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Competency of Prospective Chinese Mathematics Teachers on Mathematical Argumentation and Proof

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Abstract: The paper reports about a study on the knowledge of prospective mathematics teachers from China (Hong Kong and Mainland China) with regard to argumentation and proof as well as beliefs about using preformal and formal proofs in lessons based on a study design developed by Schwarz (2013). The study pointed out that the prospective teachers were generally strong in algebraic reflections and operations, but they lacked reflections on mathematical argumentation and proofs and their adequate teaching, for example, how to provide adequate feedback about proving approaches. Furthermore, differences concerning the acceptance of preformal proofs between prospective teachers from Hong Kong and Mainland China were found out in this study, which may be related to differences in the mathematics curriculum in these two regions. Implications for teaching proofs in geometry and algebra are discussed.

Keywords: Mathematical proofs; Teacher education; China; Geometry; Algebra

Introduction

Teacher's professional knowledge is receiving increasing attention in studies of curriculum and pedagogy. Past research has shown that Chinese teachers have certain strengths in teaching. On the one hand, there is empirical evidence that Chinese teachers in general possess strong mathematics subject knowledge (Ma, 1999), amongst others, because there is a long tradition of emphasizing basic knowledge and skills among their students (Fan et al., 2004; Watkins & Biggs, 2001; Wong, 2006, 2007). However, current international education reforms call for a paradigm shift from "produce" (content) to "process" (ability), and it is of interest to evaluate whether these strengths can be preserved with the new objectives of this reform. In particular, reforms in mathematics education around the world have turned increasing attention on "higher order thinking skills," such as reasoning, communication, proofs, and argumentation (Wong, Han, & Lee, 2004) in school and teacher education. However, the study by Schwarz et al. (2008) comparing the competencies of prospective mathematics teachers from three regions, namely Germany, Australia, and Hong Kong, found that these teachers were unable to execute mathematical proofs adequately, even when the proofs focused content from lower secondary level. Possibly these prospective teachers have not received sufficient preparation for teaching proofs, even though they have studied the required mathematical background, such as undergraduate mathematics in teacher education institutions. In this study, we addressed the same issue by investigating the competencies of prospective teachers from a *similar* cultural background (Chinese): Hong Kong (HK) and several cities of Mainland China (MC). This kind of study comparing the professional knowledge of different groups of prospective teachers from a similar cultural background provides a complementary approach to cross-cultural studies. Studies on the professional knowledge of prospective teachers are relevant for teacher preparation systems all over the world bearing in mind the issue of the "preparation gap" of teachers, as stated by the Mathematics Teaching in the 21st Century study (Schmidt et al., 2011), or the IEA-study Teacher Education and Development Study in Mathematics (TEDS-M) (Tatto et al., 2012). Both studies point out the insufficient preparation of prospective teachers in the light of their future professional tasks. Pre-service programs that prepare China prospective teachers to help students to acquire higher order thinking skills such as proving are a crucial topic, which need to be addressed by empirical research.

Theoretical Framework

Deductive reasoning used to prove mathematical propositions or theorems is an essential skill that students must develop, yet it is difficult to teach this type of

reasoning. The difficulty lies in the fact that, before proving, conceptual understanding of the ready-to-be-proved proposition is essential. In-depth studies carried out in recent decades show how well-prepared prospective teachers understand certain topic-specific mathematical concepts, such as the definitions of functions and composite functions (Meel, 1999; Sanchez & Llineares, 2003), how crucial their knowledge of proof by mathematical induction is (Styliandies, Styliandies, & Philippou, 2007). Most, if not all, of these studies show those prospective teachers generally lack a complete understanding of the underlying meaning of such topic-specific concepts and have difficulty teaching them (Selden, 2012). To prove a proposition, teachers often begin with an idea or property by drawing rough sketches or preparing tables of numbers. For example, to determine the curved surface of a circular cone, the teacher may fold a sector and not a triangle. This demonstration, obviously not a proof, shows students that the curved surface area of a circular cone is not equal to the area of a triangle. However, writing out explicitly a proof of this property is harder for both teachers and students. In general, a deductive proof requires the teachers to show a "chain of well-organized deductive references that uses arguments of necessity" (Hanna, & De Villiers, 2008, p. 3).

