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Relations between Subject Matter Knowledge and Pedagogical Content Knowledge: A Study of Chinese Pre-Service Teachers on the Topic of Three-Term Ratio

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Abstract: This study examined the connection between subject matter knowledge (SMK) and pedagogical content knowledge (PCK) among a group of Chinese pre-service mathematics teachers teaching three-term ratio. Both video-based interview and task-based interview approaches were employed to investigate six pre-service teachers' (PSTs) conceptual understanding of ratio and their PCK on teaching the topic. The results suggest that the PSTs had an unstable and inconsistent understanding of the concept of ratio, which influenced their presentation of the concept of three-term ratio. Those who possessed multiple understandings of this concept tended to be more flexible when choosing different representations. Some implications for future studies on investigating the relationship between SMK and PCK and for teacher education were discussed.

Key words: subject matter knowledge, pedagogical content knowledge, the concept of ratio, pre-service secondary mathematics teachers

Introduction

Studies have repeatedly found that students from East Asia outperformed their Western counterparts in international assessments such as TIMSS and PISA (e.g., Mullis, Martin & Foy, 2008; OECD, 2013). It is believed that the "curriculum gap" is not the sole explanation for the performance discrepancies between West and East, and that the "preparation gap" of teachers, as confirmed by the results of the IEA Teacher Education and Development Study in Mathematics (TEDS-M) (Tatto et al., 2012), is a fundamental concern.

Teachers' professional knowledge has been regarded as one important indicator of a teacher's competency. In his most cited papers (1986, 1987), Shulman defined multi-categories of teachers' professional knowledge including Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). In particular, he defined PCK as knowledge that "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching… it is of the particular form of content knowledge that embodies the aspects of content most germane to its teachability". This professional knowledge is further refined by other scholars into different categories (cf. An, Kulm, & Wu, 2004; Tatto et al., 2008, Ball, Thames & Phelps, 2008). Among these categories, SMK and PCK are two foundational ones. Despite the fact that SMK and PCK are usually treated as two separate categories, the interrelationship between the two is very important especially for the subject of mathematics, as Kant observed, "pedagogy without mathematics is empty, mathematics without pedagogy is blind" (as cited in Park, 2005). Ma's (1999) study also highlighted the implicit relationship between mathematics knowledge and mathematics teaching. In her comparative study between the United States and China, she found that Chinese teachers possessed profound understanding of fundamental mathematics (PUFM) that facilitates them to conduct more effective teaching than their American counterparts. However, in the literature, little evidence has been provided to support the connection between SMK and PCK.

A decade ago, Even (1993) studied a group of US prospective secondary mathematics teachers' PCK and SMK on teaching the concept of function. Despite the limitations, Even's study was an initial attempt to investigate the relationship through a qualitative approach, and posted some interesting hypotheses, for example, incomplete concept image of function might influence those prospective teachers' limited pedagogical reasoning. The study reported in this article investigates a group of Chinese mathematics pre-service teachers' SMK and PCK in the context of teaching the concept of ratio, and explores the interrelationship between their SMK and PCK. This article attempts to contribute to the literature through addressing Chinese pre-service teachers' competency in teaching knowledge and providing evidence to support the relationship between SMK and PCK in mathematics teaching.

Literature Review

As early as 1902, Dewey seemed to have discerned the unique SMK that teachers possessed, which is different from that of scientists due to their different concerns. Scientists were concerned with new findings, whereas teachers were concerned with how SMK was to be transferred to students in an appropriate and effective manner. Teachers have a responsibility to convey pedagogical representations to students in a comprehensive manner and, according to Dewey, teachers should consider "psychologizing" the subject matter (Dewey, 1902, cited in Wilson, Shulman & Richert, 1987). Since 1986, when Shulman first proposed the concept of PCK, this type of knowledge has been regarded as an important element in mathematics teachers' teaching knowledge base and one important indicator of teachers' professional competence (An, Kulm, & Wu, 2004; Schmidt et al., 2007; Tatto et al., 2009). PCK has been defined in terms of its different components. For example, Lim-Teo et al. (2007) defined the following four aspects of PCK: a) a teacher's own understanding of mathematical structure and connections; b) a teacher's knowledge of a range of alternative representations of concepts for the purpose of explanation; c) a teacher's ability to analyze the cognitive demands of mathematical tasks on learners; and d) a teacher's ability to understand and take appropriate action for children's learning difficulties and misconceptions (p.240). In this framework, individual teachers' understanding of mathematics structure and connection is seen as a basic requirement of content knowledge. An, et al.'s study (2004) described PCK as comprising three components: knowledge of content; knowledge of curriculum; and knowledge of teaching. Of the three, knowledge of teaching was felt to be the core component of PCK.

Both PCK and SMK are important categories of mathematics teacher's professional knowledge. These two categories of knowledge have been found to interact with effective teaching. SMK is grounded in core teaching activities and influences teachers in making decisions about content-specific instruction, such as designing a task or posing a meaningful question for student exploration. In contrast, PCK is regarded as a tool or vehicle for teachers to deliver the content knowledge in their mind to pupils in a comprehensive manner. To some extent, a teacher's capacity for selecting an appropriate way to convey mathematics ideas ultimately relies on what Ma (1999) called "profound understanding in subject matter knowledge" (p.120), also referred to as "flexible subject matter understanding" (MacDiarmid, Ball, & Anderson, 1989). A most cited model of mathematical knowledge for teaching (MKT) developed by Ball and her associates (Hill, Ball & Schilling, 2008, p.377) includes both PCK and SMK as two cornerstones. In the MKT model, both PCK and SMK consist of several sub-categories. SMK is defined as Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK). Different from CCK, SCK is special knowledge "that allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Hill, Ball & Schilling, 2008, 377-378).

