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Teaching and Learning with Concrete-Pictorial-Abstract Sequence – A Proposed Model

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Abstract: The Concrete-Pictorial-Abstract (C-P-A) sequence is a key instructional strategy for the development of primary mathematics concepts in Singapore. However, the way to go about teaching and learning with the C-P-A sequence is unclear. As a result, the benefits of this sequence cannot be fully capitalized. This paper aims to (1) expound from learning theories of Bruner, Dienes and Piaget, and literature on representations, how learning takes place with representations and (2) using the insights gleaned, propose how teaching with representations looks like. It concludes with segments of a series of classroom lesson plans crafted using the proposed model of teaching and learning with representations for the concept of equivalent fractions in Primary Three.

Keywords: Concrete-Pictorial-Abstract; Representations; Equivalent Fractions

Introduction

The Concrete-Pictorial-Abstract (C-P-A) sequence is the key instructional strategy for the development of primary mathematics concepts in Singapore (MOE, 2007, 2012). This sequence is evident in both the National Syllabus and varied textbooks adopted in schools. Figure 1 shows two examples of this sequence being employed in one of the Primary Three mathematics textbooks.

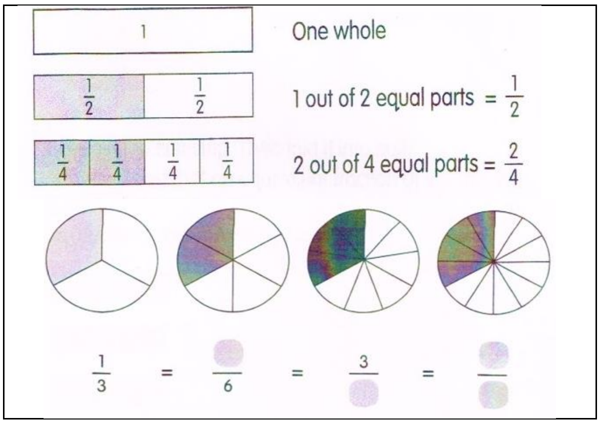


Figure 1. Examples of C-P-A sequence in one Singapore Primary Mathematics textbook.

In the textbook, pictorial representation (P) in the forms of rectangular and circular models are used to introduce the concept of equivalent fractions (i.e., $\frac{a}{b} = \frac{c}{d}$). The parts of the area in the rectangular or circular models, representing respective fractions, depict the mathematical idea of equality. The child learns that $\frac{1}{2} = \frac{2}{4}$ because the parts of the area in the rectangular or circular models they each represents is equal. With more exposures and practices with similar pictorial depictions, the child eventually makes sense of the concept of equivalence between and among fractions. Soon a reduced reliance on pictorial representations is observed while the operation with mathematical symbol or abstract (A), $\frac{a}{b} = \frac{c}{d}$ is more prevalent. The child is said to have successfully undergone the pictorial to abstract (P-A) sequence and is deemed to have acquired the concept of equivalent fractions.

Apart from abovementioned pictorial representations in textbooks, concrete representations (C) such as fraction discs and fractions strips are provided by the Ministry of Education (MOE) to all schools in support of C-P-A instructional strategy. The Ministry aims, through the use of different representations and teacher's guidance, to enable children to uncover abstract mathematical concepts by constructing meanings and understandings (MOE, 2012). However, this process from initial exposure of representations to eventual acquisition of the mathematical concept is not explicated in the document and in local literature. The guidance to be offered by a teacher to facilitate a child to progress through the C-P-A sequence is not spelt out. It cannot be assumed that mere presentation of such representations during mathematics lessons or the execution of the learning sequences illustrated in textbooks will lead to an automatic acquisition of the desired mathematical concepts. Without an understanding of the purpose and function of external representations and the learning process a child undergoes during the C-P-A sequence, teachers risk the possibility of underutilizing external representations (Flevares & Perry, 2001) or employing them as motivational rewards and games (Moyer, 2001). On the contrary, if teachers become aware of the underlying principle and mechanism by which external representations can enable a child to acquire mathematical concepts, they can provide appropriate guidance to a child and optimize the use of external representations.

The purpose of this paper is to offer some theoretical exposition of the learning process that unfolds during the C-P-A instructional sequence and to propose a model of the accompanying teaching facilitations that guide a learner to achieve mathematical conceptual understanding. Through this model, teacher can assign a learner's learning process experienced during the C-P-A sequence to the proposed stages and thereby assess the conceptual development of the child. The explication of the stages allows teachers to identify and monitor where the child is in the learning process and thereafter prescribe follow-up actions to bring the child towards mathematical conceptual acquisition. This paper follows a discussion of learning with external representations from which the proposed model is conceived and an illustration of its implementation in a Singapore Primary Three classroom on the teaching of equivalent fractions. To facilitate the discussion, concrete (C) and pictorial (P) in the C-P-A sequence shall henceforth be regarded as external representations and abstract (A) is regarded as mathematical symbols.