There exist in the extensive literature on argumentation and proof in mathematics education several classification schemes for proofs (for an overview see Hanna, & De Villiers, 2012). In the German tradition on mathematics education, the classification by Holland (1996) is often referred to. He distinguished between three levels of argumentation and proof: level of *argumentation,* level of *content-related conclusion,* and level of *formal proving;* however, these levels are difficult to distinguish accurately. Holland (1996) defined proofs on the level of argumentation as oral arguments, verbal narrations, figures, or concrete models of illustrations and demonstration. The combination of the level of argumentation and the level of content-related conclusions, on which proofs are written down and in addition more or less are noted stepwise but do not meet the criteria of formal proofing (ibid.), leads to the concept of preformal proofs (Blum, & Kirsch, 1991). Preformal proofs hinge on a prerequisite understanding of ready-to-be-proven propositions through proper observation and visualization. Blum and Kirsch (1991) claimed that preformal proofs may be sufficient for learners with weak cognitive ability. This suggests that preformal proofs are sufficient for less capable mathematics learners or those who do not need rigorous understanding of mathematics in their future studies or career.

On the other hand, preformal proofs may not work properly even for very simple problems, such as to justify whether a pair of given straight lines on a plane is parallel or not. Thus, formal proofs remain essential for proving mathematical

propositions, although preformal proofs may be used to stimulate learners' cognitive activity and derivation skills that will be required in the later stages of formal proofs.

Teachers' beliefs about the importance of formal proofs may reflect parts of their competence (e.g., Törner, 1998). This study also investigated beliefs of prospective teachers about the importance of preformal and formal proofs. In general beliefs about mathematics as a scientific discipline, beliefs about teaching and learning mathematics, beliefs about schooling of mathematics in general, and beliefs about teacher education and professional development can be seen as the main components of the belief-oriented aspect of professional competence. These beliefs may be influenced by different cultures and educational ideologies. Hong Kong is strongly influenced by the British tradition, and it has undergone the so-called "down with Euclidean geometry" development. On the other hand, Mainland China has been influenced by Russian educational positions, leading within mathematics education to a situation, where Euclidean geometry is still highly valued. This study will shed light on the question, whether these differences in the educational environment have led to differences in the knowledge of proving and the beliefs on proofs by prospective mathematics teachers in Hong Kong and Mainland China. This study will extend the findings of Schwarz et al. (2008) in order to understand the competency of mathematics teachers from these two regions concerning proofs and proving.

Research Question

We can now formulate the main research question of this study as follows:

> What are the differences and similarities in the knowledge and skills on argumentation and proofs (preformal and formal) of mathematical propositions held by prospective teachers of Hong Kong and Mainland China?

Methodology

Design of the study and test instrument

The design of the study including the test instrument was developed by Schwarz within his PhD study (2013). The test instrument and its evaluation procedure have already been used in Germany, Australia, and Hong Kong (Schwarz et al., 2008). The test instrument contains amongst other tasks about formal and preformal proving. The participants were asked to comment on the respective proofs and offer responses to hypothetical students' argumentations and proofs. Under a theoretical perspective, the distinction of the knowledge facets refers to the framework of the

MT21 study (Schmidt et al., 2007). The following two kinds of teacher knowledge were investigated by the study (cf. Bromme, 1995; Shulman, 1986).

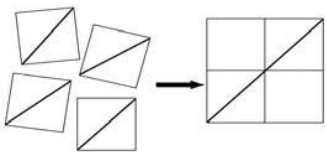
(1) Mathematical content knowledge (MCK): based on logical reasoning and proofs and modeling in real-world contexts. This requires cognitive activity, mathematics subject matter knowledge (SMK), and an understanding of the complexity of school mathematics for students.

(2) Pedagogical content knowledge in Mathematics (MPCK): knowledge of implementing the curriculum and judgment of students' ability to work on certain mathematical problems, diagnosis of students' mistakes, and their capability of understanding abstract mathematics concepts, and teaching related activities.

