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Example Sequences to Help Students Recognise and Apply Formula: The Singapore Situation

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Abstract: The research on the use of example sequences is scarce in the field of mathematics education. We surmise that this phenomenon has to do with the wrong assumption that the work of careful sequencing of examples is an unimportant part of a teacher's practice. We base the research reported in this paper on a case study of Teacher Beng Choon – the principles she used to design example sequences for formula recognition and application. These principles were incorporated into an online survey to examine the extent in which other Singapore mathematics teachers also used them for their design of example sequences. We used the findings to develop "The Singapore Portrait". This is translated into a framework for professional development in the craft of example sequencing.

Keywords: example sequences, formula application, cognitive load theory, variation theory

Introduction

The use of example sequences in mathematics teaching has a very long history. The form of use that is familiar to us today – that of having a series of similar exercises with minor variations in order to help students gain fluency of underlying common method through multiple practices – emerged very early in human history. Among the earliest is the Jiuzhang Suanshu (translated conventionally as "Chinese Nine Chapters of Mathematical Art") purportedly dated as far back to many centuries BC. Interestingly, over many centuries since, and after numerous reform efforts towards changes in mathematics instructional practices, the use of example sequences as a way to help students learn mathematics remains common in many classrooms across the globe.

However, there is relatively little rigorous research conducted to investigate the design principles that teachers adopt which would result in quality example sequences – very often, and as far as our experiences go as teachers in Singapore schools, it is taken for granted that teachers know how to sequence examples; or, that this aspect of instructional work is so inconsequential that it does not matter how it is done. This was attested by the most recent International Commission on Mathematical Instruction (ICMI) report on task design – which would apply also to example sequencing:

> A vital component often missing in curriculum innovation documents is the vivid exemplification that is necessary to show exactly what tasks might look like and how they relate to improving teaching and learning … . Despite the recent growth spurt of design studies within mathematics education, the specificity of the principles that inform task design in a precise way remains both underdeveloped and, even when somewhat developed, underreported.
>
> (Kieran, Doorman, & Ohtani, 2015, pp. 73-74, emphases added)

The work that is reported in this paper aims to go into the "specific" and the "precise" with respect to the Singapore situation, within the narrower domain of example sequencing and only within the narrower context of formula recognition and application. Despite its seemingly narrow focus, we think it is still significant as "applying formula" remains an abiding popular image of doing mathematics in Singapore classrooms – as in classrooms in different parts of the world (e.g., Flores, Koontz, Inan, & Alagic, 2015; McCloskey, 2014). But since "formula" as a language is virtually non-existent within the formal discipline of mathematics, we adopt this working definition: steps to follow based on a prescribed procedure. While formula is also usually associated with an algebraic equation (such as $A = L \times B$, as in area of rectangle equals the product of its length and breadth), we take here a broader interpretation to include other mathematical results (such as geometrical theorems) that are not usually captured in equation form.

We proceeded with our research on this focus in two stages: (1) In this first stage, we did an in-depth case study of an experienced and competent teacher (Teacher Beng Choon) to draw out – and we think this grain-size of analysis is necessary to examine the "precise and specific" – the principles she adopted in her example sequencing for teaching formula recognition and application. The outcomes of the study is submitted to a journal article (Leong, Cheng, Toh, Kaur, & Toh, Accepted) and so the details will not be repeated here. We will nevertheless summarise the findings in the next section of this paper as it served as the background for the Stage Two study reported here. (2) In the second stage, which is the focus of the empirical work reported in this paper, we seek to examine the extent to which the practice of Teacher Beng Choon in example sequencing was utilised by Singapore secondary mathematics teachers. In other words, we aim to build a "Singapore portrait" of example sequencing practices and principles that began with a case of Teacher Beng Choon and which was then broadened to include the views of other Singapore teachers.

We should forewarn readers that this article is not structured in a conventional way. You will not find a distinct section on "literature review". This is because both stages of the research sit on the same literature foundation – and since it has been given in Leong et al. (Accepted) it will not be repeated here. Nevertheless, a section similar to literature review can be found in the penultimate section of this paper where we subject the Singapore portrait to interaction with the extant literature – with a view of refining it for the purpose of crafting a provisional framework of professional development on example sequencing in the context of formula application. The ultimate aim towards principles that translate to professional development work is the motivation for this research in the first place.

Teacher Beng Choon

Beng Choon was identified as an experienced and competent secondary mathematics teacher. "Experienced" is defined as having taught the same mathematical course at the same level for a minimum of five years; and "competent" selection is based on recognition by the local professional community as a teacher who is effective in teaching mathematics.

In the interviews with her – we conducted three interviews which were at "before", "middle", and "after" of a module on "Differentiation" that she taught – she frequently mentioned that her design of materials was to help students "recognise the form". This focus of hers provided the impetus for our examination of her example sequences for recognition and application of formula. [Throughout the module, "the form" are also formulas in the sense that we use in this paper].

We studied in-depth her design and use of two example sequences – the first on the formula of $\frac{d}{dx}(x^n) = nx^{n-1}$ and the other on Chain Rule. We found that the common principles she applied across these two formulas can be summarised as:

(1) To build confidence in students of their ability to apply the formula, the exercise items are sequenced such that the first item requires easiest recognition of form, and gradually increasing in difficulty, with the items considered the most difficult at the end of the set of exercises;
(2) The surface forms of the items are varied in order to help students discern the underlying invariance;
(3) Where appropriate, items are inserted within the set to help students connect to concepts (in this case, differentiation as finding gradient) – that were introduced earlier and which would be required later in the topic;
(4) "Recognise the form" is a prerequisite to applying the formula. There may be nested forms within a form that correspond to various other formulas. The items are crafted so that the teacher can flexibly attend to different combinations (or levels) of forms during the lesson according to the difficulties students face in correctly applying the formula.

