The Mathematics Educator
2014, Vol. 15, No. 2, 31-49

The Honors Dichotomy: Characterizing Students In A US High School Honors Precalculus Class

Marissa Rakes
Naperville Central High School
USA

Alan Zollman
Indiana University Southeast
USA

Abstract: There is strong societal pressure to place students in honors courses. Are students placed in honors courses truly gifted? This mixed-methods study examines beliefs and conceptual understandings of students enrolled in an honors precalculus mathematics class in a United States secondary school. Forty-five students were surveyed, classified into one of two groups, and then four students in each group were interviewed in-depth. Results found a dichotomy between students. One group was concerned with conceptual understanding, making connections, divergent thinking and multiple representation of mathematics. In the second group students were good exercise doers, calculating correct answers and memorizing procedures. Thus teachers need to differentiate lessons as much as possible to challenge all students to reach their full potential.

Keywords: honors, gifted, advance placement, mathematics, diversity

Introduction

If one defines “gifted” by an IQ of 115 and above, then approximately 5% of the population in the United States traditionally is considered “gifted” (Zollman, 2007). Although not large in number, gifted learners do make up an important part of America’s educational system. Many schools offer accelerated and enriched classes to meet the needs of gifted students and help them to achieve their potential. However, in an age where every child is considered a winner when they participate in a competition and nearly every parent believes that their child is gifted, is true giftedness lost?

At the typical Midwestern high school where this study was conducted, 18% of all mathematics classes offered are honors classes – intended for gifted students. If Advanced Placement Calculus and Advanced Placement Statistics are included in this total, nearly 26% of classes are intended for gifted students. This represents a large number of students if only 5% of the population is truly gifted. Our study examines the beliefs and conceptual understandings of high school students enrolled in an accelerated and enriched mathematics class. What are some characteristics that distinguish those students who are truly gifted learners of mathematics from good exercise doers?

Through observation it is clear that students enrolled in honors mathematics classes are not all the same. Some students are *good exercise doers* (as defined by Greenes, 1981, and Zollman, 2007). They prefer to solve problems using the method shown by the teacher and are comfortable memorizing formulas and successfully doing procedures in mathematics. These students generally are successful solving problems only when an example of a similar problem has already been shown to them. They become frustrated when the teacher asks them to expand their thinking or asks them to work through an unfamiliar method without telling them a procedure.

However, a few students in honors classes are quite different. They are able to make connections on their own and solve non-routine problems without teacher assistance. These students have a need to understand why the mathematics they are learning makes sense. It appears that in many high school honors classes we find numerous good exercise doers and a few “gifted” students.

A teacher of high school honors mathematics classes can recognize who are truly gifted learners, although defining what specific characteristics makes a gifted learner is difficult. One often describes these students as having solid fundamentals and being good estimators. They can think logically and systematically about problems, use a variety of problem solving techniques, and they can solve problems in multiple ways, often thinking outside the box and using their own ingenuity (R. Porter, personal communication, March 28, 2012). These students also have a willingness and ability to study topics at a more complex level. They have a desire to prove that things work, and they want to see and make connections between different disciplines (T. Gebbie, personal communication, April 3, 2012).

Background Literature

Research likewise describes characteristics of gifted learners. Greenes (1981) identifies different levels of giftedness and argues that not all students who perform well in school should be considered gifted. Many students are successful academically due to their hard work and diligence, and often they are incorrectly identified as gifted. Greenes labels these students *good exercise doers*, and even though they achieve a level of competence in mathematics, they do not exhibit characteristics of gifted students. There is also an indication that there are different levels of mathematical giftedness possible in students. *Highly gifted students* can apply their knowledge and transfer it to different situations. They can solve non-routine problems, learn content quickly and retain it, and they exhibit higher levels of reasoning. Another level of giftedness can be described as *extremely gifted*. These students develop a mathematical maturity beyond their years and can perform at the level of older students with little instruction. They also can deal with very complex and highly sophisticated problems. Children who are mathematically gifted excel in mathematics and are able to utilize multiple strategies to solve problems, often finding more than one solution to a problem, and they enjoy complex mathematical challenges. Students also can be classified as *mathematically creative*, producing insightful solutions to problems and new and imaginative approaches to mathematical inquiries (Sriraman, 2005; Zollman, 2007).