Concrete-Pictorial-Abstract Sequence

The C-P-A instructional sequence has its roots from Bruner's proposition of enactive, iconic and symbolic representations of cognitive growth (Leong, Ho & Cheng, 2015; Teng, 2014; Wong, 2015). According to Bruner (1964, 2006), conceptual learning begins with an experience from actions undertaken (enactive) that was subsequently translated into images of the experience formed (iconic). With an accumulation of enactments and their corresponding iconic representations, links are formed to connect some of the representations into a collective structure. The criteria for selection into the collective structure is governed by a certain rule derived from organizing common attributes found embedded in those qualified representations. This rule is the piece of information that informs what one representation and those in the collective structure mean. Eventually this rule ascends above the enactive and iconic representations to stand exclusively by itself and is denoted by a symbol (Bruner & Kenney, 1965). Bruner's proposition is coherent with other theorists such as Piaget and Dienes, who believed that the learning passage starts from active engagement or experiences with concrete situations (Bart, 1970; Dienes, 1971; Ginsburg & Opper, 1988; Wadsworth, 1984) before moving to a recognition of an invariant property among the concrete experiences (Skemp, 1987). Dienes recommended three stages for concept learning beginning with unconscious play in concrete situations (construction) followed by the realization of something meaningful among the plays (transition from construction to analysis), ending with a moment of insight and understanding into the meaningfulness (analysis). According to Bart (1970), these stages parallel that of Piaget's developmental stages:

> To Dienes, the Piagetian notion of cognitive development being ordered about three stages – the sensorimotor, the concrete operational, and the formal operational is mirrored in the formation of every concept. (p. 360)

A common thread among the three theorists is an existence of parallel stages, analogously sequenced from concrete to abstract though they may differ in each specific. Bruner & Kenney (1965) provided a similar guideline for mathematical learning:

> "The problem sequences were designed to provide, first, an appreciation of mathematical ideas through concrete constructions using materials of various kinds for these constructions. From such constructions, the child was encouraged to form perceptual images of the mathematical idea in terms of the forms that had been constructed. The child was then further encouraged to develop or adopt a symbol to describe his construction." (p.51)

With increased exposure to multiple materials and their corresponding perceptual images of the mathematical idea and its adopted symbol, the learner is led to a point of total "emptying" of the multiple materials and their perceptual images to function solely with the adopted symbol. The learner has not only arrived at a point of acquiring the mathematical idea in its mathematical symbol, but has also developed a storage of its concrete images.

In short, the learning sequence of a new mathematical concept follows through a progression from an enactment on concrete objects or an experience, to perceptual images of both the enactment of the concrete objects and experience, to the adoption of the mathematical symbol. In this paper, a distinction is made between such mathematical learning and an arrival at the state of learned. The state of learned is the point of independence from enactive and iconic representations with operations mainly of abstract symbols (A). This independence from the enactive and iconic representations does not mean that these varied representations are forgotten or discarded. The stage of learned basically serves as a landmark for teaching and learning to signify the arrival at a point in the learning process where learner can proficiently use abstract representations at ease in outward written and verbal expressions. To achieve the state of learned, a continuous exposure to multiple concrete objects is necessary in order for extensive linkages to be formed among the corpus of stored enactive and iconic representations. The transition from mathematical learning to learned is unclear and in the words of Bruner (1964) the "greatest thicket of psychological problems" (p.70). But clearly, mathematical learning process involves first, a progression from enactment to iconic formation to symbol adoption and second, a continual exposure to enactment-iconic-symbolic cycle leads to a formation of links among the representations into a collective structure. It is when the mathematical symbol becomes the dominate representation in this collective structure that the state of learned is reached.

This enactive-iconic-abstract progression centered on Bruner's work is expounded from the perspective of learning. That is, the progression described how a learner functions in the process of cognitive growth. Singapore's C-P-A instructional sequence on the other hand, is crafted from the perspective of teaching. That is, the way teachers are recommended to tailor their instructional sequence, beginning with the provision of concrete representations of the mathematics concepts, to the pictorial representations before proceeding to the abstract symbol. Nonetheless, they are essentially two sides of the same coin. Both teaching and learning has a common goal – the acquisition of the desired mathematical concept by the learner. How teaching take place essentially should be tailored accordance to how learning takes place. In another words, "teaching is subordinated by learning" (Gattegno, 1987). But the current inadequate understanding of the C-P-A instructional sequence limits teachers from maximizing the use of different representations.

The learning theories discussed earlier provided insights that learning using different representations goes beyond a learner's experience of manipulating representations. It includes making different forms of connections between the physical representations and corresponding images formed. The role of the teacher is not confined to a mere selection of appropriate representation. More importantly is the facilitation of the teacher at each learning stage of the C-P-A instructional sequence. Together with the literature examined and the reflections of the author as a teacher, an attempt is made to unpack the probable accompanying teaching actions that corresponds to the learning process (see Table 1).

Table 1
Learning process and its corresponding teaching sequence

Learning Process	Teaching Actions
Enactive (Manipulation of concrete object/ an experience)	Selection of concrete objects to be manipulated, and Facilitation of the act of manipulating
Iconic (Perceptual images of forms of Mathematical idea)	Facilitation of making connection of iconic form and mathematical idea
Symbolic (Abstract symbol)	Facilitation of making connection to abstract mathematical symbol

The objective behind episodes of enactments with concrete objects is to first develop an awareness and appreciation of the mathematical idea associated. The required instructional facilitation therefore can be seen in twofold; the choice of a suitable concrete object to depict the mathematical idea (see section on External and Internal Representations) and how learners are to be led from the act of manipulating with the concrete object to the awareness and appreciation of the mathematical idea associated (see section on Interactions). As learner proceeds to the iconic level, continual references to perceptual images in the mind and its association to the mathematical idea facilitate the formation and strengthening of linkages in the schematic structure (see section on Interactions), until the eventual adoption of the desired abstract mathematical symbol. Teacher facilitation during learner's process of association-making and the eventual leap from concrete mode to abstract mode of representation is not explicated in the C-P-A instructional sequence. There is essentially an "insufficient attention paid to the way in which this enactive experience is related to the symbolic representation of that experience" (Post, 1981). So prior to the exposition of teaching facilitation, the notion of representations observed in the perimeters of teaching and learning requires clarification.