Due to space restriction this paper is limited to two groups of items referring to one situation about geometry and one on algebra, as shown in Figures 1 and 2.

Proposition 1: *If the length of the side of a square is doubled, then the diagonal will also be doubled.*

Preformal proof: *We use square tiles of the same size. Using four tiles to make one square gives* a *square with a side length that is twice the length of the square tiles. We can see immediately that each diagonal is twice the length of the diagonals of the square tiles because it comprises two diagonals of two tiles put together.*



Questions:

(a) *Is this argumentation a sufficient proof for you? Please give a short explanation.*
(b) *Please formulate a formal proof for the statement about squares and diagonals.*
(c) *What proof would you use in your mathematics lessons?*

Explain your position.

(d) *Can a preformal proof be sufficient as the only kind of proof in a mathematics lesson? Please explain your position.*
(e) *Please state the advantages and disadvantages of formal and preformal proofs.*
(f) *Can the preformal and the formal proof for the statement about the length of diagonals in squares be generalized for any rectangle? Give a short explanation.*
(g) *What do you think is the meaning of proofs in mathematics lessons in secondary school?*

Figure 1. Geometric Proof (from Schwarz et al., 2008, p.795).

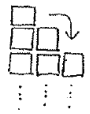
Proposition 2: *If you add three consecutive natural numbers, the result will always be divisible by 3 without any remainder.*
Students gave the following ideas to prove the proposition above:

Peter:

3+4+5 = 12 and 12 ÷ 3 = 4
9+10+11 = 30 and 30 ÷ 3 = 10
14+15+16 = 45 and 45 ÷ 3 = 15

So that works always and the statement is right.

Agnes:



By doing so, you can see that the result is always divisible by 3 without remainder.

William:

If n is the first of the three numbers, then the other numbers are n+1 and n+2. So the sum is n + (n+1) + (n+2) = 3n + 3= 3(n+1). So you can see, that the result has always to be divisible by 3.

Ben:

You can't say, if the statement is true, because there are infinite natural numbers, since for every number exists always another one, which is larger by adding 1. And so you can never say, if you proved it for all the others.

Questions:

(a) *Imagine that, as a teacher, you should formulate a response for every idea, which assesses the idea on the one hand, and provides motivation for further exploration or correction. What would you tell each of the students?*
(b) *What kind of idea of a proof becomes apparent in each of the students' statements?*
(c) *How would you prove the proposition in a Form 3 lesson?*
(d) *Is it possible to generalize this statement in the following way?*

If you add k consecutive natural numbers, the result is always divisible by k without any remainder.
Please write your answer below and give a short explanation for it.

Figure 2. Algebraic Proof (from Schwarz, 2013, p. 241).

Participants

The participants were prospective teachers in their third or fourth year of undergraduate study in education or a postgraduate diploma in education with mathematics as the major subject. They will teach in secondary schools after graduation. The sample consisted of 222 prospective teachers from various cities

coming from different parts of Mainland China such as Beijing, Shanghai, and Guangzhou. Although the future teachers came from different cities, it has to be considered that the curriculum of mathematics teachers' training is quite unified across Mainland China. Sixty nine prospective teachers from Hong Kong participated in the study. The data were collected from November 2007 to June 2008. The sample can be characterized as convenience sample and the findings may not be generalized to these two regions.

Data analysis

Different scoring rules and rubrics were used to code the responses of different items. The levels of items that tested knowledge were rated using 3 points (*high*, *average*, or *low*) or 5 points (-2, -1, 0, +1, +2). In the case of the question about formulation of formal proof, the highest score of +2 was given to a mathematically correct and complete proof that is characterized by the following elements: (i) quantities are assigned by referring to a sketch or a verbal description; (ii) mathematical theorems used in the proof (e.g., Pythagoras theorem, Mid-point/intercept theorem) are mentioned; (iii) each step of the proof is connected through structuring elements (e.g., equal and implication signs, verbal conjunctions like "therefore" or "from that"). Figure 3(a) shows a correct and complete description of a proof using Pythagoras Theorem, with score +2, whereas 3(b) is an incomplete proof with score 0, because there was no justification of which pair of similar triangles was considered and what was the final step of the proof.