Mathematical creativity is an important aspect of mathematics, but unfortunately it is often neglected in the classroom. “The essence of mathematics is thinking creatively, not simply arriving at the right answer” (Mann, 2006, p. 239). In an educational system that often rewards accuracy and speed, it is difficult to measure creativity (Mann, 2006). It is also difficult to define. Sriraman (2005) offers a definition of mathematical creativity in students as a process that results in insightful solutions to a problem or related problems or the formulation of new questions that allow an old problem to be viewed from a new angle, thus sparking the imagination. Even though the terms “gifted” and “creative” are not completely synonymous, there are many similarities between mathematical giftedness and mathematical creativity.

Another definition of creativity in mathematics can be described in terms of the following:

> flexibility, shown by the student varying the approach or suggesting a variety of methods; elaboration, shown by the extending or improving of methods; fluency, shown by the production of many ideas in a short time; originality, which is the student trying novel or unusual approaches; and sensitivity, shown by the student criticizing standard methods constructively. (Haylock, 1987, p. 62)

Haylock (1987) also describes fixation in mathematics as a hindrance to creativity. This occurs when a person's thinking becomes set in a certain direction, even if this line of thinking is inappropriate for a given situation. Two types of fixation that are common in mathematics are algorithmic fixation and content universe fixation. The first type occurs when a person continues to use a successful algorithm, even when it becomes inappropriate in a given situation. The second type can happen when a person self-restricts the range of fundamentals that may be used in a given problem. Haylock argues that fixation must be overcome in order for creativity to emerge.

Another way that creativity can be assessed is on the basis of tests for divergent thinking, in which a person is given a problem with many solutions or a situation with many possible responses. They are evaluated on their fluency, or number of responses, flexibility, or categories of responses, and originality, or the infrequency of their responses when compared with others. Divergent thinking is contrasted with convergent thinking, where only one correct solution must be obtained (Haylock, 1987). Both types of thinking are important in mathematics, even though convergent thinking is most often assessed in school. Mathematically gifted students should be able to exhibit both types of thinking.

Methodology

This study sought to identify elements of mathematical giftedness and creativity in high school students. Forty-five mathematics students were surveyed for this study. These students were enrolled in an Honors Precalculus class in a large suburban high school in Illinois. Honors Precalculus is a weighted class, in which an A earns a 5.0 in a traditional 4.0 grading scale, a B earns a 4.0, and so forth. This course is designed mainly for high school juniors who have accelerated in mathematics and who have scored highly on a cognitive ability test that measures quantitative mathematics ability and mathematical reasoning skills. There are some highly gifted freshmen and sophomores who are enrolled in the class, as well as some seniors who have entered into the honors mathematics program later in their educational careers. The course is handled generally in a more rigorous way compared with regular level Precalculus, the course normally taken by seniors. The expectation in Honors Precalculus is that students have the ability to intuitively grasp the concepts being taught and that they are able to think creatively when confronted with various problems.

Quantitative measures

The survey questions (see Appendix A) were designed to elicit feedback from the students regarding their approaches and beliefs about learning mathematics as well as their interest in mathematics. Students were asked to identify through a 5-point Likert scale how important it was to them to find new ways to solve problems other than the teacher showed them, to understand why a solution works; and to see connections between new concepts and those that were previously learned. They also were asked if they were satisfied by merely memorizing a procedure, how important memorization is in mathematics, and what they believed is the difference between an honors mathematics class and a regular mathematics class.

After the surveys were collected, two groups (Group A or Group B) of students were identified. Students were placed into these groups based on three different measures. First, students were sorted into two groups based on their current grade in the class. Students in Group A had an average grade of an 85% or higher in both semesters of Honors Precalculus. Students in Group B had an average grade of below 85% in both semesters. The second measure was teacher evaluation, and two students were removed from Group B based on their teacher's impression of their intuitive ability in mathematics and their ability to make connections and solve new problems on their own. Finally, students were sorted based on their answer to question 4 on the survey, which stated, "I am completely satisfied if my teacher shows me a new mathematical procedure, and I am able to memorize it and use it to correctly calculate solutions to problems." This question was chosen because we deemed that it represented the best division between students who believed that mathematics was about getting correct answers using procedural computation and students who believed that understanding was more important in mathematics. Students who answered that they agreed or completely agreed with the statement were left in Group B, and students who answered that they disagreed or completely disagreed with the statement were left in Group A. Students with an opposite answer were removed from their respective groups. No student was removed from one's group based on an answer of undecided. After all the sorting was completed, nine students remained in Group A and fifteen students remained in Group B.