External and Internal Representations

Extensive research (Goldin, 1998; Janvier, 1987; Kaput, 1987; Kaminski, Sloutsky & Heckler, 2009; Lesh, Post & Behr, 1987; Moyer, Bolyard & Spikell, 2002; Wong 1999, 2006) studied the range of external representations in varied forms of concrete and iconic adopted during instructions. Areas examined primarily fall under the system of representations which includes virtual manipulatives of the 21st century technologies, and quantity of representations such as the use of one versus the use of more than one representation during instructions for the learning of either mathematical concepts or mathematical skills.

Issues on the inclusion of varied types of representations and their categorization are not the focus of this paper. But an understanding of representations in the forms of external or internal is salient. External representations refer to physically embodied, observable configurations (Goldin & Kaput, 1996). External representations of fractions in this paper include manipulatable fraction discs, fraction strips, circular and rectangular pictorial models, math symbols and spoken language. Internal representations refer to mental configurations of a child (Goldin & Kaput, 1996). Such representations are objects of introspection from within the child. They are observed, deduced and inferred through the child's gestures, actions, behaviour and verbal utterances. For instance, the concrete fraction discs that a child manipulates as an external representation will essentially be stored, in the child's mind, as a corresponding internal representation, in the form of a perceptual dynamic imagery of physical act on the external representation (Goldin, 1998). The concrete fraction discs are external representations whereas the perceptual image of physical act resides in the child is an internal representation. However, this perceptual dynamic image of enactment can be harnessed as an external concrete recollection in the form of experience-based script (Lesh, et. al., 1987) evoked by the teacher to start the lesson the next day. In this instance, the explicit use of the recollection as part of the planned instruction brought the internal recollection into visibility as an external representation. Another similar example is learner's prior knowledge of equal sharing harnessed during the introduction of the concept of fractions. Prior knowledge of sharing resides as part of a schema, an internal representation, of the learner. The teacher brings to foreground this prior knowledge as an instructional tool or more accurately, as an external representation to relate to the concept of fraction. In both cases, the switch of representations in the learner's mind from internal to external representation is orchestrated by the teacher.

Though issues on the selection of appropriate external representation are important, the author is more concern with the mechanism that leads external representations to support the learning progression from enactive to iconic and symbolic. The mechanism supporting the learning progression is posited in terms of interactions. During episodes of learning, many interactions occur. For example, in one simplistic classroom scene, the learner handles and manipulates the external representations of various forms, the learner listens to the teacher who is the more knowledgeable one (Vygotsky, 1978), the learner engages in social discourse with his classmates in class (Cobb, Yackel & Wood, 1992; Pape & Tchoshanov, 2001; Hershkowitz, Hadas, Dreyfus & Schwarz, 2007; Ozmantar & Monaghan, 2007) and the learner makes sense and build meaning within himself (Thomas, Mulligan & Goldin, 2002; Simon, Tzur, Heinz & Kinzel, 2004). While the interactions among the learner, teacher and classmates are important, it is not within the scope of the paper. The focus of this paper is to explicate the possible interactions that take place when a learner, individually, manipulates external representations, makes sense of the external representations and constructs mathematical knowledge cognitively. Nestling on the theoretical foundations of Bruner, Dienes and Piaget's cognitive development, a micro-level analysis of the interactions in terms of external and internal representations is proposed next.

External and internal representational interactions

The representations denoted in the C-P-A instructional sequence refers to those, in the perspective of teaching, externally embodied. However, for each externally representation presented, there exists, from the learning perspective, a mapping of a corresponding internal imagery of the physical object for manipulation and/or internal imagery of the act of manipulation, residing in the learner. This is a type of representational interaction (see Figure 2), between external representation (fraction discs) and internal representation (imagery of fraction discs and imagery of the act of manipulation). Such image having (Pirie & Martin, 2000) forms the basic cognitive building blocks in the process of mathematical learning and lays the foundation to the initial formation of a collective structure known as schema (Ginsburg & Opper, 1988; Hatano, 1996; Sweller, Merrienboer & Paas, 1998; Wadsworth, 1984).

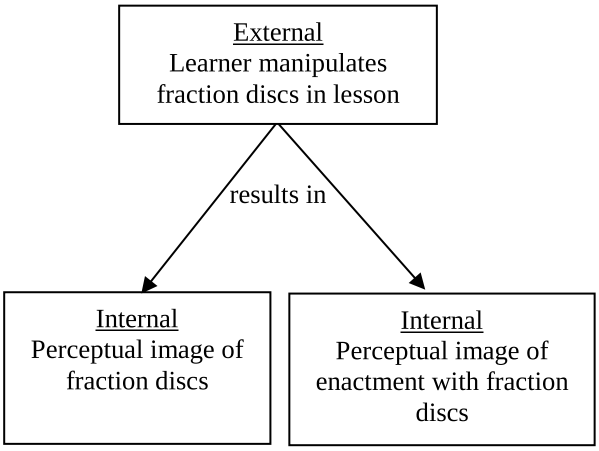


Figure 2. External-Internal Representational Interactions

The reverse, internal-external representational interaction is plausible (see Figure 3). For example, the learner applies experiences of the act of manipulation on fraction discs represented internally to a set of concrete fraction strips physically presented in class. In so doing, this internal-external representational interaction leads to another external-internal representational interaction, where the perceptual image of the enactment of fraction strips is formed internally. Each internal representation formed is liken a knowledge node in the mind. Repeated manipulations of the external representations lead the learner to become aware of the vertical relationship between the external objects and the act of manipulating (Goldin & Kaput, 1996). Internally, a connection between the individual knowledge nodes is established (see section on Internal-Internal Representational Interactions). Continuous manipulation strengthens the internal connections and results in a compression (Gray & Tall, 2007) of the related knowledge nodes into a single collective schema that can act as a single element in the working memory (Sweller, et. al., 1998).

