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Algorithms Of "The Four Operations" As Mathematical Generalizations Of Human Thought

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Abstract

In a limited way, the algorithm of the four operations is interpreted as any peculiar rule of procedure. However, in a broader sense, it can be viewed as a mathematical generalization of human thought. The latter is possible by perceiving the connections among addition, multiplication, subtraction, and division. In this article, I describe how a group of pre-service mathematics students made sense of the algorithms of the four operations as generalizations of human thought.

Introduction

Current belief is that mathematics is created by humans in an attempt to understand the environment and communicate among themselves (Jacobs, 1994; Kline, 1985; National Council of Teachers of Mathematics, 1989). In the classroom, connections in mathematics play an important role in fostering conceptual understanding of mathematical topics that students learn. Connections are possible among several mathematics topics, between mathematics and other subjects that students learn, and between mathematics and everyday life experiences of students (NCTM, 1989). Having gained conceptual understanding through connections, students can relate more confidently to more complex mathematical ideas and attempt to make insightful generalizations.

Establishing connections among the algorithms of the four operations and helping students perceive an algorithm, in a broader sense, as a mathematical generalization of human thinking, instead of the limited interpretation as "any peculiar rule of procedure" (Boyer, 1991, p.228), should be one way to make mathematics meaningful to students (Anku, 1996).

In this article, I describe how in a pre-service mathematics class to prepare teachers for the elementary school level, we established connections between the algorithms of the four operations of addition, multiplication, subtraction, and division and general human thinking. Then, we used numbers to illustrate the meaning of the algorithms of the four operations. Furthermore, we extended our understanding of the algorithms from using numbers to using letters, thus providing a smooth transition to algebraic ideas and mathematical generalizations for the meaning of these operations. Finally, we showed a connection among the algorithms of the four operations and speculated on possible further connections with other mathematical ideas. I have used very simple numbers in the examples provided because I am more interested in the meanings derived from the examples, but not in the complexity of the examples.

Algorithms of the Four Operations

First, we sought answers to the following questions: What is the nature of the algorithm? That is, what aspect of human thought does the algorithm of each of the four operations depict? How is the algorithm used in mathematics? How is the algorithm represented using symbols? The purpose was to make sense of the meaning of the algorithms of the four operations.

Addition

The meaning of addition (or sum) that we agreed on was *to put together and find the total (how many?)*. The *total* that is obtained after *putting together* depends on the *nature* of the things (numbers) being put together. For example, the addition of two and three, written as 2 + 3 means putting together 2 and 3 *things* and finding the total, which gives 5 (five). In this case the nature of the numbers put together, which is that of positive integers, determines the nature of the total, 5. Now, the addition of two and a half, written as 2 + 1/2, a mixture of a positive integer and a fraction, gives a mixed number. Sometimes, the number line was used to reinforce the meaning of addition.

We put these numbers together horizontally, vertically, or diagonally (in fact, in any directions) and reversed the orders (of the numbers, not symbols), and obtained the same totals. Implicitly, we were establishing the commutative law of addition. Similarly, we put numbers together differently to establish the associative law of addition and the identity of addition. Gradually, we perceived addition algorithm as a generalization of human thought of putting things together and finding the resulting total.

Multiplication

For purposes of speed and simplicity, the multiplication algorithm was perceived as the human thought of *repeated addition.* For example, 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 would be read as putting 2 together 10 times and it is written as 10 x 2 which gives a total of 20. So, in general, putting *a* things together *b* times would be written as *b x a* to get a total which can be represented by *c*. Now, by analogy, since 2 x 10 also gives 20, we reasoned that *a x b* must also give *c*. Implicitly, we were establishing the commutative law of multiplication. Here again, we put numbers together differently to implicitly establish the associative law of multiplication. Sometimes, we used the number line to reinforce the meaning of the algorithm. We then perceived the algorithm of multiplication as representing a human thought of repeatedly adding things.