Qualitative measures

All students were asked if they would be willing to be interviewed individually. Four students from Group B volunteered. Of all the students who volunteered from Group A, four were randomly selected. Thus the same number of students from each group was interviewed. The interview questions (see Appendix B) were designed to investigate each student's ability to think mathematically, using multiple representations, as well as a part of their mathematical creativity in coming up with new ways to explain problems for which they already knew the answer. In the interviews, students also were asked to explain how they learn and understand mathematics. These data were analyzed for commonalities between students in each group to attempt to find characteristics that distinguish students in Group A from students in Group B and ultimately to look for characteristics of mathematical giftedness and creativity.

Results

Quantitative findings

Table 1 shows the averages of student responses on the first seven questions on the survey. Each response was given a numerical value on a scale of 1 to 5, with strongly disagree receiving a value of 1 and strongly agree receiving a value of 5. Each student's responses were recorded, and an average for each question was found among the students of each group.

Table 1
Mean Student Response on 5-Point Likert Survey

		Group A	Group B
1.	Correct answer is most important	3.56	3.47
2.	Best solution is quickest	2.89	3.93
3.	Important to find new ways to solve	4.11	3.00
4.	Satisfied with memorized new procedure	2.00	4.33
5.	Unsatisfied unless I understand new procedure	4.56	3.47
6.	Always want to know concept connections	3.78	3.40
7.	Frequently want to know more	3.56	2.80

Notes: Likert Scale of "1" strongly disagree to "5" strongly agree.
Group A: 9 students: 85%+ grade average, teacher identification as gifted, unsatisfied with memorization.
Group B: 15 students:< 85% grade average, teacher identification as not gifted, satisfied with memorization.

Besides question 4 that was used to determine the groupings a difference in averages greater than 1 was found on questions 2, 3, and 5. Besides simply circling their opinion on a Likert scale that ranged from strongly disagree to strongly agree, students were given the option of writing an explanation for their circled answer if they wanted. Students in Group A were much more likely to write an explanation than students in Group B.

On question 2, “The best solution to a mathematics problem will be the one that is quickest and most efficient,” students in Group A were more likely to disagree with this statement than students in Group B. Two students in Group A wrote about their disagreement, stating, “Slower methods of solving lead to a greater understanding of that problem and all related problems,” and “It will be better if we can figure out a unique or maybe complicated way that can help us with further study.” One student in Group B responded by saying, “easy to follow when learning new concepts it’s better to stick to solutions that are easier to understand.”

Question 3 asked, “When learning a new mathematics concept, it is important to me to try to find new ways to solve problems that my teacher may not have shown me.” Students in Group A were more likely to agree with this statement than students in Group B. Two students in Group A wrote, “When I do problems, I try to find more than one way to do that problem, then I use that to check myself,” and “Making connections to old concepts helps me to understand the new concept better.” Another student in Group A responded to question 3 by writing, “It’s fun to!” Students in Group B responded to question 3 by writing, “Depends on if I like the way it is initially taught to me. If not, I’ll come up with a different way,” “I want to know if it’s right or not,” and “My teacher is smart and usually teaches us the right way to solve a problem.”

Student responses on question 4, “I am completely satisfied if my teacher shows me a new mathematical procedure and I am able to memorize it and use it to correctly calculate solutions to problems,” included one student from Group B who said, “I am mostly happy, but I’d like to understand why it works also.” Student responses from Group A included, “I can’t learn mathematics by just memorizing, it must make sense,” “Memorization only truly works if the concept is understood,” “I hate memorizing things – when people show me a procedure I need to understand how it works,” and “There needs to be a proof with the procedure.”

Question 5 asked, “When I learn a new concept in math, I am unsatisfied unless I understand why the solution works.” Students in Group A were more likely to agree with this statement than students in Group B. Only one student in Group A wrote a response to this question, and she said, “It helps me to understand the concept better." No students in Group B wrote a response to question 5.

In addition to answering the Likert scale questions on the survey, students also responded to three open-ended questions. The first question was, "How important is memorization in mathematics? Explain." Of the 9 students in Group A, 1 student said that memorization was important (11.1%), 4 students said it was not important (44.4%), and 4 students said it was somewhat important (44.4%). Of the 15 students in Group B, 11 said that memorization was important (73.3%), 1 student said it was not important (6.7%), and 3 students said it was somewhat important (20%).