We noticed that Principle (1) corresponded closely to the practical applications of Cognitive Load Theory (e.g., Sweller, 1998; Sweller, Kalyuga, & Ayres, 2011); and Principle (2) is in line with Variation Theory (e.g., Marton & Booth, 1997; Runesson, 2005). Both of these theories will be given lengthier treatments towards the end of this paper. We mentioned their points of convergences at this juncture to highlight the interesting finding that Beng Choon's practice does not – despite our initial attempts during analysis to – fit into just one of the theoretical molds; rather, hers was an integration of relevant applications of these (and perhaps other) theoretical traditions. We should qualify that we make no claims that Beng Choon was knowledgeable of these theories. We merely state that her principles approximate the applications of these theories. Indeed, This is acknowledged by Marton and Pang (2006): "We are not saying that such patterns of variation and invariance cannot be brought about by teachers who are ignorant of the framework because it is impossible to teach without using variation and invariance, and many teachers often intuitively create the necessary conditions for mastering the specific object of learning they are dealing with." (p. 217).

Beng Choon drew upon Principles (1) and (2) consistently in the sequencing of items. To her, it is important to consider gradation of item difficulty as a way of taking into consideration the cognitive load a task would pose to students. The gradation allows students to enter into the set of tasks with minimal cognitive load, and as they become more familiar, the cognitive demand is gradually increased with each succeeding item in the sequence. At the same time, the items – taken together as a set – are deemed by her to present necessary variation of surface forms in order for students to experience a "pattern of variation and invariance". Represented in diagrammatic form, we view these two principles as feeding into Beng Choon's deliberate consideration in the construction of her item sequences, as shown in Figure 1.

In terms of the outcomes she intended from the carefully-sequenced examples, her main goal was to help students "recognise the form" because she saw it as a prerequisite to applying the required formulas (Principle 4). But apart from this goal, she also slipped in connections to other ideas insofar as that they were easily derivable from the recognition of the related form (Principle 3). These principles were also included in the representation of Beng Choon's overall design conception in Figure 1.

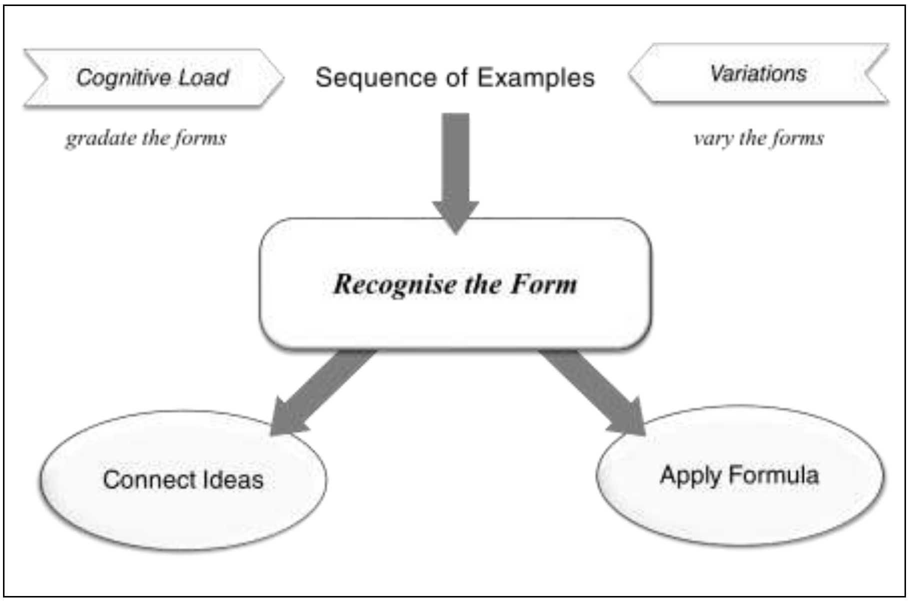

Figure 1. A model of Beng Choon's design considerations

A point of clarification is perhaps needed here with respect to the layout of Figure 1. While "connect ideas" and "apply formula" were also part of Beng Choon's design principles, we choose to place them as outcomes (that is, at the bottom of the figure and not at the top together with "Cognitive Load" and "Variations") (a) because logically, "apply formula" proceeds from "recognise the form" and hence the direction of the arrow linking them; (b) since the practice of Beng Choon to "connect ideas" took place when she actually applied the formula during her in-class teaching, we associate them jointly as outcomes of formula recognition.