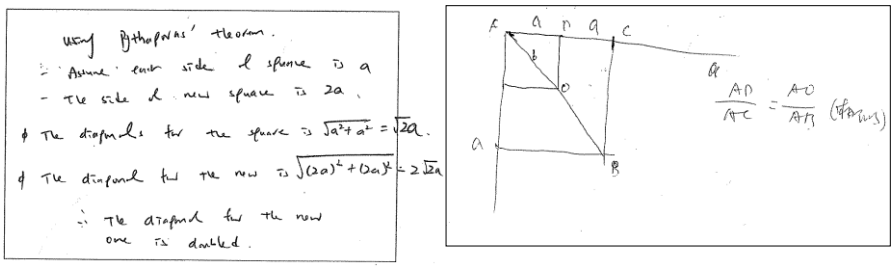


Figure 3. Sample (a) Score +2, (b) Score 0.

Belief items that tested the preference to use preformal proofs or formal proofs in lessons were coded as *acceptance* or *rejection*, also with the same 5-point scale. The item about advantages or disadvantages of formal and preformal proving was coded on a 3-point scale (-1, 0, +1). The category "Other" was used to cover responses that were not decisive in stating the points, irrelevant, spurious, misinterpretations of the questions or situations, crossed out, or left blank.

Five experts (3 mathematics educators, 1 graduate student, 1 research assistant) separately coded 10 sample answers, and discussions were held to resolve discrepancies between the different coders. After the coding manual did not contain any more ambiguities, the coding was carried out by the research assistant. When ambiguity came up in single students' answers, the research assistant then called for a meeting to discuss the case and come to a consensus. The coding was controlled by one mathematics educator in detail.

Results

The results are reported by individual items.

Item 1(a) The prospective teachers (PT) tended to reject the preformal proof. To them, preformal proofs were not sufficient. They believed that a written algebraic algorithm and a geometric deduction written out in steps were necessary as a formal proof of a geometric proposition. For HK's PT, this may be related to the influence of the New Mathematics curriculum in which formality and abstraction was much stressed. Drafts, diagrams, or tables should not be part of any proof. This is further reinforced by high-stake examinations, in which only final answers are given marks. As reported by Wong et al. (2002), HK students tend to write down only correct and formal derivations as solutions to avoid losing marks for wrong statements.

Item 1(b) The proofs given were scored on the 5-point scale. Most of the proofs used Pythagoras Theorem (63 out of 67 from HK), with a few involving the Mid-Point Theorem and properties of similar triangle. The choice of Pythagoras' Theorem indicates that these PT were familiar with proving mathematical statements using algebraic derivation and calculation. This may be the result of the heavy emphasis of both the MC and HK syllabuses on these areas (CDC, 1999, 2007; Leung & Li, 2010).

Item 1(c) About one third of HK PT preferred formal proof compared to only 18% of MC's. Thus, they would use formal proofs in their lessons, and may treat preformal proofs as auxiliary explanation and argument. About 32% of HK and 22% of MC would consider the students' capability to understand as the reason for using a particular type of proof in the lessons. This does not necessarily indicate that PT believed formal proofs to be harder to understand than preformal proofs, or vice versa. Rather, it reveals the idea of *it all depends on students' cognitive capability*. An interesting finding was that 23% of MC PT put the executing order (sequence) of the delivery of the proof as the main concern, whereas only 4% of HK PT mentioned this. Formal proofs usually have a format that is easy to follow if one is

satisfied with a stereotypical solution. With preformal proofs, a teacher has to express his or her way of thinking or offering scaffolding, which is more difficult to articulate. In the early 1990s, the Hong Kong Examinations and Assessment Authority called for standardizing the reasons given in proofs (e.g., "the base angles of an isosceles triangle are equal"), but other educators had argued against this standardization so that candidates should be given full marks as long as the reason is sound. For large scale examinations, inevitably, markers opt for unified, mechanical ways of marking, which rule out other positions. As undesired consequence of these examinations one can observe that this undesirable teaching practice of paying insufficient attention to preformal proofs has become influential in many HK classrooms.