Some responses on the importance of memorization from students in Group A included, "Not at all, if you understand how something really works you don't need to memorize anything," "Memorization is not as important as understanding the method. However, once a concept is understood, you can memorize it to make it a faster process. Understanding is for learning, and memorization is for shortcuts," and "I don't think memorization should be important in math. People should be able to work out a way to do a problem. Memorization makes it easier and faster to do problems." Some examples of student responses from Group B included, "Very. I think you need to memorize the basic foundations of ideas to then apply them in problems," "I think it's necessary to memorize the basic things because all complicated things are built off of the basics. Knowing the basics will allow you to solve harder problems more quickly and easily," "Not important, we should always have formula sheets, you can always Google it in real life," and "Really important, most of mathematics is equations so you need to memorize to do most of math."

The second open-ended question was "What do you think is the difference between an honors mathematics class and a regular mathematics class?" The three most common responses involved the pace of the class, the depth of the material, and the focus on understanding. In their responses, 11 out of 15 students in Group B (73.3%) stated that an honors mathematics class moves at a faster pace than a regular mathematics class while only 2 out of 9 students in Group A (22.2%) mentioned the same. Four out of 15 students in Group B (26.7%) stated that honors mathematics classes went more in depth than regular mathematics classes compared with 3 out of the 9 students in Group A (33.3%). Only one student in Group B (6.7%) said that in an honors class you had to understand the concepts while 3 students in Group A (33.3%) said that honors classes focused on understanding why something works rather than only focusing on how it works.

Qualitative findings

In addition to the survey data, four students from each group were interviewed individually. After students answered each question, the interviewer continued to push students to find another way to explain the solution until the student was unable to think of any other methods. The first question was "What does sin $\frac{\pi}{4}$ mean? Can you explain it in three different ways?" Two students in Group B were able to represent sin $\frac{\pi}{4}$ in four different ways, one was able to represent it in three different ways, and one student represented it in 3 different ways, but one of them was incorrect. In Group A, one student came up with four different ways, two students came up with 5, and one student represented sin $\frac{\pi}{4}$ in 9 different ways.

On the second question, "What does tan $\frac{\pi}{2}$ equal? Can you show me why this is true?" three students in Group B stated that tan $\frac{\pi}{2}$ is undefined, while one stated that it equals zero. All four students in Group A recalled that the ratio was undefined. Two students in Group B were able to explain the reason in two different ways, and two students could explain it one way, although the method one student used was incorrect. In Group A, one student explained why the ratio was undefined in two different ways. One student explained it in three, and two students explained it in four different ways, although one method was not fully formulated.

Question 3 asked, "For a given angle θ, what does $sin^2\theta + cos^2\theta$ equal? Can you show me why this is true?" All eight students knew that $sin^2\theta + cos^2\theta = 1$. One student in Group B verified that the equation was true by substituting a 30^o angle in for θ. She also attempted two more proofs but was unsuccessful. The other three students in Group B were unable to explain why $sin^2\theta + cos^2\theta = 1$. They all explained that they were trying to remember which trigonometric identities were on the formula sheet that they had, but they were unable to recall them and so could not complete a proof. In Group A, one student was able to prove the identity in one way, although she attempted three different ways. Two students completed three proofs, and one student was able to use seven different arguments to explain why the identity was true, although some of them were not completely formulated.

Student Ben. Ben is a senior student from Group B who switched into the honors curriculum after freshman year when he took regular level Algebra 1. He described his mathematical learning by saying, "I try to memorize equations the best that I can so that when I see a part I can go back and plug things in. I don't really think about why things work, and I think that's a downside." When asked why it is a downside, he stated, "I don't always understand others because they have the background knowledge and explain it in a different way than I would." He describes his approach to solving unfamiliar problems by relating "I look for familiar parts and plug in what I know and see if that helps me understand it and keep going." In his response to the first question explaining $\sin\frac{\pi}{4}$, Ben drew an isosceles right triangle in the first quadrant of a graph that had legs of length $\frac{\sqrt{2}}{2}$ and a hypotenuse of 2. He then stated that $\sin\frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{2} = \sqrt{2}$, but then decided "that's not right. It doesn't look right." He later was able to calculate the correct numerical value for $\sin\frac{\pi}{4}$. In the second question, Ben recognized that $\tan\frac{\pi}{2} = \frac{1}{0}$, which is undefined because "you can't divide by zero." When asked why, he responded, "nothing goes into zero." Ben was unable to come up with a proof for the Pythagorean Identity $sin^2\theta + cos^2\theta = 1$. He kept "trying to think of different trig identities" so that he could "get both sides of the equation equal to the same thing." He was ultimately unsuccessful, stating "I feel like there's a proof to this that was shown during class."

