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An Analysis Of Problem Types

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Abstract

Students need to be aware of the different types of patterns which are produced from the attempt to solve problems of different types. An analysis of problems is suggested, depending on the sequences which are produced in the solution of the problem. A similarity with the analysis of Order and Chaos by Davis and Hersch (1981) is considered, and an example of a problem where more than one solution is possible, is discussed.

We are frequently told that Algebra develops out of the search for patterns. Problems on sequences which it is claimed will give rise to pupils identifying mathematical patterns can be classified in various ways. In the teachers guide to "Crime and Proof" (1993), I attempted one such 4-fold classification. This I find has much in common with the analysis of Davis and Hersch (1981) who in a chapter entitled "Pattern, order and chaos" describe the aesthetic delight one can get from seeing order come out of chaos. I now attempt to provide a link between the two classifications.

Order out of Order

The college band marching in rank and file is such an example, and in mathematics we have the sort of patterns which show, for example, that the sum of the cubes of the first n natural numbers is the square of the sum of the first n natural numbers. Other examples would be the regularity in 5/37 = .135135135... This corresponds to a type of problem where the sequence, which is obtained in attempting to solve the problem, appears to have an easily recognizable pattern which can be expressed in terms of a formula for u_n, the n^{th} term. This formula may be explicit, as in the case of the sequence

$$1, 5, 14, 30, 55, \ldots \text{ where } u_n = n(n+1)(2n+1)/6$$

or implicit as in the case of the sequence

1, 2, 3, 5, 8, ... where $u_n = u_{n-1} + u_{n-2}$, ($u_1 = 1$, $u_2 = 2$)

We have however to justify why the formula applies to the problem which gives rise to the sequence; this justification may constitute the proof. In the first case the problem might be

*"How many squares are there on an n*n chessboard?"*

and in the second case,

*"In how many ways can a 2*n pavement be paved with a sufficient supply of 2*1 tiles?"*

Of course, an implicit formula (such as that for the above Fibonacci sequence) may be made explicit, but this may be difficult. As McGregor and Stacey (1995) have pointed out, in order to make effective use of the "pattern" approach to teaching early notions of algebra, we need to have a good supply of sequences where the underlying pattern is not of an "implicit" or recurrence nature. Students should also be aware that there are sequences which need different formulas depending on the parity of n; the first example has a single formula, but the equivalent problem for a triangular board needs two formulas, n odd/even. The problem

*"How many edges of unit squares must be removed from an n*n grid so that no squares remain?"*

needs two formulas, but in the equivalent triangular case, only one formula is needed.

Chaos out of Order

This happens too often! As mathematical examples, we have the expansions of $\sqrt{2}$ or π. This corresponds to a type of problem where a sequence appears to have an easily expressed formula for u_n. But we find that when we try to confirm the formula which works for the first few cases which we have considered, it fails to work for these later examples. We may either modify our conjecture or reject it completely, when we will have to go in search of another conjecture.

We should of course relate a sequence to the problem which gives rise to the sequence. If we are considering a problem frequently described as the quest for the "Lost region" - a circle has marked on its circumference n points which are joined up in pairs. What is the maximum number of non-overlapping regions so formed? We can see that an early conjecture which we might make - number of regions = 2^{n-1} does not relate to the problem. This is an example where "pattern spotting" really does not help us to solve the problem which gave rise to the sequence.

Duffin and Simpson (1994) analyze how we deal with experience which conflicts, or appears to conflict with, previously held views. Three main types of experiences are identified:

A **natural** experience is one which is easily fitted into the learner's existing mental structures. In the case of the "lost region", n = 5 is such an experience; it is easily assimilated into our conjecture for the cases n = 1, 2, 3, 4 and the conjecture is not changed. For a learner with a deep understanding of indices, the idea that $a^0 = 1$ is an example of a "natural" experience. These natural experiences have a tendency to reinforce current thinking; the more confirming instances of a conjecture you can get, the more convinced you are of its truth.