Since the purpose of this study is to inquire the *extent* to which the design practices of Teacher Beng Choon were also used by mathematics teachers in Singapore, the major methodological challenge is in the translation of the findings of Beng Choon's case into a research instrument that is suitable for data collection at a larger scale. In particular, we needed to encode the main principles of Beng Choon's design work in the area of "recognise and apply formula" into survey items that teachers can respond to in terms of the extent to which they also use these principles. We also think of this challenge as one that bridges the case study and the survey design. That is, research on "what teachers really do in their instructional practices" can be broadly classed as: (1) case studies based on analysis of actual work of teachers, including video analysis. Due to logistical resources demanded, realistically, only a few cases can be studied this way. This threatens any claims of representativeness of the findings to the wider jurisdiction; (2) survey designs that target large sample size for representativeness. This overcomes the logistical constraints but usually compromises the "resonance" with ground experiences. The results of the research are generally 'distant' and irreconcilable with "what teachers actually do". The methodological work presented in this study can thus be seen as a bridge between these two traditional paradigms of research: to recall, for Stage (1) we began with a case study to derive the characteristics – the summary of which was described in the preceding section, and in Stage (2) we used these characteristics to craft survey items that are experience-near to teachers. This second stage will be the focus for the rest of the empirical part of this paper.

Method

The survey consists of three sections: (1) the example sequence (which was directly derived from her work in the $\frac{d}{dx}(x^n) = nx^{n-1}$ formula) in which Teacher Beng Choon demonstrated the use of her design principles; (2) survey items that spelt out the design principles; and (3) space for qualitative responses. These three sections are arranged in a single page as shown in Figure 2. The reason for limiting it to a page is to present this set of components as a single unit of focus – to heighten the sense in the teacher that his/her response is targeted at design work on "Formula". [When the respondent moves to the next page, the focus shifts to another unit of Beng Choon's instructional materials. The units we present in the overall survey is shown on the top-left panel of Figure 2. For this study, we report only the page on "Formula".

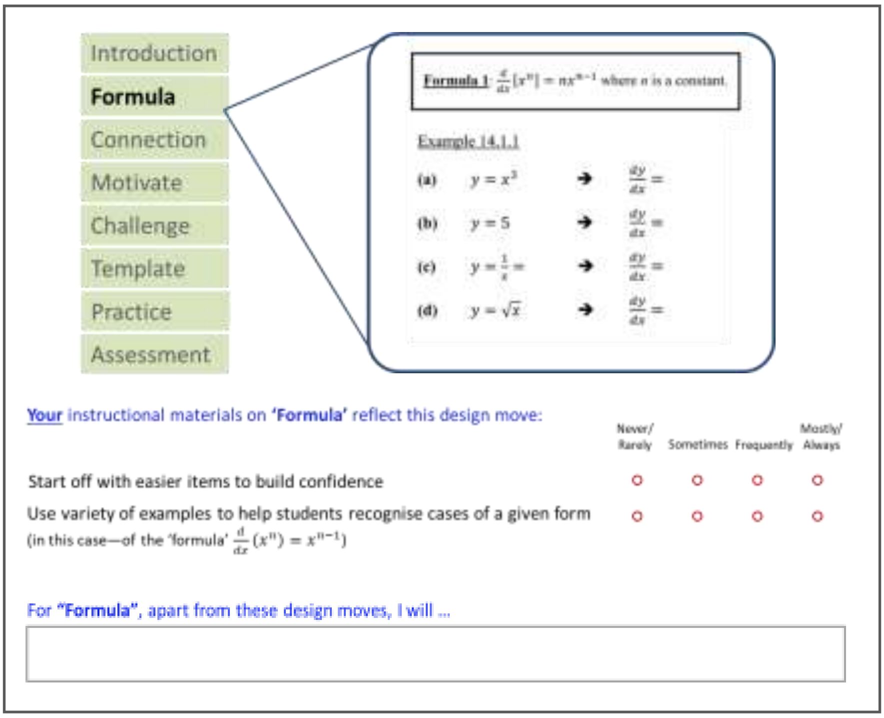

Figure 2. Page on "Formula" from Survey

Section (1) at the top of the page is to provide teachers with the grounded context in which they can resonate with. It is an extract taken from a section of Beng Choon's actual instructional materials for her Year 9 class. The topic, the arrangement, the stating of a 'formula', and the sequence of examples that follow form a routine we think teachers who are taking the survey would be familiar with. It is meant to direct their mental reference to this context so that their responses to the survey items would be grounded to a piece of instructional material that is near to their own experiences. We surmise that this would heighten their understanding of the intended meaning of the items, and hence their construct validity. Argued in another way, if we suppose this section is not included in the survey – as in most surveys reported in the literature – teachers would find it much harder to lock-in to a specific instructional experience in which to give a meaningful response.

The two survey items in Section (2) reflect the two key design considerations that Beng Choon utilised as shown in the two channelling arrows in the top part of Figure 1. We chose a 4-point Likert scale as we think it would provide a sufficient range for teachers' response; at the same time, it is designed to alleviate the middle option bias. As to the design ideas represented by the lower half of Figure 1, we think it would be too leading to include them as response items. We will examine the open responses in Section (3) of the survey for evidences of these and other design considerations.

The survey was set into an online mode for the ease of administration and collection of data. The settings were retained in the online version – that is, each page appeared as a page in the online interface. The responses in Section (2) were compulsory – in that, if response for any item was not entered, the responder will be prompted before he/she can proceed to the next online page; the response to the open qualitative comments in Section (3) was optional. As the topic that was covered in Beng Choon's materials are classified under Additional Mathematics, it is appropriate that teachers who do the survey be familiar with the contents of Additional Mathematics. In the earlier part of the online system, the responder was asked to indicate the main mathematics subject they teach – the choice was among Mathematics (Express), Mathematics (Normal), and Additional Mathematics. Only those who selected the last choice were automatically channeled to the survey questions in this study. A total of 156 teachers participated in these survey questions. Out of these, 70 wrote comments under Section (3). These numbers may look small compared to the number of secondary mathematics teachers in Singapore. But if one considers that the 156 teachers were from 65 secondary schools (out of a total of 151 secondary schools in Singapore) and that the 70 teachers who wrote the comments were from 38 secondary schools, then we may justifiably claim a high degree of representativeness of Singapore secondary mathematics teachers.