Student Kelly. Kelly is a junior student in Group B who approaches mathematics by trying "to memorize because if I know what the basics are it helps me," and then she can "go and apply it to other problems." Kelly said, "If I try to figure out how something works it takes up too much time (on tests). If I have to go back and figure out why something works it takes too much time, so I memorize it and then apply it to the problems." Kelly was able to describe sin $\frac{\pi}{4}$ by describing the ratio of the vertical segment to the hypotenuse in a right triangle drawn in a coordinate plane, by referring to a 45-45-90 triangle and the side ratios, and as $\frac{1}{\csc\frac{\pi}{4}}$ because "they are inverses of each other." She could explain that tan $\frac{\pi}{2}$ was undefined, again "because you can't divide by 0" since "if you have something, you can't divide it into 0 parts because then you wouldn't have anything. You can't have 0 parts if it's there." When asked for another explanation why the ratio was undefined, she asked, "Does it have something to do with the graph maybe? I don't know." For the last question, Kelly responded, "It all equals 1. I know that much – the proof of it? I don't know if I remember. If I had my trig sheet in front of me I would know how to do this." After thinking for a while, she explained that she was "picturing my trig sheet, something about multiplying angles together. I'm trying to find some way where they can cancel out and get 1."

Student Adam. Adam is a sophomore in Group A and was accelerated faster than the typical honors mathematics student. When learning new mathematics he likes "to know where the new formula came from because it usually came from something we learned when we were younger. You can put them together and there's a new function behind it. It's interesting to see new uses for old things." He described that he wants to know why something works because "for all you know it might not be true without the proof behind it." Also, "when you learn how to derive it you can come up with the formula again. If you just learn by memorizing you can forget it." For question 1, Adam explained why $\frac{\pi}{4}$ radians could be represented by 45$^{\text{o}}$ since a circle has 2π radians in it and $\frac{\pi}{4}$ is $\frac{1}{8}$ of 2π. After prompting, he described why a circle has 2π radians by saying, "the radius is the same as the arc length for a radian, and there are 2π of those (in a circle)." Adam explained sin $\frac{\pi}{4}$ visually, numerically, and as having the same ratio as cos $\frac{\pi}{4}$ and sin $\frac{3\pi}{4}$, all with explanations. He explained that the tangent ratio in question 2 was undefined after getting a ratio of $\frac{1}{0}$ using a unit circle since there would be an "infinite number of answers because 0 times anything equals 0." He also attempted to explain an approach based on the fact that there would not be a triangle created when drawn on a unit circle. He described what a graph would look like and referenced the asymptotes as an indication that the ratio was undefined, and he finally rewrote tangent in terms of sine and cosine to achieve the same result as his first explanation. For question 3, Adam wrote a proof using the Pythagorean Theorem and a triangle in a unit circle; he used polar representations for x and y to complete another proof, and he also explained a method using vectors and the dot product.

Student Erin. Erin is a freshman student in Group A and was enrolled in Honors Precalculus, a highly unusual class for a freshman to take. She represented sin $\frac{\pi}{4}$ in nine different ways, including using a variety of trigonometric expressions that would yield the same numerical value, by representing a triangle graphically using both a polar and an imaginary coordinate system and explaining what the ratio would mean in each case, and by describing the ratio with a visual emphasis as both cutting a square in half diagonally and as a right triangle inscribed in a circle with a diameter of $\sqrt{2}x$. For question 2, the basis of all her arguments for the ratio being undefined rested on computing a ratio with a denominator of 0, but she was able to represent the tangent function using a right triangle and the opposite and adjacent sides, sine and cosine ratios, and polar coordinates; she also was able to give an argument based on a circle circumscribed about a right triangle and how that would not work for an angle of $\frac{\pi}{2}$. In question 3, Erin gave six arguments and began another for why $sin^2\theta + cos^2\theta = 1$. These included two versions using the Pythagorean Theorem, two versions using other trigonometric identities, a verification by substituting a value in for theta, a proof using the area of two rectangles based off of a proof for the Pythagorean Theorem that she had seen once, and another attempt at an argument using area. For all of the questions, Erin generated solutions so quickly that the interviewer could barely keep up with her. She explained her prolific production of solutions by saying that she tries to "think of everything I can remember that has to do with the problem. Then I improvise on it based on what I know." She also said, "I like to think geometrically, and I visualize what the problem would look like."