The idea of SCK seems to be promising in helping us to think of how the special elements of SMK different from school mathematics knowledge makes PCK different in quality. Past studies show that some special elements in SMK facilitate mathematics teachers to teach mathematics content more meaningfully. For example, Hiebert and Lefevre (1986) advanced the notion of conceptualizing procedural knowledge. For example, when understanding the symbol "+" as limiting to a manipulation function, it is possible to acquire the knowledge of symbols without understanding the symbols' meaning; however, with SCK, it facilitates teachers to represent the symbol "+" more meaningfully and conceptually (Leinhardt & Smith, 1985). In addition, we argue that knowledge of mathematics definitions (or concepts) is an important component of SMK. As pointed out by Ball, Thames, and Phelps (2008), "teachers need to know the material they teach; they must recognize when their students give wrong answers or when the textbook gives an inaccurate definition." (p. 399). Mathematics definition plays a significant role in teaching a concept in an abstract setting (e.g., how the property of simplifying ratios works). Mathematics definitions are crucial in teaching and learning mathematics (Tall & Vinner, 1981; Vinner

1991). As stated in the Common Core State Standards for Mathematics (CCSSM, 2010):

> Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning… By the time they reach high school they have learned to examine claims and make explicit use of definitions. (Mathematical Practice "Attend to precision, p. 7)

Understanding mathematics definitions is important for teachers not only due to the importance of definitions in mathematics but also due to teachers' special role in practice. In this article, we focus on teachers' understanding of the mathematics definition of both two-term and three-term ratios because ratio and proportion play a significant role for students in transiting from additive thinking to multiplicative thinking, and understanding ratio and proportion is also crucial to understanding rational number which is reported as a serious obstacle in the development of mathematical thinking of children in the literature (e.g., Berh, et al, 1992; Hart, 1981). For example, Hart (1981) found that most secondary students participating in "Concepts in Secondary Mathematics and Science" (CSMS) research programme only possessed a low-level understanding of ratio and proportion. They tended to see ratio as an additive operation rather than multiplicative, and used the methods of doubling, halving, and both doubling and havling to "build up' to an answer to tackle all ratio problems.

Another interesting aspect of ratio is that there seems to be no agreement on the meaning of terms, especially for ratio and rate (Lamon, 2007). For instance, is ratio a number or an ordered pair? Is ratio the same as fraction? Is ratio different from rate or are they the same mathematical concept? As the community of mathematics educators calls for students' conceptual understanding of mathematical concepts (Kilpatrick, Swafford & Findell, 2001), the meaning of the terms should be clarified first. Also, since the differences between those meanings provide different support for students to approach problems, the meaning of the terms deserves attention from mathematics educators. For instance, if ratio is defined as a fraction, it is hard for students to deal with quantities which are not in a part-whole relationship. This will jeopardize students' ability in applying the concept to solve mathematical problems which are multiplicative in nature (and not in a part-whole relationship). Cai and Wang (2006) argued that Chinese teachers took a different approach to teaching ratio compared to U.S. Teachers. For instance, U.S. teachers emphasize the importance of the knowledge of equivalent fractions in understanding the concept of proportion, the ability of simplifying fractions, and the ability to convert fractions to decimals in calculating the value of the ratio. In contrast, Chinese teachers treat ratio as a separate concept and require their students to connect ratio with fraction and division.

Some studies have demonstrated that there is a connection between SMK and PCK. Even (1993) studied the SMK of pre-service secondary mathematics teachers from the U.S. and its interrelations with PCK in the context of teaching the concept of functions. Even's study involved two phases. The first phase aimed to capture the general picture of prospective teachers' PCK and SMK by a questionnaire with open ended questions. The second phase was an in-depth interview. The purpose of the interview was to ask subjects to explain what they did on the questionnaires and why. The analysis was focused on prospective teachers' understanding of two essential features of the modern view of function, that is, arbitrariness and univalence. The arbitrary nature of functions refers to both the relationship between the two sets on which the function is defined and the sets themselves. The arbitrary nature of the relationship means that functions do not have to exhibit regularity, or be described by any specific expression or particular shaped graph. The univalence requirement means that for each element in the domain there is only one element (image) in the range. Developments in mathematics have changed the concept of function from a curve described by a motion (17th century) to the modern conception of a function, namely, a function f from A to B is defined as any subset of the Cartesian product of A and B, such that for every a belonging to A, there is exactly one b belonging to B such that (a, b) belongs to f. The results of Even's study show that many pre-service teachers did not have a modern conception of function, they did not seem to appreciate the arbitrary nature of function, and only a few could understand the univalence requirement of function. Even claimed that insufficient SMK might account for why those pre-service teachers adopted teaching strategies that emphasize procedural mastery rather than conceptual understanding. Even's study is an important study because it was the first time when the interrelationship between SMK and PCK was investigated. The questions designed for exploring prospective mathematics teachers' concept of function in Even's study were very well constructed, yet only a small portion of questions were specifically designed for exploring prospective teachers' PCK. The PCK questions only focused on the analysis and explanation of hypothetical students' misconceptions. Even's study demonstrated the subjects' SMK and PCK, yet was not able to provide strong evidence for the connection between PCK and SMK. In addition, Even only described SMK and PCK in a general way. For example, Even interpreted SMK as "knowing how and knowing why".

