The Mathematics Educator
1996, Vol. 1, No. 1, 36-44

An Integrated Approach For Teaching Mathematical Problem Solving

Fong Ho Kheong

Abstract

Mathematics teachers require different styles of teaching mathematical problem solving to cater to different ability pupils in schools. The paper discusses an integrated approach of teaching problem solving which involves application of some theories: working-backwards heuristic, task analysis, productive thinking and metacognition. A speed problem was used to illustrate how these theories were used to tackle the problem. This integrated approach serves as an alternative approach for solving mathematical problems.

Problem solving in mathematics is one of the main concerns to many mathematics teachers. Teachers who have taught mathematics would have experienced difficulties to help some children solve mathematical problems. They may be puzzled to find that they could use some strategies to teach a certain group of pupils in problem solving but failed to apply the same strategies successfully for another group of pupils. The main purpose of this article is to provide some guidelines for teachers, who have this sort of experience, to consider an integrated approach to help children solve mathematical problems. To begin with, we need to define what is problem solving.

What is a problem and problem solving ?

It has been said that "one person's problem is another person's exercise and a third person's frustration" (NCTM, 1969). Thus, the word 'problem' seems to cause some confusion to teachers in the classroom. There is a need to address this 'problem' issue before the problem solving issue could be examined.

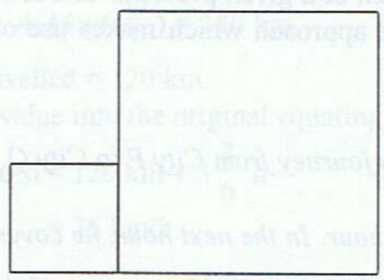
In this article, problem solving in mathematics is meant to cover a wide range of problems: verbal and non-verbal problems, and routine and non-routine problems. The following are two examples of verbal and non-verbal routine problems.

(a) Jane has 15 beads more than Kathy.
If Kathy has 6 beads, how many beads has Jane?
(b) 9 + 4 = □ Find the number in the box.

Referring to the NCTM's statement, one could infer that both the questions stated above are considered as problems under certain circumstances. For example, when the computation question 9 + 4 = □ is given to a 6 year-old girl, who has some lessons on counting by addition but has not mastered it, she may not have the right strategy to obtain the correct answer. In her case, she might not have mastered the algorithm to perform the computation. Hence it is a problem to her.

Kantowski (1981) viewed problem solving from a different perspective. According to her, a problem exists if the situation is non-routine, that is, when a problem solver does not have an algorithm at hand that can generate a solution. The following is an example of a non-routine problem.

"The figure is formed by 2 squares. The side of each square is a whole number. If the area of the figure is 58 cm^2, what is its perimeter? (Fong, 1990)



The problem is considered to be non-routine because in most mensuration problems concerning perimeter or area, the sides of figures are usually given. In this case not a single value of the side is given. Using the area and perimeter formulae to solve this problem will not lead to a solution. The problem solver is required to apply some specific strategies such as 'listing method' to tackle this problem. The following shows a solution using the non-routine strategy: 'listing-method'.

List of some whole numbers:

1 2 3 4 5 6 7 8

Squaring each number:

1 4 9 16 25 36 49 64

Examining the square numbers, notice that the sum of 9 and 49 is 58. Thus the areas of the two geometric squares are 9 and 49 respectively, and the sides of the squares are 3 and 7 respectively.
Then the perimeter of the figure is 2[(3+7) + 7]= 34 cm.

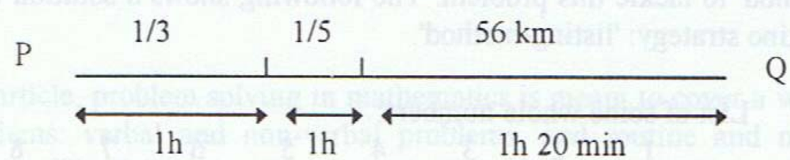
The issue in this article does not focus on whether a problem is routine or non-routine but the main concern is to determine how children could be taught to solve this type of mathematical problems. Problem solving is a complex issue which involves methods of instruction and cognitive processes in human minds. The following paragraphs discuss some teaching and learning theories and heuristics in problem solving: working backwards heuristic, task analysis, productive thinking, constructivism and metacognition.

Working-Backwards Heuristic

The working-backwards heuristic has often been applied by most problem solvers for solving various types of routine mathematical problems. This approach involves a lot of self-questioning and answering activities. Using this heuristic, problem solvers will often begin their working by stating the question requirement of a given problem. The following is an example of a speed problem and the approach which makes use of the working-backwards heuristic.