For responses in Section (2), we assign the scores of 1, 2, 3, 4, to the responses of "Never/Rarely", "Sometimes", "Frequently", "Mostly/Always" respectively. For responses to Section (3), we initially coded them based on their surface descriptions, such as particular mention of the case of $y = 5$, the use of notations other than $\frac{dy}{dx}$, and the need to derive the formula from first principles. This coding process yielded ten categories. This resulted in too many categories with small quantities under each category. We therefore sought to combine the categories, taking into consideration the underlying instructional goals of each comment, and the attempt to match the insights provided by Beng Choon's design work, especially those indicated in the lower portion of Figure 1. This re-coding process resulted in three categories: "Examples", "Proficiency", and "Concepts". The first category refers to comments specific to the sequence of examples as illustrated in Section (1) of the page; "Proficiency" comments are those who target at helping students learn better in particular aspects of formula recognition or/and application; and the last category refers to comments about conceptual development related to the formula. Since these categories are not pairwise mutually exclusive, where appropriate, the comments are multi-coded.

Findings

Under the Methods Section, we explained how the survey was designed – through the contextual groundedness of each page – to increase teachers' likelihood of correct textual interpretation when responding, thus increasing the construct validity of the instrument. There is, however, another common concern associated with a survey of this nature: the Social Desirability Bias which threatens the validity of interpretations that we may draw from the findings. In other words, the question is: Since the survey items in Section (2) of the page describe characteristics that are socially desirable within the cultural system of professional teaching, would it not be 'leading' and accentuate the likelihood of teachers indicate highly in these characteristics, even though they may not actually carry out these principles in their own practice? For this reason, we examined some items of the survey to check for evidence of such a bias in the teachers' responses. The details of this particular investigation is found in Leong, Cheng, and Toh (2019). In brief, we looked at items that are most likely to result in such a bias; we then looked at the scores assigned to these items. If the Social Desirability Bias was strong, then we would expect a high positive correlation between the two. As it turned out, for these items, the scores were among the lowest in the survey. There is thus no evidence to suggest that such a bias was a threat to validity.

Further, we also inserted two items that would act as validation items – to detect signs of unthinking responses. They were couched in such a way that they clearly describe practices that thinking professional teachers would not carry out regularly: "Insert *numerous* tasks of such tasks [of high cognitive demand] for *every* lesson"; and "Use the [shown] template for all subsequent examples". The score for these items were 2.22 and 2.13 – the lowest scores among the eighteen items in the whole survey. Based on these preliminary tests, we think that, on the whole, we can take the teachers' responses with a high degree of trustworthiness as reflecting their resonance (or lack of it) in their own design principles with that illustrated by Beng Choon.

After removing these validation items, the remaining items in Section (2) of the whole survey are given in Table 1. The two items which are the focus of these study are bold in the table and labelled as Item A and Item B.

Table 1.
Items in Section (2) of all the pages in the survey

Page	Item	Mean score
Introduction	Provide opportunity for students to explore at the point of introducing new ideas	2.94
	Build connection	3.27
Formula	Focus on students practice on a specific skill	3.41
	A. Start off with easier items to build confidence	**3.61**
	B. Use variety of examples to help students recognise cases of a given form	**3.54**
	Deliberate choice of cases	3.44
Connection	Repeatedly – in the course of the topic – go back to reinforce link	3.40
Motivate	Motivate the learning of a new method	3.26
	After motivation, demonstrate the new method	3.38
Challenge	Use tasks of high cognitive demand at appropriate junctures	2.96
	Use opportunity to practise skills taught in previous topics	3.01
Template	Provide a 'template' for students to fill in as a scaffold	2.54
Practice	Provide opportunity for students to repeatedly revise materials	3.34

	In revision materials, include skills/ideas learnt a few lessons ago, and not just what was immediately learnt	3.12
	Some of these items are assigned as homework	3.32
Assessment	Provide opportunities for students to evaluate their own learning	2.25

The overall mean score for all the items is 3.18. Notice that the mean scores for Item A and Item B are the highest across all 16 items in the survey. This means that among all the design principles listed, the teachers found they were most able to resonate with "start with easier items …" (the 'lower cognitive load principle') and the "use variety of examples …" (the 'variation principle'). Also the scores of 3.61 and 3.54 may be interpreted to mean that, on average, the frequency of the teachers' use of these principles in their design of instructional materials were between "Frequently" and "Mostly/Always", in fact, closer to the latter. Broadening from these two items to the four items under the "Formula" page, we can also see that the means of these items are also the highest among other page categories. It seems that formula – its recognition and application – received the most attention and agreement as to the extent the descriptions reflected the teachers' actual practices. But these numbers do not tell us more about *how* the teachers apply these principles into their example-sequence design. For this, we need to turn to the open comments in Section (3) of the page. As mentioned earlier, these 70 comments were categorised into "Examples", "Proficiency", and "Concepts".

Examples

22 responses from 18 schools are coded under this category. The sub-categories are presented in Table 2.