A **conflicting** experience is one which cannot be accounted for by the learners' existing mental structures. The conjecture fails to account for the case n = 6 in the "Lost region" problem. We then realise that there is a flaw in our thinking. In order to explain why $a^0 = 1$, we have to have a new explanation, possibly in terms of the division rules for indices such as $a^n / a^m = a^{n-m}$, and so $a^3 / a^3 = a^0$, and any quantity divided by itself gives 1. A number of different responses are possible in this case. The old way of thinking (the conjecture) can be destroyed. Or, a conjecture can be seen as valid only in a limited series of cases, or subject to certain restrictions, and there are other areas of experience where it does not apply. The "Inquiry training" programme developed by Suchman (1964) has as its basic idea that if the learner's environment is "responsive" to his actions, he will "invent" the cognitive structures necessary to integrate the information that is made available. The teacher is of course an important part of the learner's environment, and potentially a very important part because he is aware of many facts that are not known to the learner, and cannot be conveniently arranged for his immediate observation. The problem is to make use of the teacher's experience without interfering with the learner's creative activity. His solution is to use the teacher primarily as a source of facts rather than of structures relating to those facts. Since the teacher will almost certainly be predisposed to deal with the latter, he is forbidden to do so during an inquiry session; he may respond to questions which have been framed in such a way that they may be answered by a simple yes or no. This is rather like a game of 20 questions, except that the number of questions is not limited. An inquiry session may begin with a short film designed as a **"discrepant"** event - the structure of the event which is depicted in the film is discrepant with the cognitive structure that the student has available for its interpretation. Assimilation of the input requires accommodation of the cognitive structure. The example given by Suchman is taken from science, but there are many examples from mathematics which could be used. The discrepancy of course must not be too great, or else the learner will give up.

An **alien** experience is one where the learner cannot find a point of contact with an existing mental structure. For some pupils who are fixed in their view that indices must involve multiplication, the idea of a^0 might be such an experience. Because there is no structure which is contradicted by the new alien experience, the learner can happily move on as though nothing had happened; even though to an observer there may be obvious contradictions present.

Chaos out of Chaos

There is a definite place in the curriculum for students encountering a type of problems, which while they may be easily stated, may be difficult or impossible to solve. A popular example arises from the "polyomino" problem which is as follows:

"In how many ways can n unit squares be joined together edge to edge ?"

And the sequence is

1, 1, 2, 5, 12, 35, ...

"Polyominoes" were probably invented by S. W. Golomb in 1953.

If equilateral triangles are used to make "Polyiamonds", the resulting sequence is

1, 1, 1, 3, 4, 12, 24, ...

"Polyiamonds" were probably first discussed by T. H. O'Beirne in 1961.

If regular hexagons are used to make "Polyhexes", the resulting sequence is

1, 1, 3, 7, 22, ...

The pattern in these sequences is also unsolved.

We may refer also to the study of **cellular automata** which was developed by Ulam. Grids used can consist of squares, equilateral triangles or regular hexagons. Some of the cells on the grid are marked; these consist of the first generation of cells. Following generations of cells are then produced by the application of simple rules. Very simple rules can produce complex structures. The number of new cells created in each generation and the total number of cells "alive" in successive generations produce sequences in which it is difficult to detect a pattern. Examples are given by Wells (1995). Ulam's first example used a square grid. The first generation consists of a single square marked 1. The rules are: a new square is added at each stage to every vacant edge of a square in the previous generation, except that a new square which would have an edge in common with 2 or more squares of the previous generation is not added. The sequence for new squares at each stage is

1, 4, 4, 12, 4, ...

and the sequence for total number of squares is

1, 5, 9, 21, 25, ...

In both cases it is difficult to detect patterns.

Barrow (1992, p.163) describes this category of sequences as **"incompressible"** - there is no way of abbreviating the sequence, and to describe the sequence we have to list it.

Order out of Chaos

This, of course, is what we are striving for. As an example we take a polygon with n sides whose vertices are randomly chosen (chaos) and we replace it with a polygon whose vertices are the midpoints of the first polygon; when this procedure is repeated, an ellipse-like convex figure usually is produced (order). Clearly there are many examples from Chaos theory itself - such as Sierpinski's Gasket and the Chaos Game. This corresponds to a type of problems where a sequence arising from a problem such as the "Interchanges problem" does not appear to exhibit any recognizable regularity. On further reflection on the problem, we see a pattern. The problem is:

> *"43 people are standing in a circle. An interchange is 2 people changing places. What is the smallest number of interchanges needed so that everybody in the circle will have at least one new neighbour?"*

Note that to express the pattern as a formula for u_n, we may need to invent new symbolism, such as [x] or {x}; this, again, may be difficult.

There is also the remarkable **"Order and Chaos"** example :

> *The Sierpinski Gasket.*
> *Given an equilateral triangle; the (recursive) procedure is "The midpoints of the sides are joined, producing as a result four new triangles, each with sides half as long as the original triangle. Ignore the central triangle, repeat the procedure for the 3 remaining triangles. Continue the process: visualize and describe how the figure changes. Imagine the process continuing "without end". This produces what is called a "Sierpinski gasket" (or triangle).*

This is an ordered procedure.