Item 1(d) The responses were coded on the 3-point scale with lower score showing poor or baseless argument for preformal proofs as the only type of proofs to be used. About 53% of HK PT scored low compared to 36% of MC PT, indicating that HK PT had poorer didactic reflection.

Item 1(e) There was an even distribution of levels of competency of PT to give the advantages and disadvantages of formal and preformal proofs across the 3-point scale. Those who scored low in this item may know merely "how" rather than "why".

Item 1(f) There was an even distribution of competency to generalize the proposition to rectangles among the PT on the 5-point scale. In Figure 4, the answer on the left scores +2 as the candidate correctly identified the generalization and gave a correct proof, while the one on the right was completely wrong (score -2).

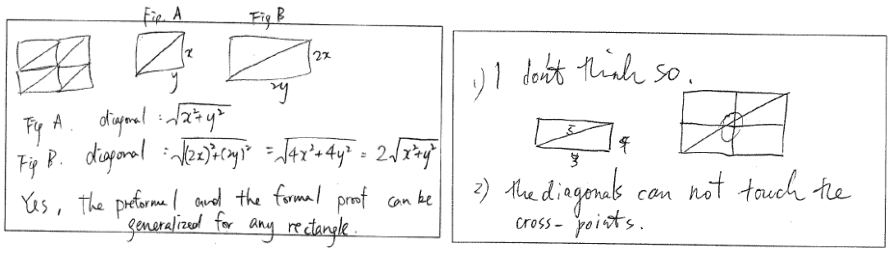


Figure 4. Sample Answers for Generalization coded as 2 and -2.

Item 1(g) Slightly more than half of the PT in each region could describe the meanings of proof, which were scored at +1 point such as the example in Figure 5: the PT gave a positive response and considered one of two areas: arguments from *mathematics didactics referring to the image of mathematics/problem solving*

techniques, or to *the ability of the students.*

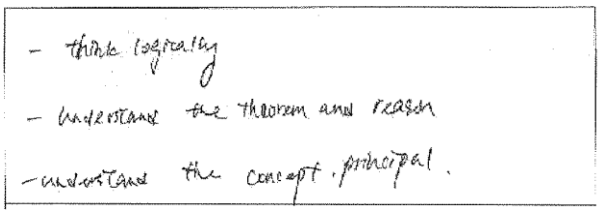


Figure 5. Sample Answer Coded as +1.

Item 2(a) One measure of teachers' PCK is their ability to identify students' misconception and to motivate students' further exploration and correction by providing them with proper feedback. Prospective teachers from both regions had weak competency in giving proper feedback (only 12% and 11% of MC and HK PT scored 1 or 2). Furthermore, 20% (HK) and MC (37%) of the PT gave responses that could not be scored.

Item 2(b) This item measures ability to make adequate judgment about student work, another important measure of teacher competency. The result was slightly better than for item 2(a), but PT from both regions still had weak knowledge in making judgments about proof attempts (only 22% and 32% of MC and HK PT scored 1 or 2). However, more than 40% of the responses from each region could not be coded. These PT did not realize that Peter's attempt is a mere verification using a specific example, Agnes' sketch of three columns of small squares is a preformal proof, and William's attempt contains the essential part of the proof by Mathematical Induction. All the PT did not accept Ben's attempt as a correct argument, nor a preliminary explanation of a proof.

Item 2(c) Although not stated explicitly, we expected the PT to refer to the four examples when they described the proof to use in their lesson. Most of the PT chose the attempts by Agnes (preformal), William (Mathematical Induction), or a combination of these two. Most PT (59% from MC and 85% from HK) chose to teach either a purely formal proof or proofs of both forms. Only 2% of MC and 1% of HK PT intended to use a preformal proof. These findings suggest that prospective teachers tend to use algebraic deduction and derivation to justify the truth of proposition.