Table 2.
Responses under the "Examples" category

Sub-categories	Number of responses
Include more of the same	5
Include more cases	12
Include non-examples	3
Exclude some cases	1
Re-order the sequence	1

"Include more of the same" refers to more examples for each case; "include more cases" refers to other cases of the formula that were not included in the list of examples shown in Section (1) of the page. The "other cases" mentioned include other real indices such as fractional indices and negative indices, using other letters of the alphabet instead of just y and x in $\frac{dy}{dx}$, and the inclusion of an example with a real-world context. "Include non-examples" refers to a deliberate insertion of examples where the formula shown is not applicable, thus highlighting the conditions in which the formula is applicable. The one response coded as "exclude some cases" mentioned that the example of $y = 5$ should be removed as it strictly does not exemplify the form of x^n; rather, it is more suitable when done under the form of ax^n (presumably at a later stage of the unit). Similar to this observation, the other response on "re-order the sequence" mentioned that since $y = 5$ does not conform to the case of x^n, it is more suitably placed at the end of the sequence of examples to act as a link to the next form of ax^n.

Proficiency

31 responses from 25 schools are coded under this category. The sub-categories are presented in Table 3.

Table 3.
Responses under the "Proficiency" category

Sub-categories	Number of responses
Opportunity for student practice	9
Worked solutions for students' reference	3
Explicit demonstration of steps by teacher	4
Memory helps	5
Provide scaffolds in the notes	4
Check for errors	5
Combine with other formulas	1

"Opportunity for student practice" refers to a focus on the students – that, in order for them to gain fluency with formula recognition and application, there ought to be time given for students to practise with similar cases. Some advocated that the practice be given after each case, some stressed the importance of more practice at the beginning with easier cases for confidence-building, and others mentioned the importance of homework for more practice. “Worked solutions for students’ reference” refers to actual full solution steps to be printed in the notes so that students can follow more easily as they apply similar steps. The “explicit demonstration by the teacher” includes the need for teachers to emphasise key steps such as visually ‘bringing down’ n to be the coefficient, and the reduction of the power by 1. “Memory helps” include asking the students to write down the formula, the recitation of the formula, and the verbalisation of the process. At face value, these are for the purpose of helping students remember the formula and hence apply it correctly. “Provide scaffolds” refers to adjustments within the notes to help students in the working steps – such as classifying the example set into categories of easy, intermediate, and challenging, and providing the space for re-writing of $y = 5$ into $y = 5x^0$ so that students can more easily recognise the form and apply the formula. “Check for errors” refers to a conscious process of correcting or anticipating the mistakes made by students – whether it is initiated by the teacher or through checks among students. The last category of “combine with other formulas” is interesting as the responder saw proficiency with this particular formula being accentuated when suitably placed within the context of other formulas: “usually teach some rules at the same time so that students can also be exposed to interleaving skills” (extract of the teacher’s response).

Concepts

33 responses from 15 schools are coded under this category. The sub-categories are presented in Table 4.

Table 4.
Responses under the “Concepts” category

Sub-categories	Number of responses
Develop formula from related concepts informally	10
Derive formula from first principles formally	12
Link application of formula to related concepts	11

The first sub-category refers to helping students see inductively how specific cases of functions – linear, quadratic, and cubic – result in the gradient functions (such as $x \to 1$, $x^2 \to 2x$, $x^3 \to 3x^2$), and then generalising to the formula. This differs from the second category in that “derive formula from first principles formally” refers to a more rigorous derivation process that starts from the definition of limits – and that the gradient of the function at a point is the 'limit case' of the gradients of chords emergent from that same point. While these two sub-categories were about development towards the formula, "link application of formula to related concepts" refers to how one ought not to stop at the application of formula to obtain the answer, but also to link it to the graphical interpretation of $\frac{dy}{dx}$ as finding gradient, particularly for the case of $y = 5$ where the link is most obvious. One respondent pointed out a link we found particularly interesting: "Include $(x^3 + 1)$ and its graph to discuss why it has the same gradient function as x^3. The use of different representations provides a visual of what the procedure could mean and imply" (Extract of a response to Section (3) of the page). That teachers attended to the need to make connections in their instructional materials is further strengthened by the relatively high mean score of 3.40 for the "Connection" page of the survey (see Table 1).

The Singapore Portrait

When we examine these findings in the light of Beng Choon's practice as described earlier, and especially with the features highlighted in Figure 1, we see that there are substantial overlaps: the second highest mean score for Item B, together with the principles for selecting examples in Table 2, indicate teachers' commitment to variation of examples (top right of Figure 1). The highest mean score for Item A on "start with easy …", coupled with the scaffolding techniques advocated in Table 3, point to how the teachers also attended to "cognitive load" considerations (top left part of Figure 1). Moreover, the emphasis in the techniques listed in Table 3 was in helping students gain proficiency in the application of the formula (bottom right of Figure 1). Finally, it is clear from the types of comments listed in Table 4 that a substantial number of teachers were concerned not just with formula-application but also with how it connected with related concepts in the topic (bottom left of Figure 1). In summary, the results of the survey show that the case of Beng Choon's design of instructional materials for formula recognition and application has representative utility to a large extent among Singapore secondary mathematics teachers.