Discussion

Responses to the survey questions suggest that the students in Group A were much more concerned with understanding why mathematical solutions make sense than students in Group B. Some Group B students expressed a belief that there is only one correct way to solve a problem and that the purpose of mathematics is only to use formulas to calculate solutions. Students in Group A referenced making connections to other concepts and looking for multiple solution methods. They also seemed to see the value of emphasizing understanding, as they recognized that the honors mathematics classes that they have taken make this a priority. Students in Group B did not seem to view honors classes with this lens, mostly remarking on how they moved at a faster pace and expected students to catch on more quickly. On the topic of memorization, students in Group A emphasized that memorization can have value in situations where shortcuts or quick solutions are required, but conceptual understanding is more valuable. Many of them also wrote about the connectedness of mathematics and about how most required formulas can be derived when they are needed. The survey questions show that the mathematical beliefs of the students in Group A are corresponding to characteristics of mathematically gifted students.

The interviews of the students in Group B showed evidence of fixation, in particular of content universe fixation. Most of them focused on the unit circle definition and explanation for trigonometric functions. The exceptions were one student who referenced right triangle trigonometry in question 1 and two students who used other trigonometric identities to represent $\sin\frac{\pi}{4}$. The only completely valid and fully formulated explanation students in Group B were able to give for question 2 involved the $\frac{y}{x}$ ratio of tangent in the unit circle representation. For question 3, all four of the students in Group B unsuccessfully attempted to remember the proof that had been done in class, showing a fixation on methods taught by the teacher.

This semester, the teacher has mostly focused on a unit circle interpretation of trigonometry, so it is understandable that students would remember that representation. However, the inability of many students to break from that interpretation shows fixation.

In contrast, students in Group A showed evidence of divergent thinking in their interviews. They were able to come up with many different interpretations and representations for different trigonometric expressions, not all of which involved a unit circle. This is especially evident in Erin's interview, as she was able to make connections to multiple different areas of mathematics. Students in Group A seemed to realize that there was more than one correct answer. They also were more able to break their content universe fixations. Both the break from fixation and divergent thinking are evidence of mathematical giftedness.

Tall and Vinner (1981) use the term concept image to describe the cognitive structure that a person builds up related to a particular topic. The concept image can include properties, processes, and visual images. It matures over time, and it is always adapting as more stimuli are added to the person's structure. A related idea is the concept of example space, referring to the class of examples that can be brought to mind when considering a concept or technique. This also includes the whole space of examples that a person is aware that they would be able to generate in a particular situation (Mason, 2008). The students in Group A seemed to have a more fully developed and mature concept image of trigonometric functions and identities, and they have a more developed and varied example space. They were able to give multiple representations and proofs for each expression rather than persisting with the unit circle approach for all explanations. The concept images of students in Group B likely contained links primarily to the unit circle, and they likely had limited example spaces that contained the unit circle representation and little else.

Conclusions

Using the literature definition (Greenes, 1981; Zollman, 2007) the students in Group B are good exercise doers. Most of them were able to correctly answer the questions in the interview using the methods that had been learned in class. In general, they believe that the method taught by the teacher is generally best and are not concerned about finding new methods. They are happy to calculate correct answers and memorize procedures, and they generally perform well in school. In contrast, the students in Group A show evidence of mathematical giftedness, and even some evidence of mathematical creativity. They showed flexibility, fluency, and originality in their solutions to the interview questions. They also showed a belief in the connectedness of mathematical concepts and the importance of achieving understanding in a subject. They want to move beyond completing exercises and want to delve into what mathematics is truly about.

More research is needed to fully understand how students similar to those in Group A and Group B think about and learn mathematics. This study focused on multiple representations and beliefs about mathematics, but there are many other avenues that still need to be explored, such as the ability of students to solve complex and non-routine problems, fluency, flexibility, and mathematical creativity. More information will assist teachers as they guide their students learning mathematics.