Similar to Even's (1993) study, the study reported in this article also focused on SMK and PCK and their interrelationship for prospective secondary mathematics teachers. For PCK, in addition to focusing on analyzing students' misconceptions, we also asked pre-service mathematics teachers questions regarding how to teach the topic of ratio. For SMK, we focused on teachers' understanding of the mathematics definition of both two-term and three- term ratios. Similar to the definition of function, the definition of ratio has changed and developed through history. The analysis done by He (2013) on 14 old mathematics textbooks indicates that throughout history there is no consistent meaning of ratio in mathematics curricula. Ratio is mainly defined as "number/fraction/percentage" or "division" in mathematics curricula. For instance, in Fish (1874), ratio is defined as "the relation between two numbers of the same unit value, expressed by the quotient of the first divided by the second." (p. 383). In Potter et al. (1952), ratio is defined as "a comparison of two numbers and is usually written as a fraction." (p. 298) Later in history, other meanings of ratio appeared in the curricula, and efforts were made to distinguish ratio from other concepts such as fraction. For example, in Usiskin and Bell (1983), ratio is defined in terms of "ordered pair". Due to the multiple definitions of ratio, studying the concept of ratio offers rich opportunities to explore teachers' various understanding of ratio and how it is connected to the ways they teach ratio. Understanding the definition of ratio could be either CCK or SCK in teaching this topic depending on the depth of questions. In this current study, we regard understanding mathematics definition as one part of SMK. This study aims to concentrate on the connection between student teachers' understanding of the definition of ratio and the relevant PCK in teaching a topic on ratio. Three research questions are addressed:

1) What PCK do pre-service teachers (PST) in our sample possess on teaching a topic on three-term ratio?
2) What is those PSTs' understanding of the concept of ratio?
3) In what way is those PSTs' SMK on the concept of ratio connected to their PCK on teaching three-term ratio?

Methods

The PCK data presented in this article were selected from the first author's Ph.D. dissertation on investigating the PCK among lower secondary pre-service mathematics teachers from a normal university in Hangzhou, a city in eastern China (Ding, 2014). The instruments for the study consist of a survey of PSTs' PCK, and a qualitative component involving follow-up interviews for PSTs to further elaborate their understanding of the PCK items in the survey, plus three video-based interviews. Ten PSTs participated in the follow-up interviews, and six out of them also joined the three video-based interviews. The participants did the same interviews twice, first at the beginning of their third year, and the second time was at the end of their fourth and final year. For the purpose of this article, only data regarding the six PSTs' second round of the video-based interview on the topic of "three-term ratio" are presented. In addition, in order to address the relationship between PCK and SMK, an additional task-based interview with a focus on investigating PSTs' SMK on the topic of ratio was conducted among the six PSTs after their second round of PCK interview.

The context of the research site

Normal universities (colleges) in China are mono-purpose institutions, mainly responsible for preparing future teachers at different levels since 1998 (Li, et al., 2008). Usually, the preparation of future secondary school teachers is conducted through a four-year undergraduate program delivered by a discipline-specific department. Normal universities offer future secondary mathematics teachers a teaching certification allowing them to teach mathematics in all secondary school grades after graduation. The selected teacher preparation program in this normal university in Hangzhou is a four-year B.Sc. (major in mathematics) program labelled shi fan (师范), which means to nurture future secondary mathematics teachers. (Hangzhou, the capital of Zhejiang province, is a major city located in the Yangzi River

Delta region.) The program consisted of required and elective courses in the mathematics department and other faculties, depending on students' individual needs. The required mathematics courses are different types of advanced mathematics subjects, including analytical geometry, higher algebra, modern algebra, real analysis; pedagogical content knowledge courses include mathematics specific teaching methods; and pedagogy courses include psychology and education theory. In addition to course work, student teachers need to spend one week on teaching observation in one school in the second semester of their third year, and an intensive 10-week teaching practicum in the first semester of the final year.

Participants

The criteria for recruiting PSTs to participate in the qualitative part of the study are based on their performance in the PCK survey, gender, classes[2] , and the willingness to participate in this study. 101 PSTs who enrolled in this selected program in the Year of 2008 were recruited for the survey at the beginning of their third year; in June 2012, they were recruited again at the end of their teacher education program. For the qualitative part of the study, fifteen out of the 101 PSTs were selected for follow-up interview, and the selected PSTs varied in terms of their performance in the PCK survey (5 rated as Low, 5 rated as Intermediate, and 5 rated as High), gender and classes. Only ten out of fifteen agreed to join the follow-up interviews. Six out of the ten PSTs (Zhi, Yuan, Pei, Jing, Han and Fang[3]) were recruited for three video-based interviews subject to their willingness. Table 1 provides the background information and suggests how these six PSTs differed in their background.

[2] Here the term "class" does not refer to social rank but to a cohort of students meeting regularly to study the same subject. In China, for purposes of more efficient enrollment management, university students who enroll in the same year and who pursue the same major, usually are split into different groups (or classes). For example the "82" refers to the specific class. "8" means that the students were enrolled in the university in 2008 and "2" refers to the second class.

[3] All Pei, Han, fang, Zhi, Jing and yuan are pseudonyms.

Table 1
Background information of the six PSTs in the three video–based interviews

Name	Class	Gender	The level of accuracy in PCK items
Pei	82	Female	Intermediate (rank[4] 48)
Han	81	Female	High (rank 11)
Fang	83	Female	Low (rank 74)
Zhi	81	Male	Intermediate (rank 42)
Jing	83	Female	High (rank 13)
Yuan	81	Male	High (rank 20)

Instruments

Only the instruments for capturing relevant data for this current study are introduced in this section. A video-based interview was conducted for capturing the six PSTs' PCK and SMK on the topic of three-term ratio. This was followed by a task-based interview to further explore the PSTs' understanding of the definition of ratio.