But the results also revealed some principles that were not covered in Beng Choon's case. If Beng Choon is a portrait of basic practices in the design of instructional materials on formula recognition and application, then the other principles listed in Tables 2 – 4 can be viewed as embellishments and variations of the Singapore portrait. For example, the sub-categories in Table 2 can be interpreted as additional considerations in example-sequencing. Like Beng Choon, these principles (such as more cases of the same and more cases that are not the same) also target recognition of cases of the formula, but there were also emphases on checking of conditions for which the formula is applicable (such as the use of non-examples). Table 3 shows that the instructional materials do not by themselves fulfill the goal of student proficiency with the formulas – the teachers were committed to actively helping students to make good use of the materials to attain fluency. To us, the results in Table 4 are particularly illuminating. It challenges the still-popular view that teachers are either proficiency-focused or concept-focused—they can do both. Not only were the teachers concerned about linking both these foci whenever opportunity presented; they advocated a *deliberate* sequencing of instructional materials that would aid in students' development of conceptual understanding alongside formula recognition and application.

Interaction with related literature on example sequencing

As mentioned at the start of the paper, our interest in learning about example sequencing for formula recognition and application does not stop at the Singapore situation; it is instructive to compare the Singapore portrait with similar research done internationally.

For this purpose, we begin with the work of Zodik and Zaslavsky (2008) as their study overlapped substantially in nature and scope with ours – it was about choices of examples made by five secondary mathematics teachers. Even though their definition of "example" was broader than ours – as it included every type of object that would serve as "an example of … a larger class" (p. 170) – their study included also examples of a more general formula, which is the focus of the study reported here.

They identified six principles in order of decreasing frequency as utilized by the five "experienced" teachers: (a) start with a simple or familiar case; (b) attend to students' errors; (c) draw attention to relevant features; (d) convey generality by random choice; (e) include uncommon cases; and (f) keep unnecessary work to a minimum.

There are obvious similarities between these principles and those exemplified by Beng Choon (Figure 1) and the lists in Tables 2 – 4. (a) corresponds to the cognitive load principle, (b) is essentially the same as "check for errors" (Table 3), (e) is a special type of "include more cases" (Table 2).

For (c), the authors further clarified that strategies to "break the pattern" (p. 175) were used by the observed teachers. In particular, they presented a case of a teacher who wanted to illustrate polynomial functions by first beginning with a sequence of linear functions before "break[ing] the pattern" by introducing a quadratic polynomial to draw attention to the degree of the polynomial. This is very similar to the rewriting prompts given by Beng Choon (see Figure 2 Item (c) and implied in Item (d)) in her notes to direct students' attention to the exponents of the functions. The other sub-categories in Table 2 (such as the inclusion of non-examples) were also directed towards the goal of drawing students' attention to specific features (including the conditions for application) of the formula.

(d) was not a principle used by Beng Choon nor did it surface as a consideration from the teachers who participated in the survey. The authors described a case where a teacher wanted to convey the generality of the example by asking students to supply 'random' values for measures of two interior angles of a drawn triangle. She then proceeded to show that – implicitly, regardless of the actual values of these measures – that the exterior angle of the remaining angle has a measure that is the sum of these measures. We think this is a potentially powerful design principle in terms of drawing upon students' thoughtful contributions and in terms of helping them see the general in the particular. But, relating back to the context of the differentiation formula and application which was Beng Choon's focus (and hence, that of the teachers in the survey), this principle may pose significant challenges: in the case where students propose a 'random' function (say, $\sin x$), they would not have the required conceptual tools to evaluate whether and why the formula would work for this function. For this reason, Rowland, Thwaites, and Huckstep (2003) cautioned that it is critical to distinguish between cases in which it is appropriate to use this principle of choosing random examples and those that could be misleading or unhelpful. In fact, Zodik and Zaslavsky (2008) described such a case used by a teacher in their study where it "would not be considered a judicious choice for this particular situation" (p. 177) – in verifying that for quadratic equations of the form $ax^2 + bx + c = 0$, the sum of roots is -$\frac{b}{a}$ and the product of roots is $\frac{c}{a}$, the 'random' case of $2x^2 + 4x + 5 = 0$ results in complex roots which students have no tools at that stage to meaningfully deal with the roots for verification.

(f) refers to choice of example such that there is no unnecessary work that would distract the students away from the "essence" of the example. Zodik and Zaslavsky (2008) provided the case of how a teacher in the study chose $\frac{1}{7}$ to illustrate recurring decimals, specifying that it has a "long enough period" (p. 177), and that choosing other examples such as $\frac{1}{17}$ or $\frac{1}{19}$ would result in unnecessary work to illustrate the idea. We might say that this is theoretically part of the cognitive load principle in Beng Choon's case and reflected in Item A – as formal cognitive load theory deals with "extraneous cognitive load" that is not inherent in the task (van Merrienboer & Sweller, 2005). However, it would be a strain based on the data alone from this study to claim that Beng Choon or the teachers in the survey had a specific principle of design equivalent to (f). In particular, this principle 'forces' designers to be clear about the 'essence' prior to the choice of examples – meaning, to articulate the specific instructional aim for a set of examples. Thus, in this interaction with the work of Zodik and Zaslavsky (2008), the Singapore portrait has found a fresh useful input that can improve the design process.

