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Compass Constructions: A Vehicle For Promoting Relational Understanding And Higher Order Thinking Skills

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Abstract

Compass constructions refers to standard procedures for constructing geometrical entities such as angle bisectors using compasses and straight edge only. These constructions are usually taught as a series of fixed steps without any justification of why the methods work. This article outlines a scheme of activities which provides links between the compass constructions required in the Singapore Mathematics Syllabus and other topics in the syllabus with extensions to problem solving.

Introduction

The secondary one mathematics syllabus in Singapore includes geometrical constructions as one topic of geometry. This topic stresses the mastery of skills and procedural steps of using geometrical instruments to construct geometric entities such as parallel and perpendicular lines, angle bisectors, perpendicular bisectors of line segments and simple geometrical figures with specified data. Since the objectives are skill-oriented, there is a natural inclination to reduce each construction to a progression of fixed steps without any explanations as to why such steps work. Furthermore, for straight-edge and compasses constructions, students find it difficult to understand why such restrictions are necessary and why protractors and set-squares are not allowed.

The issue of reducing geometry to a list of unrelated results and procedures was discussed in the section of NCTM's 1987 Yearbook entitled "Learning and Teaching Geometry, K - 12" entitled "Perspectives" and in the Geometry chapter of the Proceedings of ICME IV (1983). There was a general call for improvements in relating geometry to the real world and to problem solving as well as in communicating to learners the underlying structure of geometry as the mathematical study of space.

For learning Mathematics, Skemp (1976) makes the distinction between Instrumental and Relational understanding. Where in the former, the learner understands how to use procedures without the reasons why such procedures work, the latter form of understanding is more desirable as the learner understands the links or relationships between concepts and procedures. Making links between various topics in Mathematics thus promotes relational understanding and also higher order thinking such as justifying through a series of deductions and the organising of concepts into structures.

It is the objective of this article to explore how the topic *Compass Constructions* can be taught with appropriate links and with the underlying philosophy of a problem solving approach.

Relationship between Compass Constructions and Properties of Geometrical Figures

Research both overseas and locally has shown that secondary school students are generally between van Hiele Levels 2 and 3, i.e. they are able to discuss properties of geometrical figures and perhaps relate figures through their properties but are unable to perform a series of deductions to understand or produce proofs. For students at Level 2, if teachers are not to leave construction procedures as merely a series of steps without some justification of why they work, teachers would therefore have to find some means of providing reasons other than formal proofs.

Pegg (1987) recommends that compass constructions can be easily linked to the properties of a rhombus. An alternative approach is to link the constructions to properties of a kite. The properties of these two quadrilaterals, which justify the construction of angle bisectors and perpendicular bisectors of line segments, in turn depend upon the properties of isosceles triangles.

The following scheme of activities suggests a structured progression leading from properties of isosceles triangles to the relevant properties of rhombuses and kites to the construction procedures. The strategy is an elaboration and modification of ideas suggested in Pegg (1987). In the light of van Hiele phases of learning (see Crowley, 1987), the activities for the establishment of properties should be carried out with an investigative hands-on approach and these activities are extremely suitable for group work as students should be encouraged to guess, discuss, evaluate and communicate their ideas. Appropriate worksheets can be designed for guidance and for recording findings. After the group work, the teacher should lead the students to use accurate geometrical language in describing the properties and recording them in the worksheets. The scheme is structured according to Table 1.

Table 1: Progression of the Topics

Stage	Topic	Activities
1	Properties of Isosceles Triangles	Examining properties through paper folding or other means. Establishing symmetry properties.
2	Properties of Quadrilaterals	Establishing the properties of Parallelograms, Rhombuses and Kites. (Justification of properties through those of isosceles triangles).
3	Construction Procedures	Establishing construction procedures with justification through properties of Rhombuses and Kites.

Stage 1: Establishing the properties of an isosceles triangle

The base angles property of isosceles triangles is taught at primary levels but other symmetry properties should be explored and established here. First, students can be asked to construct different isosceles triangles on paper. To promote creativity and thinking, groups can brainstorm for different ways to construct these triangles using compasses and straight edge only. The teacher should also ask students to construct isosceles triangles on a given base as this activity is used later on for perpendicular bisectors. They then cut out the triangles and fold to establish that there is a line of symmetry which divides the triangle into two congruent right-angled triangles. The word *congruency* need not be used if the concept has not been established at this point in the scheme of work.