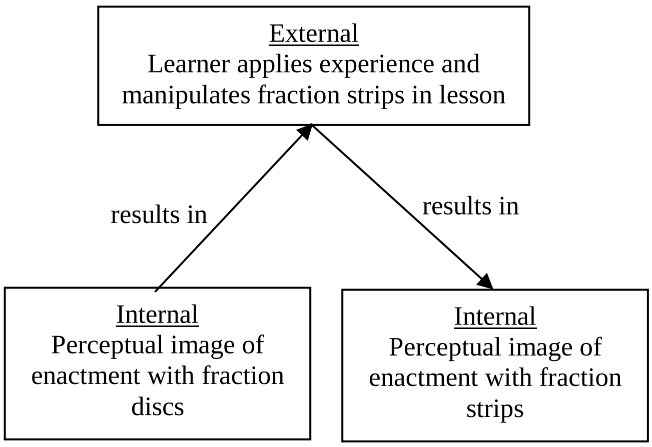


Figure 3. Internal-External Representational Interactions

In a nutshell, the enactment of external representations must be accompanied with opportunities for both forms of vertical interactions (i.e., external-internal and internal-external) to occur so as to bring about the formation of initial basic knowledge nodes that contain enactment experiences and images of the external representations.

External-external representational interactions

The abovementioned compression may not happen should the learner fail to see that both the enactment of fraction discs and strips are related. This relationship can be established through an awareness of a "horizontal" relation (Goldin & Kaput, 1996) that is brought about by external-external or internal-internal representational interactions.

External-external representational interactions can be illustrated through three different examples (see Figure 4). First, when a learner manipulates a set of fractions discs (C) to complete a given task of comparing two given fractions (A) to determine their equivalence. An external-external interaction occurs when learner picks the corresponding fraction discs of the respective given fractions for comparison to be made. The same can be said of pictorial bar models (P). Second, when learner is tasked to show a given fraction using both fraction discs (C) and fraction strips (C). This interaction between both physical objects is what Lesh et. al. (1987) coined as transformation. Third, when learner colours in the number of parts in a rectangular or circular model (P), required to fulfil the equivalent criteria of a given fractions in the workbook after having enacted the manipulating experience with fraction discs (C). This interaction here includes concrete fraction discs and pictorial models. A translation (Lesh, et. al., 1987) between two different modes of representation is required.

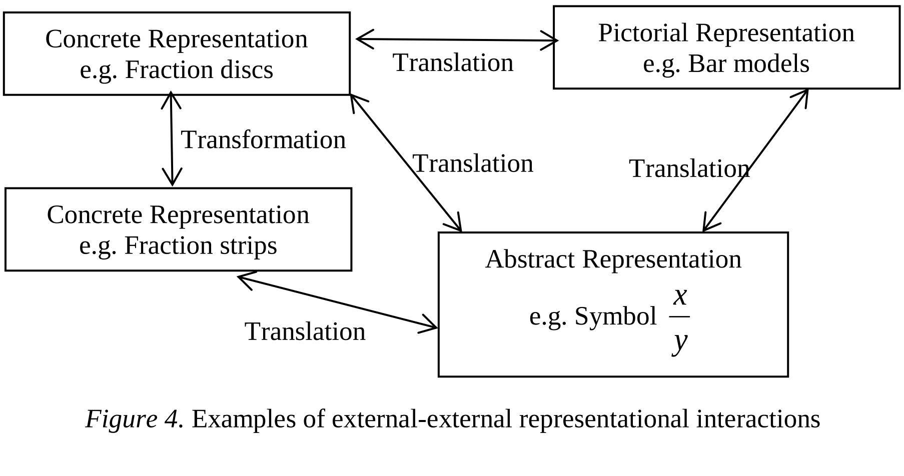


Figure 4. Examples of external-external representational interactions

Schwonke, Berthold & Renkl (2009) in their collection of students' gaze data observed that students made frequent transitions between representations when learning and concluded that more extensive visual processing of physical diagrams (external pictorial-external pictorial) was related to better conceptual understanding of concept. Hiebert (1984) suggested that it was not the use of external representation that improves mathematics understanding but the processing of both actions on the representation (C) and related mathematical symbols (A). The presentation of varied representation is only part of the learning equation. Saxe, Taylor, McIntosch & Gearhart (2005) mentioned that students need to make connections among mathematical representations and symbols because knowledge for the representations and mathematical symbols can develop independently leading to rote and mechanical learning rather than the desired conceptual understanding. It is important to create opportunities for learners to process the representations by examining the relationship derived from external-external representational interactions (Ainsworth, Bibby & Wood, 2002; Janvier, 1987; Lesh et. al., 1987). It is the derivation of the desired relationship that move learner forward in the learning process.