We next go to research on "Worked Examples". They are examples of how specific skills are applied for a particular type of task (for example, in solving linear equations in one variable) where the solutions are worked out for students' reference. This is clearly a more specific type of examples than the one used in the study here; nevertheless, there is sufficient commonality – in terms of the common goal of learning a skill or formula for proficient application – for our reference. Research in this area is mostly conducted within the field of cognitive psychologists who have an interest in specific instructional implements that would result in improved achievements of subjects in specific strands of mathematical proficiency. One such implement is the use of – and the ways of use of – Worked Examples. Our main reference for this purpose is from Atkinson, Derry, Renkl, and Wortham (2000) as they provided a synthesis of research on the design of Worked Examples within this tradition.

On intra-example features, useful instructional principles include the need to integrate different modes of information, such as diagram, text, and symbols, in a form that is easily accessible to students; however, when the example is too complicated, there is tendency for cognitive overload. In such cases, the example presentation should be accompanied with explicit methods of directing students' attention to pertinent features of the task and solution(s).

On inter-example features, the findings favoured the use of more than one example to illustrate a target formula for application; however, excessive varying of examples along multiple dimensions can lead to cognitive overload. The recommendation was that, for a set of examples illustrating a common formula application, a common problem structure such as a unifying cover story be used. As to the sequencing of practice examples and demonstrated examples, the interspersing of examples throughout practice produced better outcomes than lessons in which a blocked series of demonstrated examples is followed by a blocked series of practice examples.

Much of these features overlapped with the sub-categories listed in Table 3. For instance, the intra-example consideration of explicit references especially in the case of more complicated examples is reflected by "explicit demonstration by teacher" and "provide scaffolds in notes"; and, the inter-example feature of having a unifying cover story is demonstrated by Beng Choon's use of an overriding formula that covers the cases of all the stated examples (see Figure 2).

At the same time, we can draw upon useful specific principles that did not surface in the Singapore portrait: the deliberate insertion of diagrams and alternative notations into the example sequence (in the case of differentiation, especially for $y = 5$, the helpfulness of inserting the graphical representation apart from the symbolic representation); and careful weaving of worked examples between student exercises. However, we should also heed the warning against overloading the example sequences with multiple mathematical aims – which can lead to cognitive overload. This point relates back to Zodik and Zaslavsky's (2008) principle of "keep[ing] unnecessary work to a minimum".

Finally, we refer to the literature that stresses the importance of systematic variation of examples. The theoretical streams that contribute to this focus on variation (and invariance) are traceable by Pang, Marton, Bao, and Ki (2016) to the Variation Theory developed by Swedish and Hong Kong scholars, and the Chinese theory of variation which originated from Gu (1991). We take our reference mainly from Pang et al. (2016), as it set out to “illustrate what systematic use of variation and invariance might entail” (p. 458), which coheres with the focus of this study. As our interest is on the principles that can be applied into the design of examples, we will not delve here into the contributing theories.

Their study involved a close examination of two experienced teachers – one in Hong Kong and the other in Shanghai – in the teaching of a common topic: the addition of three-digit numbers at the Primary levels. For our purpose, we focus on the design of the examples by the teachers. In brief, the Hong Kong teacher designed a series of 5 worksheets that provided a very gradual progression, beginning first with 3-digit numbers that are whole-hundreds, making very incremental variation (while keeping others constant) across each worksheet until the last worksheet deals with the usual addition of 3-digit numbers, including the “carrying over”. The Shanghai teacher started with examples of comparisons of additions (such as 20 + 4 versus 20 + 40) to draw attention to the significance of place value. This was seen as an example of conceptual variation using the strategy of keeping some things the same while varying what is critical. The rest of her instructional work subsequently (which is of less relevance to our study here) was on the application of this place value awareness to a ‘problem’ of adding 247 and 335 using a variety of procedures (known as procedural variation). The authors observed that though the ostensible “object of learning” (that is, the addition of three-digit numbers) were the same, but due to their focus on different “critical aspects” of this object of learning, different dimensions and hence patterns of variation emerged – for example, the Hong Kong teacher focused on the link between different representations (the numeric and the diagrammatic pattern blocks), while the Shanghai teacher focused on different methods of thinking about three-digit addition.

This talk on “critical aspects” reminds us of Zodik and Zaslavsky’s (2008) “essence”. This means that, prior to the sequencing of examples, the designer is required to study carefully the essence or critical aspects of the formula –

its recognition and application. Unnecessary work that is not of essence should be avoided in the examples; but effort should then be trained on systematic variation (and invariance) on critical aspects of the formula so that students would be able to discern them.

Relating it back to the Singapore portrait, there is no doubt a conscious awareness among both Beng Choon (Figure 1 and Figure 2) and the teachers in the survey (Table 2) to provide variation in example choice and sequencing. It is however, not clear how systematic – in terms of the identification of critical aspects – the variation is conducted. One may purport that if it was indeed systematic, teachers would have considered varying the variable instead of sticking with y and x throughout – and so vary along this dimension by inserting, for example, "$u = v^4$, find $\frac{du}{dv}$". Such an analysis of critical aspects is useful in the overall planning of the instructional materials – even if the teacher should judge the insertion of such examples as cognitive overload at this stage, it can be kept in mind for inclusion at a more suitable juncture later.

Discussion: Towards a framework for professional development

We started our study by presenting the case of Teacher Beng Choon's design of example sequences for the purpose of helping students with formula recognition and application (as summarised in Figure 1). From the results of the survey, we found that her case had strong resonance in the practice of mathematics teachers from a range of Singapore schools. We also found "embellishments" to Beng Choon's case (as summarized in Tables 2-4). For convenience, we say the picture of design up to that point of investigation provides the 'Singapore portrait'.