A video clip on the topic of three-term ratios used in this study was an unpublished video collected for Hong Kong component of the TIMSS 1999 Video Study (Mullis, et al., 2000). The video clip was edited to act as stimuli for investigating PSTs' PCK and SMK on teaching the topic of ratio. The PSTs were more motivated to reveal their PCK when watching videos, especially when watching videos with unfamiliar content or with content which contrasts with what they considered as good teaching. The videos collected for the Hong Kong component of the TIMSS 1999 video study acted as a targeted source, with significant attention paid to the concept of "contrasting". Two contrasting features in the Hong Kong videos were evident: firstly, the context and content of mathematics teaching in Hong Kong were new to the PSTs from mainland China; secondly, viewing examples of mathematics teaching from a decade ago might inspire PSTs to talk more openly about their own PCK. In this selected video clip, a male teacher showed students how to derive a ratio with three terms (a, b and c) given two-terms ratios (a: b= 5:2, and a: c =3:4). When introducing the three-term ratio, he used circles and arrows to explain how to obtain the numbers 15, 6 and 20 to represent the letters, a, b and C (see Figure 1

[4] PSTs whose performance ranks among the top 30 students in the program are rated as High, those ranking between 31 and 70 are rated as Intermediate, and those below 71 are rated Low.

below). After this part of teacher's presentation, students raised questions. One student asked whether 15:6:20 could be simplified. The teacher in the video clip only nodded and did not respond to the question in a straightforward way. Instead, he illustrated under what circumstances a three-term ratio could be simplified.

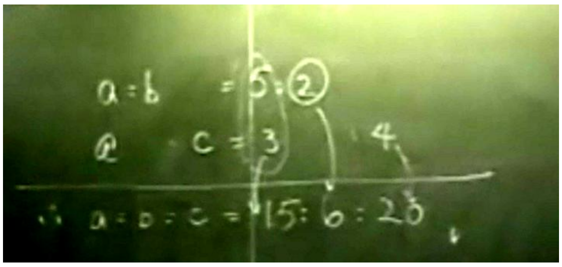

Figure 1. The teacher's way of explaining the process of calculating the three-term ratio

The six PSTs were invited to answer PCK and SMK questions based on the content of the video clip. PCK questions were designed to capture PSTs' strategies in designing several segments of teaching this topic, including the representations used in deducing a three-term ratio based on two two-term ratios, and the explanations to students' questions. For SMK, not only did we look at PSTs' understanding of two-term ratio, but we also looked at their understanding of three-term ratio and ratio equivalence. The reasons why we looked beyond two-term ratio is because it is easily connected to fraction and division, thus by asking the PSTs the meaning of three-term ratio and the equivalence of ratio, more insights on PSTs' understanding of the definition of ratio can be gained.

In addition to asking SMK questions related to this video clip, a task-based interview was conducted as a supplement to further investigate PSTs' understanding of the definition of ratio in terms of two-term and three-term ratio and ratio equivalence. Different from the SMK questions in the video-based interview, which were designed in a specific teaching context, the task-based interview investigated PSTs' understanding from different perspectives. The tasks consisted of a series of questions with a similar structure yet constructed in different contexts. In a sample question (see Appendix 1), PSTs' understanding of the concept of the equivalence of ratio was explored in both a pure mathematics context and in a word problem context. For instance, PSTs were asked to give their definition of two- term and three-term ratio and the meaning of equivalent ratio. The two interviews were conducted separately, with the video-based interview followed by the task-based interview. The first interview lasted one hour to one and an half hours, and the second interview took about 40 minutes, subject to individual differences.

Data analysis

Data for analysis include the interview transcripts and the six PSTs' written responses. In general, the task-based interview was designed to answer SMK questions and the video-based interview was designed to mainly answer PCK questions. We analyzed the data by first looking at the six PSTs' SMK and PCK, and then attempted to seek the alignment between SMK and PCK for each PST and detect the pattern. In this paper, only four PCK questions from the larger study were selected as the focused analysis. The four questions are: PSTs' representation of a three-term ratio, explanation for why the three-term ratio can be simplified, the introduction of this topic, and approaches on how they may extend this current topic. The PSTs' responses to the four questions were based on their understanding of the topic of ratio. This may provide some potential to see a close connection among the three aspects of SMK mentioned above and those of PCK. For instance, the response to the PCK question named "explaining why this three-term ratio 15:6:20 cannot be simplified" provides various possibilities about to what extent SMK connects with PCK. Table 2 shows the analytical framework for PSTs' responses to each PCK question. This analytical framework differentiates the six PSTs' responses by categories which are kind of the summarization of the major features of each response. Similarly, table 3 shows the analytical framework for addressing SMK questions on this topic of ratio in this study. This framework for coding PSTs' understanding of ratio was developed from various textbooks' definitions of ratio, the literature (e.g., Cai & Sun, 2002; Hart, 1981), as well as these PSTs' responses. Both data from the video-based interview and the task-based interview regarding the focused aspects of SMK were coded according to the analytical framework. The individual's responses were coded as multiple categories.

Table 2
Analytical Framework for Coding PCK questions (specific examples of how the method is implemented is offered in the Results Section)

PCK question	Categorization	Explanation
The ways of representing three－term ratio	A. Supplementing more steps by using the property of ratio	Elaborating the process of enlarging the quantities by multiplying the same number based on the knowledge of equivalent ratio.
	B. "Norming" (Lamon, 1994)	Norming "means one element is a scalar multiple of another within a measure space" (Lamon, 1994, p. 95). For example, if a:b=2:5, then we can rewrite b as (5/2) a.
	C. Rearranging the representation of two two-term ratios	Considering how to rearrange the order of numbers
Explaining why the three-term ratio 15: 6: 20 cannot be simplified	A. Conceptual understanding	Explaining the idea of unit
	B. Mastery of procedures	Judging if the ratio is in the simplest form, and the ways of finding the greatest common factor (GCF)
The way of introducing this topic	A. Situated in real life problems	Designing word problems related to real life which involve ratios
	B. Property of ratio	Reviewing the properties regarding the equivalence of two-term ratio
	C. Fraction	Comparing the property of ratio with fraction, and distinguishing the differences and similarities
The way of extending the content of this topic	A. Approaching from a kind of mathematical thinking	Emphasizing the process of getting three-term ratio and the advantage of using three-term ratio
	B. Within the scope of teaching the content of this topic	Emphasizing the content regarding learning three-term ratio