Item 2(d) Mathematics teachers should be able to judge that the proposition does not hold for an arbitrary k. A counter-example is enough to refute the given proposition. Surprisingly, only 40% of MC and 57% of HK PT obtained the

completely correct score. Thus, a large proportion of these PT did not know how to disprove a proposition in a mathematically proper way.

Discussion

Affinity for preformal proofs

Competency in proving can be attained gradually in a progressive fashion. Although formal deduction is the final destination, preformal proofs are a prerequisite that can be treated as an acceptable form of analysis and explanation for a proposition. However, in this study, a high proportion of PT rejected preformal proofs as sufficient argumentation. They may have kept to the definition of deduction given in the curriculum, which states that mathematical "deduction" is to use "known conditions" and "verified properties and correct characteristics" to derive the conclusion. Thus, some of them may not accept that putting the tiles together constitutes a "geometric deduction" or the sketch of three columns of small squares in Agnes' proof constitutes an algebraic derivation, thereby rejecting them as sufficient. Unlike PT from MC, HK PT tend to focus more on the understanding of mathematics and an adequate image of mathematics, therefore were more inclined towards the usage of preformal proofs as a kind of aid in teaching. However, despite this positive affinity to preformal proofs by a significant group of HK PT, many HK PT felt that the mathematical processes such as deduction, reasoning, and logics are most important.

Another interesting finding was the low acceptance of preformal proofs among MC PT (18%). The MC curriculum gives much detail on the requirements of statements to justify a proof, such as definition, propositions, conditions, meaning of the proof, examples and counterexamples, and proposition versus inverse proposition. In the teaching and learning of proofs, stepwise and logical arguments and the mathematical reasons must be clearly stated. Thus, the preformal proof using tiles as a demonstration cannot be accepted mathematically by students and teachers who have been taught this meaning of geometric proof. Similarly, they cannot treat the proof by Peter (using one example) or Agnes (sketching three columns of small squares) in item 2 as a formal algebraic proof. Indeed, MC and HK prospective teachers did not show much difference in their affinity for preformal proof, at least for the algebraic proposition. In contrast, in the study of Schwarz et al. (2008), the HK prospective teachers were stronger in MCK, while prospective teachers of Germany and Australia were relatively stronger in PCK in using preformal proofs in teaching.

Beliefs about preformal proofs

The prospective teachers in this study tend to consider proofs with a formal logically algorithmic procedure as formal rather than preformal. For them, preformal proofs are less rigorous and are only suitable for lower achieving students, whereas a formal proof is needed for more capable students. A few prospective teachers stated that they would not exclusively use preformal proofs in their lessons. This aligns with the belief that preformal proofs are not sufficient proof of a proposition and that a more rigorous proof is needed in teaching a proposition. Derivation seems to be the main choice in teaching.

On the other hand, some PT believed that preformal proofs are conducive to learning. Indeed, preformal proofs have an essential role in the illustrative procedures of a mathematics course as a legitimate form of proof. As a kind of dual delivery of conceptual notions (Mayer, 2003), showing pictures with a written sequence of logical statements and mathematical symbols is likely to enhance learning effectiveness. Moore (1994) identified the three major sources of difficulties when students carry out a proof: their conceptual understanding, the mathematical language and notion, and where to start. The *quasi-animated* picture of tiling squares in item 1 gives a hint that leads the student to consider the property of triangles (similarity) and the relationship about right-angled triangle (Pythagoras' Theorem). Similarly, the sketch of the three columns of small squares by Agnes (item 2) gives hints that the two rows of three squares must be a multiple of three, hence divisible by three. These tiling squares or sketch play a role in the transition from a loose, initial idea to formal proof (Moore 1994), and may be a starting point for a formal proof. We may treat preformal proofs as stimuli to help learners generate insightful understanding of where to start a proof for a particular proposition. Preformal proofs should thus be taken seriously as a prerequisite component during the process of deriving a complete proof. Neglecting preformal proofs may lead mathematics learners to gloss over important ingredients of mathematical content stated in the propositions. Further research is needed to identify the extent to which teaching effectiveness might improve after the skillful application of preformal proofs.