Now label the 3 vertices of a triangle L, T, R (for Left, Top, Right). Start at any point within the triangle. A random roll of a die determines the direction of each move. Move halfway towards the vertex specified. Each "stopping point" is marked. Continue from the new point, as before. The process is called "The Chaos Game". The result produced is the Sierpinski Gasket - A "highly structured fractal is generated by infinitely many random rolls of a die".

There are other ways of classifying problems - one such is to consider the nature of the information which is given, and the nature of what is required. This has been considered by writers such as Krutetski (1976) but as an addendum to his classification I would like to refer to examples of problems where more than one answer is possible.

Varga (1987) suggests that "the traditional curriculum and teachers' habitual practices may not meet the requirements of our rapidly changing world but they have their riches; in the course of time teachers have accumulated a great deal of common sense". He distinguishes between the intended curriculum, the implemented curriculum as it is put into practice by teachers, and the attained curriculum. As an example he considers the problem

> *Somebody tells a joke on Monday to 5 people. The next day each of the 5 tells the joke to 6 people. Each of these tells the joke to 7 people on Wednesday. How many will have heard it on Wednesday?*

In an interview published in a (Hungarian) national newspaper, a mathematician complained about this problem, for his nephew had received it as homework, and had found 3 different solutions, depending on how you interpreted the text:

1. Five persons heard the joke on Monday, $5 \times 6 = 30$ heard it on Tuesday, and so $30 \times 7 = 210$ heard it on Wednesday. **Answer: 210.**

2. Those who heard the joke on Monday and Tuesday will have heard it on Wednesday, together with the 210 who hear it for the first time on Wednesday, so $5 + 30 + 210 = 245$. **Answer: 245.**

3. The person who told the joke to the first 5 people must have heard it, unless he invented it; so $245 + 1 = 246$. **Answer: 246**

The pupil was concerned that if he produced any of these solutions, the teacher might have another solution in mind, and he would be made a fool in front of the whole class because he had not got the "right" solution. His uncle concluded that "problems should be more carefully worded so as to exclude different interpretations".

Varga continues "I like giving children problems which they can interpret in different ways. Finding different interpretations is a first step towards inventing problems of their own, or towards mathematizing an open situation. Such activities are at least as important as solving ready-made problems, if learning mathematics has any goal beyond itself. In some cases - test items, contest problems - unambiguous wording may be a virtue. There is a watershed here: either you believe that the main goal of mathematics problem solving is to enable pupils to solve further, more difficult problems in order to pass tests and win contests, or you see something, maybe a great deal, that can be achieved beyond this. In the first case you will see no point in ambiguous problems or open problem situations; in the second case you will".

The pupil's problem arose not because of an ambiguous problem, but because of an authoritarian teacher, or at least a teacher who was perceived as authoritarian by the pupil. Our policy should be to make it clear to such teachers how to make use of ambiguous problems. As a result the teacher may even find it possible to suggest further interpretations.

246 seems to be the maximum number who could have heard the joke. But is 210 the minimum number?

> Suppose A on Tuesday heard the joke from B and C out of the 5 who heard it on Tuesday. He does not repeat it to those who have already heard it. Monday 5; Tuesday 29; Wednesday 29×7. **Answer: 238.**

If pupils are encouraged to use their imagination, they produce and defend their answers, and possibly come down to the answer 7. The main interest is possibly not in the minimum solution, but certainly in good, better, and still better solutions. The second example illustrates what is meant by the "attained" curriculum. 10 year olds were given the problem:

> *The area of a rectangular flowerbed is 36* m^2. *Surrounding the flowerbed along its edge is a rope with knots tied at one meter distances everywhere. How many knots are there on the rope?*

Very few pupils could solve it, or even produce an attempted solution. In a few classes, most pupils, even those with low marks, found one or more solutions, or at least worked on it in a way that made sense, even if they miscalculated the answer. The reason for this was that the "good" classes were "good" because their teachers accustomed their pupils to experimenting with "manipulatives" and using techniques such as making drawings in order to tackle problems. Some classes with low average scores on this item were quite good in routine problems and calculations. On this problem they were blocked probably because the problem was not in their stock of knowledge. They could not recall ever having met a similar problem, so they did not do anything.

Some sort of originality was required of the pupils. Do teachers feel that it is unfair to ask a question for which the answer has not been taught, or to give a problem for which the method has not been provided? Such a narrow set of goals needs to be extended. Many teachers ask "at the expense of skills?". The answer to this is "along with, not instead of". The slogan "Do non-routine work and the skills will come" is self-deception. Varga suggests that some of their textbooks, workbooks and maybe even teachers' manuals overstress originality at the cost of skills; a lack of balance will discredit sound principles.

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