The relationship derived from external-external representational interactions is brought about by the process of empirical abstraction (Ginsburg & Opper, 1988; Skemp, 1987). During this process, an invariant attribute among similar external representations is observed and identified. This invariant attribute abstracted serves as a reference that points to one or more aspect of a specific mathematical concept that a teacher intends for learners to learn (Ainsworth, 2006). The objective of this preselected aspect of the representing world (i.e., invariant attribute) is to represent externally, a certain aspect of the represented mathematical concept (Dufour-Janvier, Bednarz & Belanger, 1987; Palmer, 1977). It appears invariantly in several examples of the employed external representation for the learner to identify and abstract (Skemp, 1987; Ohlsson & Lehtinen, 1997). An external-external representational interaction facilitates the abstraction of that invariant attribute. This abstraction is prerequisite for learning and "serves as a front-end process that furnishes the mind with low-level, initial abstractions which serve as building blocks for more complex abstraction" (Ohlsson & Lehtinen, 1997, p. 44).

In comparison with external-internal or internal-external interactions discussed in the previous section, the end-product here is not just corresponding perceptual images or replicas of the reality but a "new" knowledge birthed from studying the relationship between or among external representations. This "new" knowledge shall be referred to as mathematical idea in this paper. This "new" knowledge frees the learner from a reliance on empirical references or actual manipulation of physical objects and facilitates subsequent higher-order mental processing required during the organization and reorganization of the schematic network (Ginsburg, 1988, Ohlsson & Lehtinen, 1997). In short, learner no longer has to enact the act of manipulation to retrieve the mathematical idea embedded in the physical object. This knowledge (mathematical idea) is embodied in a knowledge node, grouped with the previous schema comprising of perceptual images of the enactment and physical objects and is readily retrievable and applicable to other situations and experiences.

Internal-internal representational interactions

Another horizontal relationship is internal-internal representational interaction. Internal representations are mental configurations in the learner and an occurrence of an internal-internal interaction is inferred from forms of behavioral traces that relate to and correspond with external-internal and external-external interactions. There are two significant occurrences of internal-internal representational interactions proposed.

The result from an external-internal representational interaction is the formation of a perceptual image of the external representation. After manipulating External Representation A, a perceptual image of it (Internal Representation A) is formed in the mind of the learner. Subsequently, the learner manipulates External Representation B and similarly, forms a perceptual image of it (Internal Representation B). Both perceptual images (Internal Representations A and B) resides in the learner's mind as two isolated and unrelated knowledge nodes. As the learner reflects on the two separate learning episodes under the guidance of the teacher and classmates during classroom discourse, he is led to derive a relationship between both external representations in the form of a mathematical idea. It is mentioned earlier (See section on External-external representational interactions) that the derivation of the relationship is brought about by empirical abstraction and is a result of examining both forms of external representations rather than its internal counterparts. This explanation is incomplete. This empiricist form of abstraction where an invariant attribute is abstracted out from the varied external representations has long been argued as inadequate in explaining the learning process especially for concepts that are totally alien and unknown to a learner, who has no awareness of what attribute to pay attention to or identify (Ohlsson & Lehtinen, 1997; Ozmantar & Monaghan, 2007). On the contrary, in addition to the empiricist thought of simple identification, a deeper theoretical thought, not available by sight, must be activated to establish a relationship through analysis and synthesis (Davydov, 1990; Ohlsson & Lehtinen, 1997). In so doing, a new knowledge node known as the mathematical idea is constructed followed by formation of connections among the two originally isolated knowledge nodes of Representations A and B and mathematical idea. Collectively, all knowledge nodes are assimilated and organized into a schema.

While learner undergoes empirical abstraction via the external-external representational interaction, a corresponding internal-internal representational interaction is taking place in the mind to analyze the embodiment of the invariant attribute presented in the external representations in relation to the mathematical idea (see Figure 5). Thereafter, with the establishment of a relationship that ties up the various elements, all knowledge nodes are synthesized into a meaningful structure. The entire process of empirical and theoretical thoughts is complex and facilitated by both sociocultural discourse in the classroom and private individual reflection in the learner because it is recognized that "on one hand, the individual is the actor and the mathematical knowledge is constructed by the actor. On the other hand, the individual is an object of cultural practices, and given mathematical knowledge is internalized" (Voigt, 1994, p.187). The paper does not undermine or is inclined to the contribution of either aspect but has its primary focus on the role of teacher in facilitating the private reflection in the learner (see Section on Teaching Framework). The behavioral traces of a successful internal-internal representational interaction are therefore, the learner's exhibition of an ability to execute transformation and translation which is often regarded as a hallmark of deep conceptual understanding.

Referring back to the case of the concept of equivalence (see Introduction), an ideal schematic structure comprised of interconnected knowledge nodes of perceptual images and experiences of manipulation of fraction discs and strips, rectangular pictorial models and the mathematical idea "same size". A learner who has learnt the concept is able to adopt the abstract symbol of $\frac{a}{b}=\frac{c}{d}$ though it is possible that he or she may make frequent references in the mind with the associated concrete or pictorial representations. Nonetheless, the proposition thus far is only sufficient to explain learner's ability to function with concrete and pictorial representations (C-P), leaving the last link to abstract representation to be addressed.