"Rahim made a journey from City P to City Q. He covered $\frac{1}{3}$ *of it in the first hour. In the next hour, he covered another* $\frac{1}{5}$ *of the journey. He then took 1h 20min to travel the remaining 56km. Calculate his average speed in km/h for the whole journey." (MOE, 1995)*

First, read the question statements and draw diagram to represent the whole event described in the question.



To use the working-backwards heuristic, a problem solver will first examine the question statement and determine the unknown value to be found as follows:

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

$$= \frac{?}{1\text{h} + 1\text{h} + 1\text{h } 20\text{ min}}$$

Next, the total distance is then computed. The diagram shows that only the remaining part of the journey is given as 56 km. For the first hour and the next hour, the fractional parts of the journey are given. Probably, the problem solver will interpret and evoke proportion concept to write an equating statement comprising fractional part and distance covered at the last stage of the journey.

The last stage of the journey is given by

$$1 - \frac{1}{3} - \frac{1}{5} = \frac{7}{15}$$

$$\therefore \frac{7}{15} \rightarrow 56 \text{ km}$$

$$\frac{15}{15} \rightarrow 15 \times \left(\frac{56}{7}\right) = 120 \text{ km}$$

Total distance travelled = 120 km.

Substituting this value into the original equating statement,

$$\text{Average speed} = 120 \text{ km} \div 3\frac{2}{6} \text{ h}$$

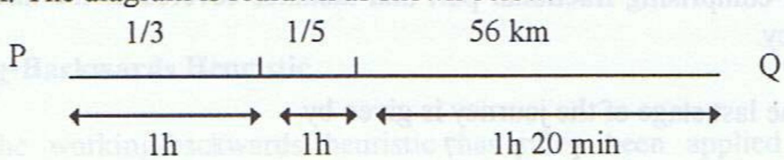
$$= 36 \text{ km/h}$$

Apparently, the working-backwards approach above begins with stating the requirement of question and stating the known and unknown values of formula. It is followed by examining other given information to see how the unknown values can be related and obtained. In the above example, the known value is the total time taken and the unknown value is the total distance travelled. The total distance can be found using proportion concept by relating the fraction parts, $\frac{1}{3}$ and $\frac{1}{5}$, and the distance of the last stage of the journey, 56km. This approach of solving mathematical problem is called the working-backwards heuristic.

Task Analysis

The working-backwards heuristic is a powerful strategy for tackling a mathematical problem. However, there are other factors which a problem solver should be equipped with in order to solve the problem successfully. Before one could solve a particular problem, one should also need to master some specific tasks related to the problem under consideration. Using the above speed problem as an example for illustration, the following are some related tasks which a problem solver may use to complete the whole solution.

(a) First of all, a problem solver is required to interpret the problem and draw a diagram to show and reflect all the given information from the question. The diagram is redrawn as shown below.



The diagram helps children to conceptualise the whole picture travelling from P to Q. The information given includes the time taken at each stage of the journey (i.e. 1h, 1h and 1h 20min). The last stage of the journey is given as 56km whereas the other parts of the journey are given as fractional parts of the journey.

(b) The problem solver is also required to recall the average formula:

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}.$$

From the formula above, he needs to find the total time taken in hours. In order to do so, he needs to know how to convert hour and minute to hour. Beside these the problem solver is also required to apply the proportion concept by equating fractional parts to distances. In the example, first find the fractional part equivalent to 56km followed by writing a proportional statement as follows:

$$1 - \frac{1}{3} - \frac{1}{5} = \frac{7}{15}$$

$$\therefore \quad \frac{7}{15} \rightarrow 56 \text{ km}$$

$$\frac{15}{15} \rightarrow 15 \times \left(\frac{56}{7}\right)$$

Notice that operation on fractions is also another task required in the solution. Multiplication and division of numbers by fraction are some essential skills to tackle the problems:

(i) $15 \times (\frac{56}{7})$

(ii) $120 \text{ km} \div 3\frac{2}{6} \text{ h}$

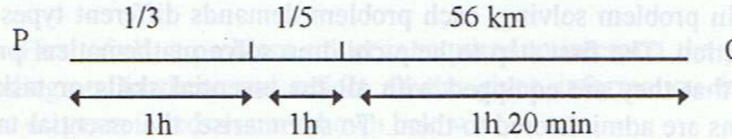
In problem solving, each problem demands different types of tasks for the solution. The first step to help children solve mathematical problems is to ensure that they are equipped with all the essential skills or tasks before the problems are administered to them. To summarise, the essential tasks required to solve the above problem are as follows:

1. Interpreting question statements and representing them with a diagram to conceptualise the whole event.
2. Applying the average speed formula.
3. Applying the proportional concept to write equation relating fractional part and distance.
4. Converting hour and minute to hour.
5. Operating fractions involving multiplication or division of whole number by fraction.