Subtraction

Subtraction as a human thought of *separating things*, finding the difference in number between things, or taking away a number of things from a given number of the same things and then finding the result, was also explored. For example, subtracting 2 from 6 is written as 6 - 2 and gives a result of 4. So generally, subtracting b from a is written as *a - b*, to give a result of, say *c*. But since 6 - 2 does not give the same result as 2 - 6, we reasoned out that in general, *a - b* does not give *b - a*. Thus, we were asserting implicitly that the commutative law (also the associative law) does not hold for subtraction. Again, the number line was used at times to reinforce the concept of subtraction.

Division

The meaning of division as a human thought of *repeated subtraction* or *regrouping into equivalent sets* to find a result, was explored. As a repeated subtraction, 6 divided by 2 which is written as 6 ÷ 2, is perceived as *how many times* 2 can be taken away from 6. This gives a total of 3 times. So in general, *a* divided by *b* is written as *a ÷ b* and is perceived as the number of times *b* can be taken away from *a* to obtain, say *c*. Obviously, 6 ÷ 2 does not give the same result as 2 ÷ 6 and therefore generally, *a ÷ b* is not the same as *b ÷ a*. Implicitly, it follows then that the commutative law (also the associative law) does not hold for division perceived as repeated subtraction.

For division as regrouping into equivalent sets, 6 ÷ 2 is perceived as the total number of things in each set if 6 things are regrouped into 2 equivalent sets (3 in this case). So, in general, regrouping *a* things into *b* equivalent sets to obtain *c* things in each set is written as $a \div b = c$. Here also, the commutative and associative laws do not hold.

Notice we drew important distinction between the results obtained from repeated subtraction and regrouping into equivalent sets. For example, for 6 apples, we were to give out (take away) 2 each time and give to a friend, then the 3 times that it can be done actually tells us about the number of friends who received 2 apples each. However, if the 6 apples are to be regrouped into 2 equivalent sets, then the result of 3 is actually the number of apples in each of the 2 equivalent sets. So, although for both cases we write $6 \div 2 = 3$, the resulting 3 represents different human thinking. The distinction turned out to be very satisfying for students.

Connecting the four operations (The "mega-connection")

To appreciate the connections within human thinking, we tried to relate the algorithms of all the four operations. As division, $6 \div 2$ gives 3. As subtraction, we get 3 as the number of times 2 can be taken away from 6, that is $(((6 - 2) -2) - 2)$. As multiplication, we write 6 x 1/2 which gives 3 as a result. And as addition, we get 1/2 + 1/2 + 1/2 +1/2 + 1/2+ 1/2 (6 times) which also gives 3. Clearly, a connection exists between the algorithms of the four operations, that is $6 \div 2$ yields the same numerical result as the number of times 2 can be taken away from 6, which in turn is the same as 6 x 1/2, which can also be expressed as 1/2 + 1/2 +1/2 + 1/2 + 1/2.

So, extending our understanding from connecting the algorithms of the four operations using numerical values to algebraic values, we see that $a \div b$ should yield the same result as the number of times b can be taken away from a, that is $((((a - b) -b) -b) \ldots -b)$, which in turn gives the same result as $a \times 1/b$ or $1/b + 1/b + 1/b + \ldots$, a times. Perceiving these general symbolic forms as mathematical representations of human thinking helped students to see the value of the algorithms of the four operations as more than a *chore of repetitive activities* that they have to *follow to get the right answer*.

Other possible connections

Students were challenged to explore possible connections between the algorithms of the four operations and other concepts in mathematics. They soon realized that ideas from geometry, algebra, statistics, probability, calculus, and many more, are all connected explicitly or implicitly, to the algorithms of the four operations. This realization generated a lot of excitement among the students. One student put it this way: "Yes, now I understand how to link one problem to another and how mathematics is actually all things put together, not separated into topics."

Conclusion

For students of this class, they found establishing the connections among the algorithms of the four operations meaningful since they had understood for the first time many of the ideas they memorized and used for several years without understanding. They realized that the algorithms of addition, multiplication, subtraction, and division, are mathematical generalizations of human thought. Emphasizing the human thought that the algorithms represent can rekindle students' interest as they do not see these algorithms as only repeated drills but as a mathematical generalization of human thinking. And establishing connections at early stages might help students experience the meaningfulness of the algorithms of the four operations.

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