Table 3
Analytical Framework for Coding SMK responses

SMK question	Categorization	Explanation
Understanding of two-term ratio a: b	A. Fraction	Making an analogy with fraction
	B. Division	Making an analogy with division
	C. A type of relationship	Quantitative relationship between two terms
	D. Unclear	Either stating that they are unclear or the explanations are ambiguous
Understanding of three-term ratio a: b: c	A. Three pair-wise fractions	It can be explained by $\frac{a}{b}, \frac{c}{b}, \frac{a}{c}$
	B. One complex fraction	A complex fraction is a fraction where the numerator, denominator, or both contain a fraction, like $\frac{\frac{a}{b}}{c}$, $\frac{a}{b}$ as the numerator
	C. A type of relationship	Quantitative relationship among a, b, and c
	D. Unclear	Either stating that they are unclear or the explanations are ambiguous
Understanding of the equivalence of ratio	A. Referring to the property of fraction	The value of a fraction stays the same if both the numerator and denominator are multiplied or divided by the same number
	B. Using the value of ratio	Divide the first term of the ratio by the second term and get the value
	C. Restating the property of ratio	The value of a ratio stays the same if both terms are multiplied or divided by the same number

The first author was responsible for coding PCK responses, and the second author was responsible for coding SMK responses. They also checked the other's coding. The discussion on the coding continued until a final agreement between the two authors was achieved.

Results

We start with a summary of the six PSTs' SMK and PCK on teaching the topic of ratio based on our analysis to address our three research questions. Table 4 provides an overview of the major types of SMK and PCK employed by the six PSTs at the final stage of their teacher preparation. The description of each type corresponds to Table 2 and Table 3.

Table 4
A summary of PSTs' SMK and PCK regarding the teaching of the topic of ratio

Name	SMK[5]			PCK[6]			
	a:b	a:b:c	Equiv. ratio	Teach a:b	Sim.	Intr.	Ext.
Han	A	A, B &C	A	A	B	B	A
Fang	A &C	A & C	B	A	B	A	B
Zhi	C	C	A	C	B	A	A
Jing	B &C	C	C	B	B	A	B
Yuan	A& D	Missing	B	A	B	A&C	B
Pei	B & C	D	C	A& B	B	A	B

As can be seen from Table 4, the fraction-type and the relationship-type of understanding of ratio are most widely adopted by the PSTs in their responses to the three categories of SMK questions. In responding to PCK questions, there are less variations in their explanations to why the three terms ratio 15: 6: 20 cannot be simplified, and their approaches to extending

[5] The three names in the second line below SMK correspond to the names listed on the first column in Table 2; because of the lack of space, some letters of a long name are omitted, e.g. equiv refers to equivalence.
[6] The four names in the second line below PCK correspond to the names listed on the first column in Table 3; because of the lack of space, some letters of a long name are omitted, e.g. Intr. Refers to introduction, Sim. Refers to simplify, and Ext. refers to extending.

the content of this topic to other topics. In the sections below, more detailed analysis is provided.

PSTs' PCK on teaching three-term ratio

The ways of representing three-term ratio. Figure 1 shows the way the teacher presented the teaching of this topic of three-term ratio in the video clip. All PSTs commented that the teacher skipped the step of explaining where the number 15 comes from (common multiple of 3 and 5). They concerned that students might be confused, and regarded the teacher's representation as procedural.

After commenting on the teacher's presentation, all PSTs were asked to show their ways of representing the topic of three-term ratio. The methods they came up with are listed in Table 2. Four of the six PSTs (Pei, Yuan, Han and Fang) adopted Method A, namely:

$$a: b = 3: 4 = 15: 20$$
$$a: c = 5: 6 = 15: 18$$
$$a: b: c = 15: 20: 18$$

Pei and Jing adopted Method B, namely, transforming both b and c into multiples of a (See Figure 2).

$$a : \frac{2}{5}a : \frac{4}{3}a$$
$$= 1: \frac{2}{5}: \frac{4}{3}$$
$$= 15: 6: 20$$

Figure 2. Pei and Jing's method

Zhi modified the teacher's representation in this video clip, and suggested Method C. Figure 3 shows Zhi's method.

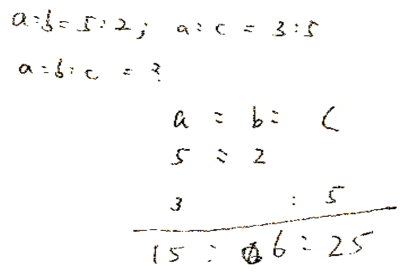

Figure 3. Zhi's method

Explaining why the three-term ratio "15: 6: 20" cannot be simplified. In this video clip, one pupil raised a question: why "15:6:20" cannot be simplified? When asked to explain this pupil's question, all six PSTs gave a similar answer: "there is no common factor among the three quantities, 15, 6 and 20." No participants were found to adopt Method A in this study, which relates to the conceptual understanding of the idea of "unit". Instead, they referred to "the simplest ratio" (in Chinese: Zui jian bi), for example, Han explained,

> *If there are no common factors among the three quantities, then the ratio is the simplest ratio; otherwise, we need to simplify. That means, we need to figure out the GCF (Greatest Common Factor)* (from Han)

It is clear that the PSTs tended to respond to this pupil with a more "standard" procedure, that is, figure out the GCF first. In the absence of GCF, then check whether the ratio is the simplest. If the ratio is the simplest, there would be no simplification.

Introducing and extending this topic on three-term ratio. As indicated in Table 3, for the introduction of three-term ratio, four PSTs – Pei, Fang, Zhi and Jing- used method A. The example given by Pei and Jing was related to ages. For example, Jing designed a problem,

> *The ratio between the age of Jack's father and that of Jack is 34:7, the ratio between the age of Jack's mother and that of Jack is 4: 1, what is the ratio among the ages of Jack's father, Jack's mother and Jack?* (from Jing)

Yuan adopted both methods A and C. He designed a detailed and concrete example pertaining to buying books which involved three persons ("Little a", "Little b" and "Little c") (see Figure 4). As shown in the third text box, the way Yuan tried to introduce the topic was by an illustration of the method of fraction.