Paradigm shift in geometry learning

Geometry has been one of the most important subjects in mathematics education for primary and secondary students in MC and HK since the beginning of the last century. Understanding Euclidean geometry and knowing how to produce deductive proofs are traditional goals for students. During the 1960s, mathematics education in HK underwent "modernization." In addition to the theories of psychologists such as Bruner, Piaget, and Gagné, the development of geometry was deeply connected with the new mathematics movement in the United States. Linear systems, calculus,

and symbolic logic were positioned as the three foundation stones in modern secondary school mathematics (Wang & Lu, 1981, p. 227). In HK, during the Modern Mathematics movement in the 1960s, along with the world trend, the learning of Euclidean geometry and formal proofs was downplayed and "replaced" by symbolic logic and abstract mathematical structures, such as groups, rings, and vector space. However, many students generally found symbolic logic and mathematical structures too abstract to comprehend, and educators did not consider Modern Mathematics a more effective way to learn argumentation and reasoning (Wong, 2001). With the popularizing of the geometry learning theory of van Hieles in the 1980s, the shift from concrete to abstract and from visual to axiomatic became mainstream in the teaching and learning of geometry. Experimental (hands-on) geometry and the role of geometric transformations were emphasized before the learning of deductive geometry. In the late 1990s, formal geometric proofs were reintroduced in HK and became popular again (Cheung et al., 2010).

On the other hand, Mainland China was never influenced by the New Mathematics movement. Learning Euclidean geometry and deductive proofs remain the two main objectives in the geometry curriculum for primary and secondary school students. As with other topics in mathematics, the two "basics" of fundamental knowledge and basic skills were emphasized in the teaching and learning of geometry in the 1980s. In the 1990s, in addition to these two basics, fostering "abilities" in geometry and the "process" of fostering these abilities were emphasized (Zhang, 2006; Zhang et al. 2008). The above findings of prospective teachers' affinity for preformal proofs in geometry are probably curriculum dependent as noted above.

Conclusion

This investigation reveals that the prospective teachers in the two Chinese regions do not treat preformal proofs seriously as a necessity in the process of creating complete proofs. Helping prospective teachers to realize the advantages of executing preformal proofs in the classroom, especially for less capable students, inevitably requires a reconsideration of the curricula of teacher training programs, such as offering courses for proofs, where proving tasks are dealt with in detail as well as ways how to treat them in classroom. Doing so will hopefully strengthen the attitude, knowledge, and skills of prospective teachers towards the use of preformal proofs for the betterment of mathematics classroom teaching.

The prospective teachers in this study regarded formal proofs as a rigorous, although not necessarily legitimate, approach to use in their lessons. Showing sequential deductive steps in proofs and arguments seems to be accepted as a more

robust method of proving geometry properties, although some of them also believed that other methods (preformal ones) may be correct. This leads to a fundamental question raised by many teachers of whether it is necessary for those who may not be ready or capable of further study in more advanced mathematics to learn the rigorous approaches to proofs. The answer to this question will affect curriculum design for mathematics teacher education, and in particular how advanced the mathematical content knowledge is required of prospective mathematics teachers before they graduate from education colleges. Subject content knowledge is vital. In this study, proving the truth of the proposition by Pythagoras' Theorem is less rigorous than using the similarity of triangles, because the sufficiency of the former is apparent (the existence of a right-angled triangle), whereas several steps of deductive arguments are needed to justify the similarity of the pair of triangles. The ability to note differences in the cognitive load on students when they work through different proofs (e.g., Pythagoras' Theorem versus the similarity of triangles in item 1; figurative representation of three consecutive integers versus three columns of small squares in item 2) reflects the level of teachers' pedagogical competency. Indeed, the readiness and willingness to use preformal proofs displays a rich image of mathematics with proofs and preformal proofs being an essential part of mathematics and its teaching processes. Such a rich image is necessary for highly productive learning and teaching processes of mathematics.

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