Stage 2: Establishing the properties of rhombuses and kites

The properties of a rhombus that the diagonals (i) bisect each other, (ii) are perpendicular and (iii) bisect the internal angles of the rhombus are in fact part of the Secondary 1 syllabus and it is recommended that they be established next. Similar properties of a kite are not part of the syllabus but can be included as an extension.

As for the case of isosceles triangles, students can be asked to construct rhombuses using only a pair of compasses and a straight edge. Using two intersecting circles (or arcs) with equal radii and joining centres to intersection points make use of the defining property of a rhombus. The students can be asked to construct several different rhombuses with different angles and sides. These are then cut out and folded along the diagonals to establish the above properties. Other properties not related to diagonals such as opposite angles being equal can also be established.

The justification for these properties can be discussed as a class. The links between these properties and those of isosceles triangles can be made as the teacher leads the students to view a rhombus as the joining together of two congruent isosceles triangles. (See Figure 1). The extension to kites (to be considered as the joining of two non-congruent isosceles triangles with equal bases as in Figure 2) can then be discussed. Questions should lead students to compare and contrast the kite with the rhombus and see how the differences and similarities lead to certain common properties and not other properties.

Figure 1

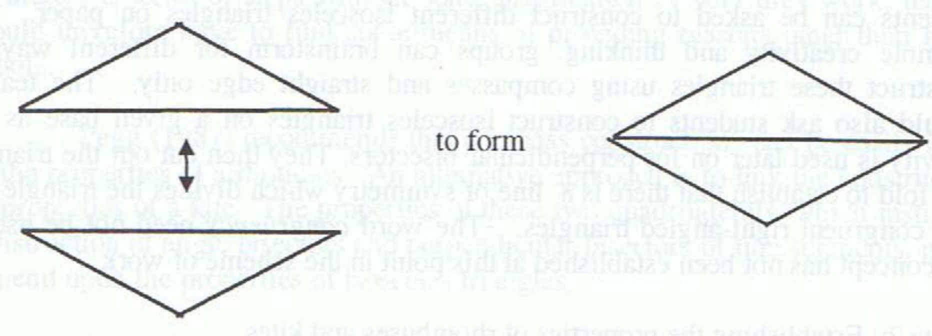
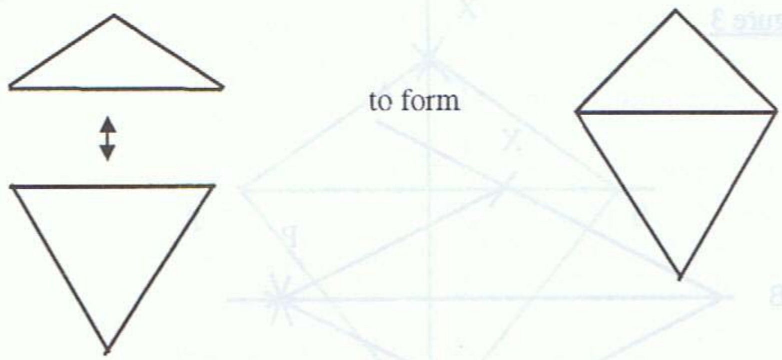


Figure 2



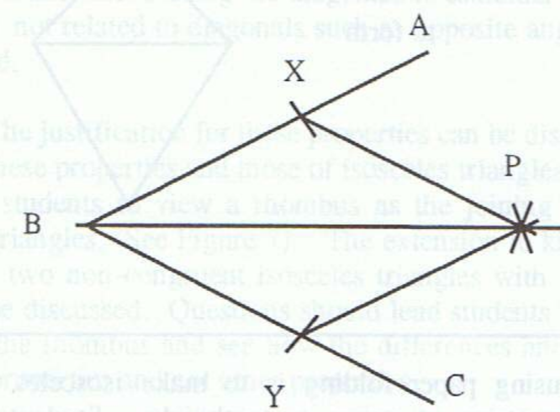
An alternative to using paper folding is to make isosceles triangles, rhombuses or kites using geostrips or straws or toothpicks. Students can use protractors and rulers to measure the relevant angles and sides. With the availability of appropriate software such as the Geometer's Sketchpad, the teacher can get pupils to construct these figures and investigate and establish these properties on the computer, again with appropriately designed worksheets for recording purposes.