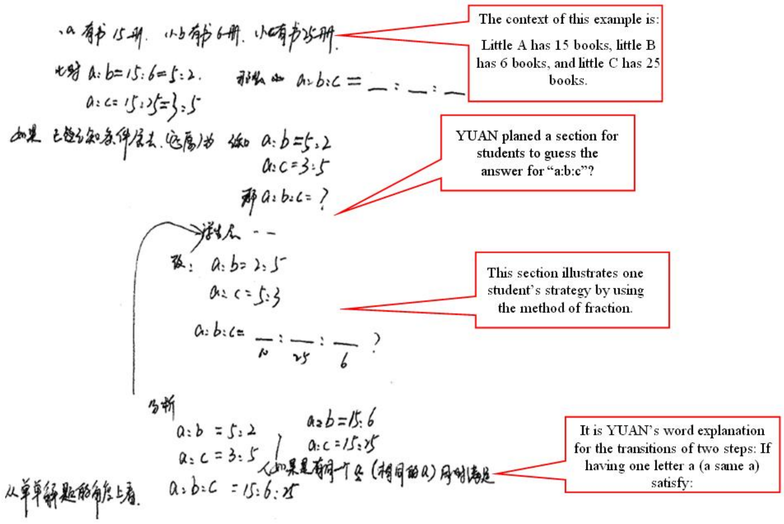

Figure 4. Yuan's example to illustrate the idea of three-term ratio

The method used by Han is Method B. She tried to review the properties of fraction which students had learnt before and help them to build some connections in terms of the common properties between the concept of ratio and fraction for learning this current topic.

In response to the interviewer's question regarding how to extend the content of this current topic for students' future learning, Zhi and Han adopted method A (approaching from a kind of mathematical thinking). Zhi responded that students should grasp the method regarding the conversion between two-term and three-term ratio, namely, a: b: c a: b and b: c , and regarded it as a mathematical thinking process. This method was coded as method A. Similarly, Han commented that three-term ratio includes more information and indicates any two-term ratios, so she would like to explore some interesting points with students. Four PSTs adopted method B (exploring the content within the scope of teaching the content of this topic), and they did not further explore beyond the lesson in the video.

PSTs' SMK on the concept of ratio

The results presented in Table 3 are mainly from responses from the task-based interview. We found three PSTs interpreted two-term ratio as fractions and two PSTs, Fang and Han, interpreted three-term ratio as either "three pair-wise fractions" or "one complex fraction". More than half of the PSTs thought that two-term and three-term ratio were "a type of relationship" but they could not further articulate what they thought the relationship was. This phenomenon is particularly prevalent for three-term ratio when it is harder to connect three-term ratio to fraction, compared to two-term ratio. No PSTs have attempted to use the idea of "unit" to interpret either two-term or three-term ratio. The "unit" idea is that two quantities measured in the same unit are in the ratio of A to B, where A, B $\neq 0$, if for every A units of the first there are B units of the second (Parker & Baldridge, 2004). For instance, the ratio of 6 oranges to 8 oranges could be 6 to 8 because we can regard 1 orange as the unit. The ratio could also be 3:4 if we take 2 oranges as the unit. The idea of unit is important for describing the meaning of equivalence of ratio. In the previous example, the ratio 6 to 8 is the same as the ratio 3 to 4 because they describe the same multiplicative relationship and the only difference is the unit we define. Due to the failure of conceptualizing the idea of unit in ratio, no participant can explain well the equivalence of ratio. Han and Zhi regarded equivalent ratio as the same as equivalent fraction. Fang and Yuan used the value of ratio to interpret equivalent ratio (essentially it is a division). Jing and Pei could not provide further explanation and they simply just called equivalent ratio a property of ratio. All these types of understanding tend to be procedural.

The connections between PSTs' SMK on the concept of ratio and their PCK on teaching three-term ratio

Some connections can be identified between the PSTs' understanding of two-term ratio and three-term ratio and their teaching representations for three-term ratio.

Firstly, for methods related to the representation, either Method A (supplementing more steps by using the property of ratio) or Method C (rearranging the representation of two two-term ratios) was based on the

PSTs' understanding of the meaning of ratio as a type of relationship (Type C- understanding of the concept of two-or three-term ratio). As shown in Table 2, five PSTs understood at least three-term ratio or two-term ratio as a type of relationship. Among them, Pei and Jing are the only two PSTs who possessed a "division" view of the concept of ratio and also endorsed teaching method B ("Norming"). It is possible that this view helped them to understand b and c as multiples of a.

Secondly, those who possessed multiple understandings of this concept tended to be more flexible when choosing different representations. For example, Jing endorsed two ways of understanding the concept of ratio as "a is to b" (Type B and Type C). The first referred to a type of relationship, while the second was a division type of understanding. For the first type, she seemed to have a vague impression of this relationship but could not explain it explicitly. Although she initially tried to describe the relationship in terms of a numerical relationship by saying 'a' and 'b' each refers to a separate quantity, she found that the coefficient of proportionality "k" could better express the relationship between 'a' and 'b',

> *For a is to b as 5 is to 2, "a" and "b" indicate different numbers. So "a" could be expressed as 5k, and "b" could be expressed as 2k. k is a real number….* (from Jing)

In addition, Jing tried to describe a division type of understanding in terms of the relationship between the lengths of segments or between the quantities of objects. Finally she tried to contrast this concept with a division and multiplier relationship through a concrete example about apples and pears.

