or in **applying basic tests** for divergence or convergence.

Basic series. The students must be told that they have to know the behavior of some basic series that were discussed in class (geometric series, harmonic series, harmonic series of order p, alternating harmonic series), and recognize them as such.

Examples: $\Sigma(-1)^n\left(\frac{3}{4}\right)^n$, $\Sigma\frac{1}{\sqrt{n}}$, $\Sigma\frac{1}{n^{3/2}}$

Algebraic manipulations. Series that are a sum or a multiple of basic series should be recognized and treated as such.

Examples: $\sum \frac{2}{n}$, $\sum \frac{1}{2n^2}$, $\sum \frac{n^2+2^n}{2^n n^2}$, $\sum \left(\frac{1}{n}-\frac{1}{2^n}\right)$

Limit of the general term. This very useful criterion for divergence is often neglected by the students although it is an easy way to show the divergence of many series.

Examples: $\sum (-1)^n$, $\sum \log n$, $\sum \sqrt{n}$, $\sum e^n$, $\sum \frac{e^n}{n^3}$, $\sum \frac{n^2}{\log n}$

The problem with this criterion is that students often misuse it in the sense that they think that a general term converging to zero implies convergence. Of course this approach should be abandoned if the student is unable to find the limit or if finding the limit of is far more complicated than using one of the criteria in the next steps.

If step 1 does not provide an answer, we proceed to step 2 for series with positive terms or for the absolute convergence of series with arbitrary terms, or to step 2' for the convergence of alternating series.

Step 2 : basic criteria for positive series. This second step, if necessary, consists of choosing and applying one of the basic criteria for convergence or divergence for positive series. If the series has arbitrary terms, this step provides us eventually with absolute convergence of the series, and hence convergence.

The choice of the test is *based on the form of the general term* of the series. We include the following criteria.

Comparison test: Can only be used if the student knows series to compare with, and is able to do estimations. The test is not so easy to apply for the students and asks for a lot of exercises. However it is very powerful. Rational terms are most easily handled by this test.

Examples: $\sum \frac{1}{n\,2^n}$ converges, since $\frac{1}{n\,2^n} \le \frac{1}{2^n}$,

$\sum \frac{n+1}{3n^2+5n+2}$ diverges, since $\frac{n+1}{3n^2+5n+2} \ge \frac{1}{4n}$ for $n \ge 2$,

$\sum \frac{\log n}{n}$ diverges, since $\frac{\log n}{n} > \frac{1}{n}$ for $n \ge 3$,

$\sum \frac{\log n}{n^2}$ converges, since $\frac{\log n}{n^2} = \frac{\log n}{\sqrt{n}} \bullet \frac{1}{n^{3/2}} \le \frac{1}{n^{3/2}}$

for n sufficiently large, as $\lim_{n\to\infty} \frac{\log n}{n^p} = 0$ for $p > 0$.

Limit comparison test: Like for the previous test this criterion can only be used if the student is able to find a series of known behavior to compare with. Very useful in case of rational terms.

Integral test: To be used if the terms have the configuration of a derivative and if the corresponding integral is easy to compute.

Examples: $\sum \frac{1}{n^{5/2}}$ and $\sum \frac{1}{\sqrt{n}}$ if these are not recognized as harmonic series of order $5/2$ and $1/2$,

$\sum \frac{1}{n \log n}$ diverges since $\int_1^\infty \frac{dx}{x \log x}$ diverges.

Ratio test: Very popular among students, although using the comparison test is in general faster and more efficient. Useful if the general term contains factorials or a combination of powers and factorials.

Examples: $\sum \frac{2^n}{n!}$ is convergent and $\sum \frac{n^n}{n!}$ is divergent.

Root test: Useful if the general term is a n^{th} power.

Examples: $\sum \frac{1}{(\log n)^n}$ and $\sum \left[\frac{n}{1+n^2}\right]^n$ converge by the root test.

In case of a series with arbitrary terms which is not absolutely convergent by step 2, we turn to

Step 2', if the series is alternating.

Examples: Leibniz's test assures the convergence of

$$\sum \frac{(-1)^n}{\sqrt{n}},\ \sum \frac{(-1)^n}{\log n},\ \sum (-1)^n \frac{\log n}{n} \text{ and } \sum (-1)^n \sin \frac{1}{n},$$

series which are not absolutely convergent.

Steps 1, 2 and 2' contain the basic techniques for deciding on the behavior of series of real numbers, and will do for most exercises in a calculus course. For more advanced courses we add

Step 3 : finer criteria for positive series, such as

- compute or estimate partial sums;
- extended ratio test, using upper and lower limits;
- extended root test, using upper limits;
- Raabe's test;
- Kummer's test, etc.

Step 3' : finer criteria for arbitrary series, such as Dirichlet's test.