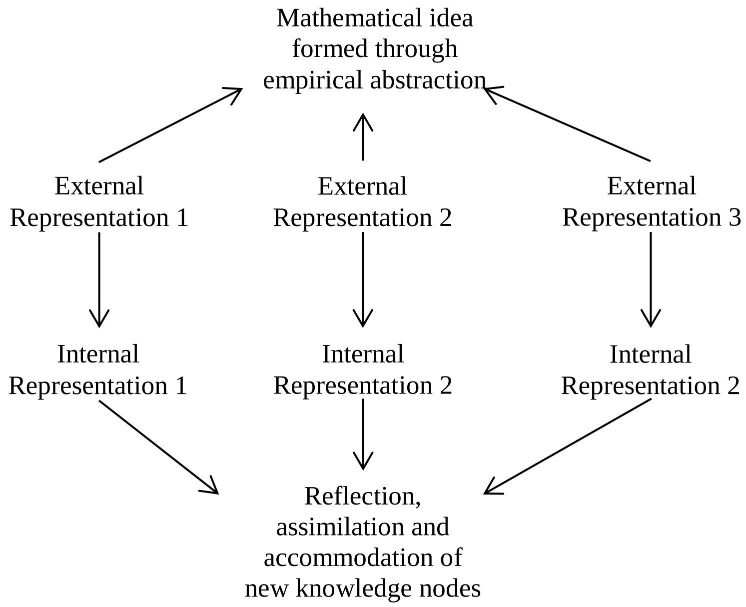


Figure 5. Internal-internal representational interactions

The second proposed occurrence of an internal-internal representational interaction is between the newly organized schema with an existing related schema (see Figure 5). Mathematical concepts are hierarchical in nature and in Singapore Mathematics Curriculum, they are introduced in a spiral manner (MOE, 2007, 2012). The concept of fractions is first introduced in Primary Two where fractional abstract symbol is defined through the mathematical concept of parts and whole. In Primary Three, the concept of fraction is extended to the concept of equivalent fractions where two or more given fractions can be equivalent through the mathematical idea "same size". Initial formation of fraction schema at Primary Two comprised of the identification and interpretation of parts and whole to form the basic fraction abstract symbol. The newly formed schema of concept of equivalence at Primary Three ought to be assimilated into this initial fraction schema of Primary Two to become one larger and more complex schema. During the reorganization process, the learner is required to engage in theoretical thoughts in order to make adjustment or accommodation in previously structured schema to assimilate the newly learned concept (Ginsburg & Opper, 1988; Sweller et al, 1998). Such demands are not merely at the level of abstraction but across levels of complex interactions (Ohlsson & Lehtinen, 1997). A failure to achieve this level of internal schematic reorganization may result in a learner functioning proficiently only at the transformation and translation between concrete and pictorial representations and unable to reconcile with or handle the abstract symbol $\frac{a}{b} = \frac{c}{d}$ and its algorithm subsequently.

C-P-A Sequence in the Light of Representational Interactions

In associating and redefining Bruner's learning stages of Enactive, Iconic and Symbolic from a learning perspective to a teaching perspective of Concrete-Pictorial-Abstract, it is evident that teaching and learning with external representations encapsulates many important interactional processes that is not spelt out in the original documents. Teachers' instructional design should include opportunities for external-internal interactions to facilitate image formation, external-external interactions to facilitate empirical abstraction of mathematical idea and internal-internal interactions to facilitate schematic connections and organizations.

The sequence of occurrence of the various interactions is not linear and highly fluid. Each interaction may happen simultaneously and recursively. Although initiated and guided by the teacher, occurrence and success of each interaction is varied and dependent on individual differences. Concrete and abstract representations (C-A) are presented simultaneously at onset to stage external-external representational interactions. Likewise, pictorial representations are presented with abstract representations (P-A). In fact, an array of combinations of C-P-A sufficed; C-A, A-C, P-A, A-P, C-P, P-C. Therefore, the C-P-A sequence in reality, should more accurately be defined as a concurrent presentation of all three representations in the classroom where the learner manipulates the concrete fraction discs, indicates the result by shading the pictorial circular models in workbook practices and writes down the mathematical statement using abstract symbols.

With this backdrop, the next section of this paper attempts to spell out the proposed teaching facilitations to equip teachers to carry out instructional lessons involving external representations to impart mathematical concepts. The teaching facilitations proposed aimed to support the representational interactions explicated in the earlier section.

The Teaching and Learning Model using C-P-A

This section introduces the proposed model and its descriptions in three parts (i) the learning goals for the learner explained through the backdrop of the proposed learning processes discussed in previous sections of this paper, (ii) the schematic connections of focus to track the learner's learning progression and (iii) the teaching intent for teachers to anchor their instructional design. An example from a classroom instruction on Primary three concept of equivalent fractions is provided to illustrate the application of the model.

The model is made up of four stages namely Guided Explication, Exploratory Familiarization, Knowledge Classification and Concept Reification. Each stage includes a teaching intent that emulates closely to specific learning goals expounded in the earlier section and the desired schematic connections to be established (see Table 2). The objective of the model is to serve as a guide for teachers using the C-P-A instructional sequence to tailor their instructional support to facilitate learner's learning process.

Table 2
Stages in proposed model

Stage	Learning goals	Representational interactions	Teaching Intent
Guided Explication	• Formation of knowledge nodes involving external representations and experiences of manipulation	External-Internal	• Provide step-by-step guidance to manipulate external representations
	• Achieve empirical abstraction to identify mathematical idea	External-External Internal-Internal	• Explicate the embodiment to focus on • Provide opportunities for reflection
Exploratory Familiarization	• Familiar with mathematical idea and external representations	External-External and External-Internal	• Provide appropriate examples and non-examples for exploration and familiarization of mathematical idea • Provide opportunities for learner to explore
Knowledge Classification	• Strengthen connections between knowledge nodes	Internal-Internal	• Provide guiding questions or prompts to guide learner to reflect
	• Organize and reorganize schematic structure	Internal-Internal	• Provide opportunities for learner to analyze, compare,

			synthesize and make justifications
Concept Reification	• Exhibit representational fluency and dominant use of abstract symbol	External-External	• Provision of opportunities to demonstrate representational fluency including abstract symbol and application of concept learned

Guided explication stage

This stage is characterized by the provision of guidance and explication of the intended mathematics to be acquired. Referring to Figure 6, the teacher provides each learner with fraction discs and demonstrates how to manipulate them (External-internal). The concept of equivalent fraction is embodied by the area model of the discs. Teacher directs learners' attention to the equal area model depicted by both sets of fraction discs and reiterate the equivalence through the outlining and shading of the pictorial pizza models (External-external). Teacher uses probing and guiding questions to direct learners to identify the invariant attribute e.g. "Look at both the pizza left over, what do you see?", and opportunities to reflect on the mathematical idea in relation to the fraction discs, its enactment and pictorial pizzas models (Internal-internal) e.g. "What do you mean by the 'same'?"