> *From the perspective of division, say if there are "a" apples and "b" pears. Assuming that the quantity of apples is three times of the quantity of pears, then we can list an equation like "a" equals "3b". If writing in the format of division, then "a" is divided by "b" equals 3. So we can introduce a new idea, say if "a is divided by b" could be written as 'a is to b', so, correspondingly, a is to b as three is to one.* (from Jing)

Thirdly, it seems that PSTs' limited understanding of ratio equivalence (no one referred to a unit-type understanding in either stage) may have led them to teach in a more procedural way of simplifying ratios. For example, all six PSTs gave a similar answer in explaining why the three-term ratio "15:6:20" could not be simplified based on procedural criteria, such as the simplest ratio and GCF. No PSTs endorsed the Type A method, that is, to explain the simplifying issue by the idea of unit, namely no whole number "unit" can be found to go into the three numbers 15, 6, and 20 simultaneously.

Discussion and Conclusion

Consistent with Even's (1993) study, our results suggest that some PSTs might have unstable and inconsistent understanding of mathematics concepts in different contexts. In the current study, four out of the six PSTs showed various understanding of the concept of ratio in the context of two-term ratio and three-term ratio. For example, they could easily adopt the fraction as the metaphor in understanding the concept of two-term ratio, because the property of fraction can easily explain why two-term ratio cannot be simplified. However, similar understanding cannot apply to understanding the concept of three-term ratio. In other words, interpreting the concept of two-term ratio as fraction or division has its limitations. For example, fraction tends to be interpreted as a part-whole relationship, but ratio can be used to describe both part-part and part-whole relationships. Also, as a number, fraction has its own rule of operation (e.g., fraction multiplication), but ratio does not. Another disadvantage in thinking of ratio as fraction or division is that it cannot be generalized to three-term ratio. Using three concepts, fraction, division, ratio, interchangeably will give students a sense that there is no definite meaning of the ratio concept. In addition, in this current study no PSTs demonstrated the idea of "unit" when they interpreted the concept of ratio. We suspect that the reason why these PSTs were unable to bring out this idea of unit is due to their lack of exposure to clear definition in their previous schooling. Their teachers' instruction of ratio might mainly focus on solving real life problems but less focused on the meaning of the concept (Cai and Wang, 2006). The way that the curriculum presents ratio is another factor. In China, the concept of ratio and its related concepts are introduced only in primary school, and more applications of ratio are taught in the later stage of study. Therefore PSTs might not have the chance to develop more in depth understanding on those concepts.

In addition, the results show that these PSTs tended to design the introduction of topics based on pedagogical consideration. In general, their PCK tended to be “immature” in terms of developing students’ conceptual understanding of the topic of ratio. When responding to PCK questions, the PSTs tended to rely more on their pedagogical knowledge or personal experience without applying their mathematics understanding. The majority of them tended to situate a problem in real life examples. For instance, the examples designed by Pei and Jing for introducing the topic of three-term ratio are related to ages. According to them, real-life examples are important because students love them, however, judged from the perspective of mathematical understanding, it is risky to use examples regarding the growth in ages in explaining the concept of ratio, because it promotes students’ misunderstanding on the relationship between two quantities in ratios —more an additive relationship rather than a multiplicative one (Hart, 1981).

This study has a number of limitations. First, this case study only includes six PSTs, thus the results are not representative of all PSTs in China. The findings regarding the connections between the PSTs’ SMK and PCK gained from this study cannot be generalized because of the small sample size, and future studies should be carried out to confirm our results. Second, the connections between SMK and PCK are not explicitly identified because of some methodological limitations. For example, PSTs’ SMK is interpreted from different data sources and defined as a combination of different types. We have not been able to identify the dominant type of each PST’s understanding of the concept of ratio. This makes it hard to draw conclusions about the connections between SMK and PCK because PSTs usually endorse one type of PCK while they may perceive more than one type of SMK. However, no comparable measurements have been developed in past studies, thus the development of valid and effective instruments to measure the connection between SMK and PCK should be a subject for future research.

Notwithstanding these limitations, this study has attempted to draw connections between six Chinese PSTs’ SMK and PCK regarding ratio, which has not been explored in past literatures. In particular, a specific analytical framework to analyze SMK and PCK regarding the concept of ratio was developed in this study, which lays a basis for future research in this content domain. In addition, this study could constitute a paradigm shift towards focusing on the connections among different domains of teaching knowledge, especially about how SMK can be applied to other domains of teaching knowledge. Our findings suggest that insufficient SMK might be one factor for undeveloped PCK. Therefore, more SMK courses are suggested for pre-service teacher education for enhancing PSTs' conceptual understanding on basic concepts. In the meantime, we should be cautious in making the conclusion that limited SMK leads to immature PCK. In this study, the inexperienced PSTs seemed not to have the awareness of integrating their mathematical understanding into the examples they designed. To them, the purpose of the examples may just be to attract students' interests. Whether PSTs have the awareness of applying SMK to their PCK is an interesting topic which is worth exploring further. The next-step in our studies will explore into the considerations and difficulties PSTs face in applying their SMK to their PCK.

References

An, S., Kulm, G., & Wu, Z. (2004). The pedagogical content knowledge of middle school, Mathematics teachers in China and the U.S. *Journal of Mathematics Teacher Education*, 7(2), 145-172.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching : What makes it special? *Journal of Teacher Education*, 59(5), 389-407.

Behr. M., Harel, G., Post, T., & Lesh, R. (1992) Rational number, ratio, and proportion In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Cai, J. & Sun, W. (2002). Developing students' proportional reasoning: A Chinese perspective. In B.H. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios and proportions* (pp. 195Y205). National Council of Teachers of Mathematics 2002 Yearbook. Reston, VA: NCTM.

Cai, J., & Wang, T. (2006). U.S. and Chinese teachers' conceptions and constructions of representations: A case of teaching ratio concept. *International Journal of Mathematics and Science Education*, 4, 145–186.