Productive Thinking

Newell and Simon (1972) postulated the production system theory which explains how information is processed in human mind. A student's ability to solve problems depends on the extent to which he is able to elicit related information from the long-term memory (LTM). Questions which require him to elicit information from the LTM are thought to be more difficult than those questions which require him to acquire information from the external source for processing in the short-term memory (Fong, 1994). The implication from this theory is that one of the factors which determine a student's ability to solve a problem is the availability of knowledge stored in the LTM. The lack of related knowledge will greatly affect the student's ability to solve mathematical problems. From the previous paragraph, a list of related tasks (or related knowledge) were identified which are essential for solving the speed problem. The theory explains that without the mastery of these tasks, a problem solver would not be able to take off when the problem is given to him.

The productive thinking is not only dealing with the 'have' or 'have not' of knowledge but also concerning with how knowledge could be evoked from the LTM. A production system is a set of related productions. Each production specifies the conditions in which an action should occur. The set of productions is viewed as the body of knowledge which is represented in the LTM of human brains in the form of condition-action rules. This theory could be explained using the speed problem given above. The diagram is reproduced as follows:



From the diagram, a problem solver recognises that some parts of the journey from P to Q are represented by fractions ($\frac{1}{3}$ and $\frac{1}{5}$). Another part of the journey is represented by 56 km. Here there are two sets of "conditions": (1) distance represented by fraction and (2) distance represented by measurement. If the problem solver's LTM contains proportional concept relating fractional units to any form of measurements then an equating statement could be evoked from the two given conditions. As a result, the proportion statement $(1-\frac{1}{3}-\frac{1}{5})$ -----> 56 km is evoked from the LTM.

The above paragraph indicates that problem solving involves production of information from memories through activation of stored information from external sources. This mental activity is part and parcel of problem-solving processes.

Discussion

Some teachers have experienced that their styles of teaching mathematics are suitable to a certain group of pupils but found futile when they are applied to another group of pupils. How could we explain this peculiar phenomenon? One possible explanation given in the previous paragraph is that some children have not mastered the related tasks which are essential for the solution of a problem. Without equipping children with the related tasks, there is no possibility that they could solve the problem. A teacher needs to change his style of teaching when he discovers that his pupils have different "mathematical" behaviours.

There are also possibilities that some children who have mastered the related tasks could not come out with a solution to the problem. Then what reason(s) could we explain for their failure in solving the problem? The possible reason is that they are not able to activate, evoke and connect the related information which is stored in the problem solver's LTM. Related information includes not only isolated facts or knowledge but also pairs of information which are logically related such as the proportion statement given in the previous section. That is, a problem solver needs to connect fraction and distance to form a proportional statement. This productive thinking procedures can only become effective when children have often rehearsed them in problem solving. This implies that deliberate training of productive thinking is required.

Another question is how children acquire and construct related pieces of information (condition-action statements) in the LTM so that they are ready for production when they are required. One possible explanation is that children learn to construct knowledge through imitation of teacher's procedures of solving mathematical problems. Not all children could construct and further reconstruct the knowledge given to them. Only the better ones could see how facts are connected. This phenomenon can be explained using the constructivism theory. Given a set of knowledge, some children can construct and retain it in their memory. They are also able to reconstruct and extrapolate the knowledge in a new form. This helps to explain why some children can produce creative solutions in problem solving.

Is there a theory of teaching children to solve mathematical problems? Discussion above seems to indicate that there is no single theory for helping children to solve mathematical problems because there are too many factors to be considered in problem solving. However, there are some theories discussed above which provide teachers with some ideas of how children could be coached to do better in problem solving. Beside providing children with essential related knowledge (based on the task-analysis concept) , they should also be given an environment involving the use of metacognitive behaviours such as reviewing of essential skills, reflecting and connecting tasks and deliberately practicing productive thinking (Fong, 1995). Teachers should also demonstrate to children the processes of productive thinking and the holistic approach of problem solving using one or more problem solving strategies such as working-backwards strategy.

To conclude, we should realise that each child behaves differently from other children. In order that we can successfully help the child solve mathematical problems, the child should be diagnosed if he/she has already acquired all the essential and related skills of the problems under consideration. Some problem solving heuristics strategies should be taught to them, e.g. the working-backwards heuristic. Productive thinking strategies should be deliberately taught to them. The child should also be taught the use of metacognitive behaviours in solving mathematical problems through constant reflection and connection of knowledge stored in the LTM.

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