Teacher poses question: Look carefully at the size of the pizza fractions left over (T to outline the left over pizzas as depicted below). What can you say?

See boardwork:

Ben's Pizza Jane's Pizza

Pizza left over $\frac{2}{4}$ $\frac{4}{8}$

Instruction to teacher:

Outline (in red) the remaining pizza to emphasize the 'size' (area) left behind.

Comparing area model of both Ben's and Jane's pizza fractions and fractional notations
Teaching focus: Different fractional notation may represent the same fractional size.

Teacher directs pupils' attention to Ben's pizza and Jane's pizza on the board.
Key Question: Look at both the pizza left over (pointing to the literal pictorial representation), what do you see? P's response: "Pizza left over by Jane is the same as the pizza Ben left over."/"Pizza Jane ate is the same is pizza Ben ate." T to get pupils to explain what they mean by 'same', what is the 'same'?

P's response: "The size of the pizza fraction for Ben is as big as/as small as the pizza fraction for Jane's.
Teacher then directs pupils' attention to the fractional notation of Ben's pizza and Jane's pizza on the board.
Key Question: Look at both the fraction written for the pizza left over (pointing to the fractional notation), what do you see? P's response: "Two fractions are different."
Linking area model and fractional notation with the idea of equivalence.
Key Question:
The pizza left over in Ben is the same in size as Jane's but the fractional notation of both of them is different. Can you explain why?
P's response: "If $\frac{2}{4}$ looks the same as $\frac{4}{8}$ in fraction pizza, then the two fractions are the same."

Figure 6. Guided Explication of Mathematical Idea

The learning goal of this stage is for learner to manipulate external representations and to construct internal perceptual images of both the external representations and its enactment as basic cognitive knowledge nodes in the mind. The teaching intent is to guide and direct learner's attention to the invariant attribute. The rationale of guided explication is to ensure that learner, who have no knowledge of the concept and invariant attribute, focuses on the mathematics depicted through the external representations. It reduces the danger of learner focusing on superficial characteristics of the external representations which increases extraneous cognitive loads (Brown, McNeil & Glenberg, 2009) and avoid learner's ambiguous interpretations of the external representations from its original intent to depict the mathematical concept (Palmer, 1977; Uttal, Scudder & DeLoache, 1997).

Exploratory familiarization stage

Stabilizing individual knowledge nodes and establishing initial connections among them are the focus of this stage. It is characterized by the provision of opportunities for exploration and familiarization. The teacher provides appropriate and well-selected examples and non-examples to strengthen learner's understanding of mathematical idea (see Figure 7). Learner work individually and reflect upon the series of enactments before participating in a whole-class discourse directed by the teacher.

<u>Lesson One - Scenario 2</u>

Ali bought two pizzas. One for himself and another for his sister, Nur.

Ali cut his pizza into 6 pieces. He ate 2 pieces.

Nur cut her pizza into 12 pieces. She ate 4 pieces.

Who ate more?

Guiding Questions:

1. Can you choose the correct pizza for Ali and Nur? Why must they be of the same size?
2. Look at Ali's pizza. Write down the fraction of pizza he ate and fraction of pizza left over.
3. Look at Nur's pizza. Write down the fraction of pizza she ate and fraction of pizza left over.
4. Study the pizzas leftover. What do you oberve of their sizes? Compare the fractions and the sizes of pizza eaten/left over. What can you see? Evidence: Ps' response in worksheet.
5. What conclusion can you make? Write it down in the worksheet. Evidence: Ps' response in worksheet.

Teacher presents another scenario (non-example) over powerpoint presentation (ppt) – see below.

<u>Lesson One - Scenario 3</u>

Raju bought two pizzas. One for himself and another for his brother, Bala.

Raju cut his pizza into 6 pieces. He ate 2 pieces.

Bala cut his pizza into 2 pieces. He ate 1 piece.

Do they eat the same amount?

Guiding Questions:

1. Can you choose the correct pizza for Raju and Bala? What must you check?
2. Look at Raju's pizza. Write down the fraction of pizza he ate and fraction of pizza left over.
3. Look at Bala's pizza. Write down the fraction of pizza he ate and fraction of pizza left over.
4. Study the pizzas leftover. What do you observe of their sizes?
6. Compare the fractions and the sizes of pizza leftover. What can you see? Evidence: Ps' response in worksheet.
5. What conclusion can you make? Write it down in the worksheet. Evidence: Ps' response in worksheet.

Figure 7. Provision of more exemplars in Exploratory Familiarization Stage

The learning goal here is for the learner to undergo more manipulation of external representations through well-crafted examples and non-examples. With repeated manipulation, learner becomes more familiar with the mathematical idea embedded in the external representations and its association with the intended mathematical concept. The teaching intent is to provide the learner with more opportunities to manipulate external representations so that he may become familiar with the mathematical idea embedded (Dienes, 1971; Ainsworth et. al., 2002; Chao, Stigler & Woodward, 2000; Clement, Lochhead & Monk, 1981), to construct and negotiate meaning via classroom discourse (Stacey, Helme, Archer & Condon, 2001; Saxe, Taylor, McIntosh & Gearhart, 2005) and to engage in self-reflection on the enactments and materials (Clancey, 1997; Ginsburg & Opper, 1988).