Common Core State Standards Initiative. *(*2010*). Common Core State Standards for mathematics.* Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Ding, L. (2014). *Changes in pedagogical content knowledge of secondary mathematics student teachers in Hangzhou during their pre-service teacher education* (Unpublished Ph.d thesis). The University of Hong Kong, Hong Kong.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, 24(2), 94-116.

Fish, D. W. (1874). *The complete arithmetic: Oral and written*. New York: Ivison, Blakeman, Taylor & Co.

Hart, K. (1981). Ratio and proportion. In K. M. Hart (Ed.), *Children's understanding of mathematics*: 11-16 (pp. 88-101). London: John Murray.

He, J. (2013). A historical examination of the treatment of ratio, rate and proportion in U. S., Mathematics Textbooks. In Martinez, M. & Castro Superfine, A (Eds.), *Proceedings of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 94). Chicago, IL: University of Illinois at Chicago.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). London: Lawrence erlbaum associates.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372-400.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Research Council, Mathematics Learning Study Committee, National Academy Press.

Lamon, S. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (S U N Y Series, Reform in Mathematics Education) (pp. 89-120). State University of New York Press.

Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Charlotte, NC: Information Age Publishing.

Leinhardt, G., & Smith, D., A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology*, 77(3), 247-271.

Li, S.Q., Huang, R., & Shin, H.(2008). Discipline knowledge preparation for prospective secondary mathematics teachers: An East Asian perspective. In P.

Sullivan and T. Wood (Eds.), *The international handbook of mathematics teacher education: Volume 1. Knowledge and beliefs in mathematics teaching and teaching development (pp.63-86).* Rotterdam, The Netherlands: Sense Publishers.

Lim-Teo, S., Chua, K., Cheang, W., & Yeo, J. (2007). The development of diploma in education student teachers' mathematics pedagogical content knowledge. *International Journal of Science and Mathematics Education*, 5(2), 237-261.

Ma, L .(1999). *Knowing and teaching elementary mathematics teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, N.J.: Lawrence Erlbaum Associates.

McDiarmid, G. W., Ball, D. L. & Anderson, Ch. W. (1989). Why Staying One Chapter Ahead Doesn't Really Work: Subject-Specific Pedagogy. In M. C. Reynolds (Ed.), *Knowledge Base for the Beginning Teacher* (pp. 193-205). Nueva York: Pergamon Press.

Mullis, I., Martin, M., Gonzalez, E., Gregory, K., Garden ,R., O'Connor, K., .. Smith, T. (2000). *TIMSS 1999 international mathematics report: Findings from IEA's repeat of the Third International Mathematics and Science Study at the eighth grade*. International Study Center, Lynch School of Education, Boston College, Chestnut Hill, MA.

Mullis, I., Martin, M., & Foy, P. (2008). *TIMSS 2007 international mathematics report: Findings from IEA's trends in international mathematics and science study at the fourth and eighth grades*. TIMSS & PIRLS International Study Center, Boston College.

OECD (2013). *PISA 2012 results: What students know and can do (Volume I): Student performance in mathematics, Reading and Science*, PISA, OECD Publishing. doi: 10.1787/9789264201118-en

Park, K. (2005). *Mathematics teacher education in east asian countries*. Paper presented in ICMI- EARCOME 3, Shanghai, Naijing, and Hangzhou, China.

Parker, T. H., & Baldridge, S. J. (2004). *Elementary mathematics for teachers*. Okemos, MI: Sefton-Ash Publishing. ISBN: 0974814008.

Potter, M. A., Neitzel, A. L., Root, D. A., & Enright, F. C. (1952). *Mathematics For Success*. Boston: Ginn.

Schmidt, W. H., Tatto, M. T., Bankov, K., Bloemeke, S., Gedillo, T., Cogan, L., . . . Schwille, J. (2007). *The Prepartion Gap: Teachers Education for Middle School Mathematics in Six Countries Mathematics teaching in the 21st century (MT21)*. MSU Center for Research in Mathematics and Science Education.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 57(1), 1-22.

Tall, D. and Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12, 151-169.
Tatto, M. T., Schwille, J., Senk, S. L., Ingvarson, L., Rowley, G., Peck, R., et al. (2012). *Policy, practice, and readiness to teach primary and secondary mathematics in 17 countries: Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-M)*. Amsterdam: IEA.
Usiskin, Z. & Bell, M. (1983). *Applying arithmetic: A handbook of applications of arithmetic*. Chicago: University of Chicago.
Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65-81). Dordrecht, Netherlands: Kluwer Academic Publishers.
Wilson, S., Shulman, L., & Richert, A. (1987). *150 different ways of knowing: Representations of knowledge in teaching. Exporing teachers' thinking*, 104-124.

Appendix

Sample question for testing student teachers' understanding of equivalent ratio
(1) Please explain the reasons why this statement "20: 8= 10: 4" is valid from a mathematical point of view?
(2) Please explain the reasons why this statement "3: 4/3= 9: 4" is valid from a mathematical point of view?
If you need to explain the two equivalent ratios to your students, which strategies will you use?

Below are two strategies used by two teachers to explain the meaning of equivalent ratio to their pupils.

Teacher A: Jack likes running around a small pond nearby his house. Usually he can run 4 laps in 10 minutes, so he can run 8 laps in 20 minutes with the same speed. We know that his speed could be calculated as 0.4 laps/ per minute, so 8: 20 = 4: 10.

Teacher B: Mary's family bought some fruits at a market. They got 15 apples and 27 pears from the market. For every three fruit, they use one paper bag to wrap them up, so 5 paper bags were used to wrap up 15 apples, and 9 paper bags were used to wrap up 27 pears, so 5: 9 = 15: 27.

Could you explain the rationale behind the two teachers' strategies for explaining equivalent ratio? Why do those strategies work? (From a mathematical point of view)Which one do you think is better, and why?

Will their methods work for three terms ratio or multiple terms ratio? Why?

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