Although not linked with constructions, a similar approach can be used especially with the Geometer's Sketchpad to construct parallelograms and examine properties such as diagonals bisecting each other, opposite angles being equal, etc. By comparing the two sets of properties, the relationship between rhombuses and parallelograms, the former being a special case of the latter, can also be discussed.

Stage 3: Constructions of angle bisectors and perpendicular bisectors using the properties of a rhombus or isosceles triangles

To teach the angle bisection procedure, the teacher can get the class to recall the properties of a rhombus or a kite as established in Stage 1. The given angle ∠ABC to bisect is then considered as an internal angle of a rhombus or the vertex angle of an isosceles triangle. It remains to construct a rhombus or a kite and draw in the diagonal which is then the angle bisector. The students can discuss how a rhombus or kite can be constructed from the given angle using the two arms of the angle as two adjacent sides as in Figure 3. Whether a kite or rhombus is formed depends on whether the equal arcs BX and BY are equal in radii to the equal arcs XP and YP. BP is the diagonal which bisects ∠ B.

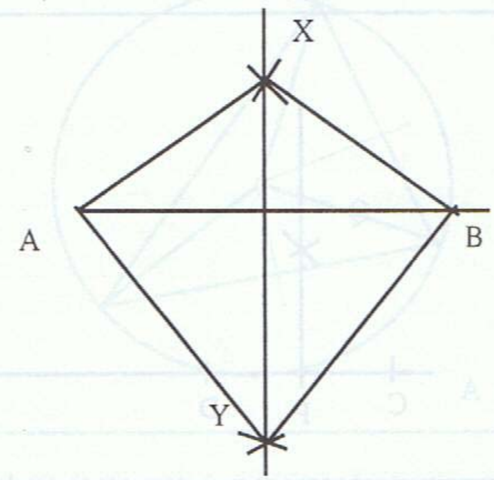
Figure 3



To construct the perpendicular bisector of a line segment AB, recall first that the diagonals of a rhombus bisect each other at right angles. The line of symmetry of a kite also bisects the other diagonal at right angles. The idea is then to construct rhombus or kite with AB as the appropriate diagonal. The teacher can get the groups to discuss how to construct such a rhombus or kite. The steps are similar to that of constructing isosceles triangles on a given base which students have carried out in Stage 1. For a kite, students should arrive at a conclusion that they are constructing two different isosceles triangles on a common base. (See triangles AXB and AYB in Figure 4.)

From the construction of the perpendicular bisector of any given line segment AB, students can be led to see that any point X on the perpendicular bisector is a possible vertex of an isosceles triangle AXB where AX = XB. This property is important in establishing the locus of a point P which is equidistant from two fixed points AB.

Figure 4



It can be seen that in the above scheme of activities, students are encouraged to make links between established properties and procedural skills. Links to future topics such as loci are also crucial in building up the structure of geometrical ideas. Such an approach promotes relational understanding as students can see the reasons behind the procedural steps and the relationship between topics and concepts.

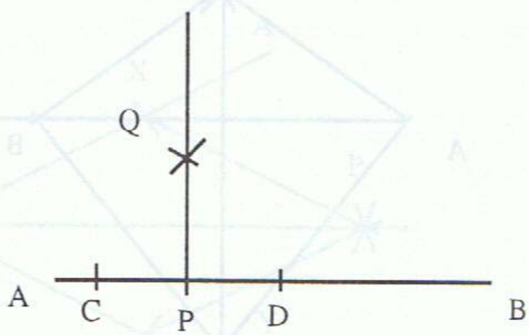
Problem-Solving Extensions

Compass constructions in general provide a rich environment for problem solving as the restriction to using compasses and straight edge only "forces" the solver to exercise higher order thinking skills such as analysis, evaluation, hypothesising, organising besides bringing to mind and applying the various geometrical properties and results he has learnt.