Knowledge classification

This stage is characterized by the provision of opportunities for constructive thinking through teacher's questioning techniques and prompts (Berthold & Renkl, 2009; Collins, Brown & Newman, 1989). This stage can be viewed in two ways depending on learner's ability. It can be seen as being subtly integrated into the stages of Guided Exploration and Exploratory Familiarization through some forms of teacher questioning that accompany the enactment and reflection processes that encourage observations and comparisons. For example, "Look carefully at the size of the fraction discs left over.", "What is the same?", "Look at the fraction written for each discs. What do you observe?", "The pizza left over in Ben is the same in size as Jane's but the fractional symbol of both of them is different. Can you explain why?", "Compare the size and the fraction symbol of each fraction given. What can you see?". Alternatively, lessons can be staged to deepen learner's understanding by activating their analyzing and reasoning abilities through 'what' strategies (Kaur, 2009).

Consolidation and introduction of mathematics terms – 'equivalent fractions'
Teacher connects all scenarios (1, 2 and 3) by highlighting in each case, the fractional notation and the size of the pizza fraction. In scenario 1 & 2, the fractional notations are different but the pizza sizes are the same. Since the fractional notations show the same size, they are equal and we call them equivalent fractions (i.e. $\frac{2}{4}$ shows the same size as $\frac{4}{8}$, therefore $\frac{2}{4} = \frac{4}{8}$. They are equivalent fractions).
In scenario 3, the fractional notations are different and the pictorial pizza sizes are not of the same size. Therefore, they are not equivalent fractions. (i.e. $\frac{2}{6}$ does not show the same size as $\frac{1}{2}$, therefore, $\frac{2}{6} \neq \frac{1}{2}$. They are not equivalent fractions). See board work:

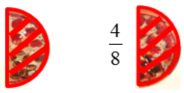
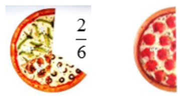
Ben and Jane	Ali and Nur	Raju and Bala
Ben's Pizza $\frac{2}{4}$ Jane's Pizza $\frac{4}{8}$ 	Ali's Pizza $\frac{2}{6}$ Nur's Pizza $\frac{4}{12}$ 	Raju's Pizza $\frac{2}{6}$ Bala's Pizza $\frac{1}{2}$ 
From the pizza fractions, we see that $\frac{2}{4}$ is the same as $\frac{4}{8}$, therefore $\frac{2}{4} = \frac{4}{8}$. We call them equivalent fractions.	From the pizza fractions, we see that $\frac{2}{6}$ is the same as $\frac{4}{12}$, therefore $\frac{2}{6} = \frac{4}{12}$. We call them equivalent fractions.	From the pizza fractions, we see that $\frac{2}{6}$ is not the same as $\frac{1}{2}$, therefore $\frac{2}{6} \neq \frac{1}{2}$. They are not equivalent.

Figure 8. Consolidation to facilitate analysis and synthesis in Knowledge Classification Stage

The learning goal here is for the learner to engage in the construction of meaning through internal-internal representational interactions. Depending on the ability of individual learner, learning progression may range from an initial making sense of the mathematical idea to a deeper examination of the relationships among acquired knowledge. The isolated knowledge nodes formed in the first two stages are now being assimilated and organized in a meaningful way, into the intended and desired schematic structure. The teaching intent therefore is to assist learner to make correct interpretations of the knowledge presented by providing and guiding learner to compare, analyze and synthesize. This stage is crucial in determining the level of understanding of the learner. A failure to interpret current knowledge correctly leads to the development of misconceptions.

Concept reification

This stage is characterized by the learner's ability to function predominantly with abstract mathematical symbols and an ability to exhibit representational fluency. This is the optimal goal for teaching and learning using C-P-A. It is necessary to clarify that this optimal stage requires the fulfilment of both criteria above because it is not uncommon to find learners who can function proficiently with abstract mathematical symbols and yet lack the desired level of conceptual understanding especially among elementary learners. In addition, the schematic structure is constantly evolving in its organization and degree of connectivity. Coupled with individual differences, a learner's schematic structure may not necessarily be stable or fully developed at the end of the instructional sequence and may continue to reorganize after the cessation of formal instructions.

Conclusion

The proposed teaching and learning model using Concrete-Pictorial-Abstract instructional sequence is designed to model after the proposed theoretical conjectures of the learning process defined in the perimeters of mental cognitive structures. The premise by which the model anchors itself is that mathematical concept acquisition is not attained merely by the presentation or goal-less engagement of external representations. To maximize the potential of external representations, the coordination between teacher's facilitation and learner's learning progression must be closely aligned. But in order to understand the role of the teacher, it is only natural to examine the learning process of the learner. The passage from the use of concrete to abstract adoption in concept formation involves some form of processing and interactions between and among external and internal representations. Successful outward manifestations of transformation and translation abilities among external representations is achieved only if the internal schematic network is established. Therefore, teachers must not overlook the crucial facilitations required during the instructional sequence of C-P-A. The four stages in the proposed model attempts to highlight the stages that learner undergoes and the corresponding teaching facilitations to scaffold the learner towards concept acquisition. Just as interactions among representations are non-linear and recursive, the teacher has to monitor the progress of individual learner and assess the stage he or she is in before tailoring the required facilitations accordingly.

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