To us, the value of the Singapore portrait does not lie merely in its answer to the pertinent question, "How do Singapore teachers design example sequences to teach formula application?" We also see it as an enterprise to tap upon the "wisdom of practice" (Shulman & Wilson, 2004) which we assume is located (perhaps deep) within the routines of practice of Singapore teachers. We see our research as a way to unearth these riches so that we can translate it into a form suitable for professional development for (especially novice) mathematics teachers. Currently, we do not know of any professional development work directed at example sequencing. This study is a first step at providing "a sound basis for designing teacher education programs that better prepare secondary mathematics teachers for judicious choice and use of examples" (Zodik & Zaslavsky, 2008, p. 168).

But we do not think such a "sound basis" for professional development should be derived exclusively from within-Singapore investigations, however culturally near and appropriate it would be – as it can also result in insularity from practices in other parts of the world. To avoid this undesirable development, we propose to incorporate these other useful principles drawn from the international literature – insofar as they fit well into the broad structure of the Singapore portrait. Thus, we conclude this paper by presenting our proposal of a Singapore-based framework for professional development in example sequencing for teaching formula application (as summarized in Figure 3).

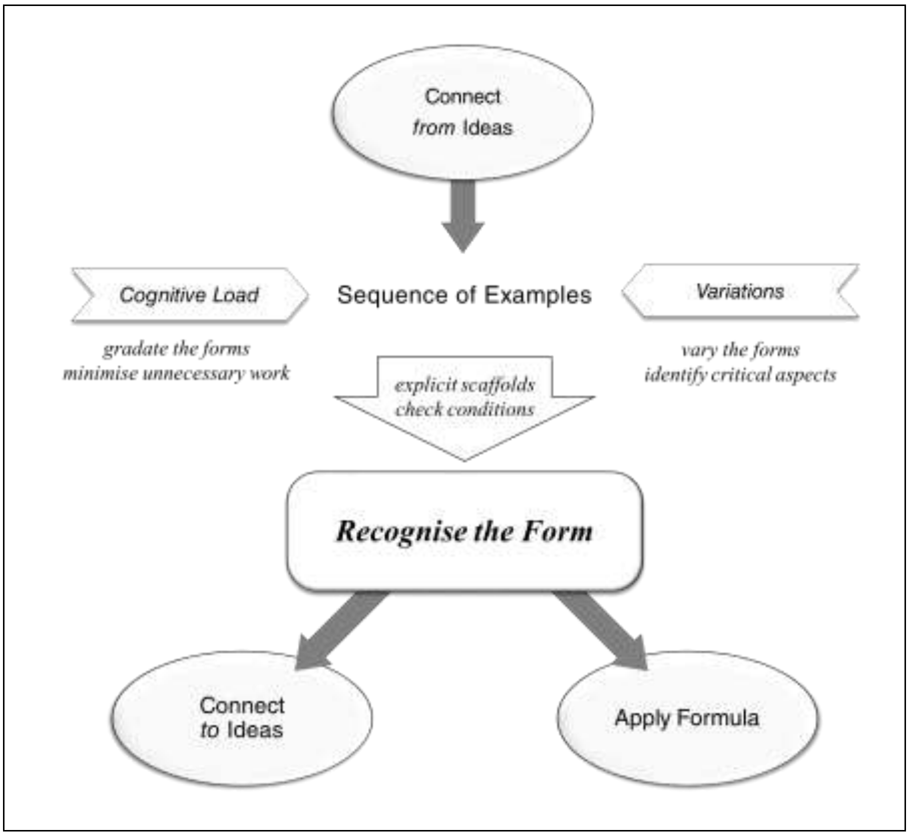

Figure 3. Singapore-based framework for professional development in example sequencing

Clearly, Figure 3 is an adaptation of Figure 1. We made the following refinements: (a) we included "connect *from* ideas" at the top to indicate the commitment by a number of Singapore teachers towards conceptual development leading to formulas (see Table 4). This commitment towards intertwining concepts and skills has widespread support from the international literature (e.g., Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; Hurst, & Huntley, 2018; Raveh, Koichu, Peled, & Zaslavsky, 2016); (b) we added "minimize unnecessary work" under "Cognitive Load" on the left and "identify critical aspects" under "Variations" on the right (as well as inserted the word "systematically" in front of "vary the forms"). These additions are respectively from the work from Zodik and Zaslavsky (2008) and Variation theorists we reviewed in the previous section. These two principles should also be seen as held in tension: focusing on the core necessities for a particular formula may means a curtailing of the number of critical aspects (hence dimensions of variations) that the set of examples can afford; (c) Based on the views of the teachers in the survey, we have inserted the facilitation arrow to include "explicit scaffolds" and "check conditions" – which we think capture the sub-categories listed in Table 3 and Table 2 respectively that are not already stressed in Figure 1; (d) "connect ideas" is refined as "connect *to* ideas" first to distinguish it from "connect *from* ideas" inserted at the top of the diagram; also, it better reflects this move as located after and linked to the application of the formula – as was the case in Bin Choo and as described by the teachers in the survey under the sub-category of "Link application of formula to related concepts" in Table 4.

Further research is needed to test this framework. This includes a tweaking of its features to suit the purposes of professional development work according to the needs of the mathematics teachers. Another area that requires validation is whether an explicit utilisation of the principles within the framework would indeed result in better outcomes for teacher learning and student proficiency.

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