As a simple example, students can be asked to construct a line perpendicular to a given line AB through a given point P on the line. The steps are simple: (i) make equal cuts C and D on line AB on either side of P, and (ii) make equal arcs from C and D to intersect at Q and finally join PQ. See Figure 5.

However, instead of telling students the steps, students are asked to solve the problem.

Figure 5

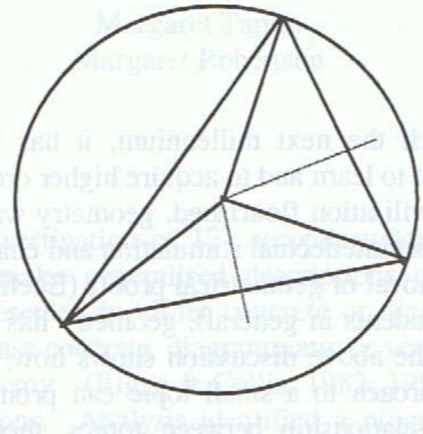


Two possible solutions students may think of involve considering this required line as the perpendicular bisector of a segment CD with P as midpoint or considering it as the angle bisector of the 180° angle ∠APB. Both views of the required line result in the same steps given above.

Another example is to provide students with a given circle and ask them to locate its centre with straight edge and compasses only. The solution hinges on the property that the centre lies on the perpendicular bisector of any chord which is again directly related to the fact that any chord with the two radii at its extreme points form an isosceles triangle. The solution is thus to draw any two chords and construct their perpendicular bisectors, the intersection point of which will be the centre of the circle.

A more difficult variation of the previous example is the problem of constructing the circumcircle of a triangle. The problem of drawing the circle first requires an understanding of the meaning of the term *circumcircle* and its relation to the given triangle. An analysis of what is required should lead to the realisation that to draw a circle, the location of the centre is necessary. Further analysis will lead to the conclusion that this centre is equidistant from the three vertices and thus lies on the perpendicular bisectors of any two of the three sides as seen in Figure 6.

Figure 6



Another problem is to require the students to construct an isosceles triangle on a given line segment as base, given that the base angles are to be 30°. The solver needs to analyse the problem and organise his steps. He will need to realise that he has to locate the vertex and to recall from the previous scheme of activities that the vertex lies on the perpendicular bisector of that line segment. He will also need to deduce that he can bisect a 60° angle to obtain a 30° angle and that he knows the steps for constructing a 60° angle using compasses only.

The reader is referred to Posamentier & Sheridan (1982) for more examples of geometrical constructions using straight edge and compasses although these may not be related to angle bisection and perpendicular bisectors of line segments.

The teacher may find that these extensions of constructions to problem solving are too difficult for his students who have just learnt the procedures of angle bisection and constructing perpendicular bisector of straight line segments. However, it is felt that the more complicated constructions which involve higher order thinking skills provide the motivation and application of the procedural skills of simple bisections. Naturally the teacher should provide a progression from simpler problems with fewer steps and less analysis to harder problems which require multiple steps, retrieval and organisation of a few previously learnt geometrical results, etc. The teacher can alternatively select only the easier problems suitable for his students. All the usual guidelines for encouraging problem solving such as providing a non-threatening atmosphere, providing guidance but not solutions, giving sufficient time, encouraging discussion and so on should certainly apply here.

Conclusion

As we approach the next millennium, it has been advocated that our students should be taught to learn and to acquire higher order thinking skills. In the days when the Greek civilisation flourished, geometry was highly regarded as the topic which provided most intellectual stimulation and challenges. However, in the present syllabus, the removal of geometrical proofs (Euclidean proofs were thought to be too difficult for students in general), geometry has been reduced to isolated and unrelated topics. The above discussion shows how, without using Euclidean proofs, a change in approach to a small topic can promote higher order mental activities and provide relationship between topics, thereby contributing to the building up of a cohesive geometrical structure. It is hoped that such a philosophy is also carried by teachers into other mathematical fields.

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