The implications for teaching practice are many. With the societal pressure to place students in honors and Advanced Placement high school courses, Group A and Group B students will be mixed in these classes. Therefore, teachers should seek to identify mathematically gifted students in their classes and notice the differences in levels of giftedness that are present. Teachers should differentiate lessons as much as possible to challenge those students who are truly mathematically gifted and not just good exercise doers. Mathematically gifted students become bored easily with repetition, and teachers should allow for and plan for creativity in the classroom. Students should be allowed to create their own solution paths and follow their own ideas at times instead of staying within narrow procedural boundaries of the curriculum. For Group B students, it also is vital to break from the well-established norm of only valuing convergent thinking in the mathematics classroom. Teachers should ask questions that reward divergent thinking as well. With some preparation and awareness on the part of the teacher, gifted students will be more likely to thrive, but all students will benefit as they stretch themselves in reaching their full potential.

References

Greenes, C. (1981). Identifying the gifted student in mathematics. *The Arithmetic Teacher, 28* (6), 14-17.

Haylock, D. W. (1987). A framework for assessing mathematical creativity in schoolchildren. *Educational Studies in Mathematics, 18* (1), 59-74.

Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted, 30* (2), 236-260.

Mason, J. (2008). From Concept Images to Pedagogic Structure for a Mathematical Topic. In M. P. Carlson, & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (pp. 255-274). Mathematical Association of America.

Sriraman, B. (2005). Are giftedness and creativity synonyms in mathematics? *The Journal of Secondary Gifted Education, 17* (1), 20-36.

Tall, D., & Vinner, S. (1981). Concept image and concept definition with particular reference to limits and continuity. *Educational Studies in Mathematics, 12* (2), 151-169.

Zollman, A. (2007). Revisiting the needs of the gifted mathematics students: Are students surviving or thriving? *Mediterranean Journal for Research in Mathematics Education, 6* (1 & 2), 139-147.

Appendix A

5-Point Likert Survey

Please answer each question honestly. For questions 1 – 7, please circle the word or phrase that most describes your choice. After each question numbered 1 – 7, there is space below the choices if you would like to explain your answer, but this is not necessary.

1. When solving a mathematics problem, the most important thing is to get the correct answer.

Strongly Disagree	*Disagree*	*Undecided*	*Agree*	*Strongly Agree*

Explanation, if desired:

2. The best solution to a mathematics problem will be the one that is quickest and most efficient.

Strongly Disagree	*Disagree*	*Undecided*	*Agree*	*Strongly Agree*

Explanation, if desired:

3. When learning a new mathematics concept, it is important to me to try to find new ways to solve problems that my teacher may not have shown me.

Strongly Disagree	*Disagree*	*Undecided*	*Agree*	*Strongly Agree*

Explanation, if desired:

4. I am completely satisfied if my teacher shows me a new mathematical procedure, and I am able to memorize it and use it to correctly calculate solutions to problems.

Strongly Disagree	*Disagree*	*Undecided*	*Agree*	*Strongly Agree*

Explanation, if desired:

5. When I learn a new concept in math, I am unsatisfied unless I understand why the solution works.

Strongly Disagree	*Disagree*	*Undecided*	*Agree*	*Strongly Agree*

Explanation, if desired:

6. I always want to learn how the mathematics concept that I am learning relates to other mathematics concepts that I have already learned.

Strongly Disagree	*Disagree*	*Undecided*	*Agree*	*Strongly Agree*

Explanation, if desired:

7. When I learn a new mathematics concept, I frequently want to know more about that concept.

Strongly Disagree	*Disagree*	*Undecided*	*Agree*	*Strongly Agree*

Explanation, if desired:

Please answer the following three questions.

8. How important is memorization in mathematics? Explain.

9. How do you approach a new mathematics problem that you have not seen before?

10. What do you think is the difference between an honors mathematics class and a regular mathematics class?

Appendix B

Interview Questions

1. What does sin $\frac{\pi}{4}$ mean? Please explain it in three different ways.

2. What does tan $\frac{\pi}{2}$ equal? Please show me why this is true.

3. For a given angle θ, what does $sin^2\theta + cos^2\theta$ equal? Please show me why this is true.

Authors:
Marissa Rakes, Naperville Central High School Naperville, Illinois, 60540, USA, mrakes@naperville203.org; **Alan Zollman** (corresponding author), School of Education (Hillside Hall 0020), Indiana University Southeast, New Albany, Illinois 47150, USA; alanzoll@ius.edu.