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Pre-service Teachers' Knowledge about Fraction Division Reflected through Problem Posing

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Abstract: This study investigated the conceptual understanding of fraction division of 72 Hong Kong pre-service teachers by asking them to pose a real life story problem. Analyses of the problems posed show that many of these pre-service teachers had incorrectly interpreted the division concept in their problems as sharing rather than measurement. A short intervention shows significant improvement in posing problems in terms of mathematical correctness as related to real life situations. This study particularly shows an increase of the correct use of the measurement concept from the pre-test to the post-test. The study also shows that problem posing could be used to measure conceptual understanding. Additional studies should be conducted to show effects of different teaching interventions regarding improving the conceptual understanding of divisions of fractions.

Keywords: Division of fractions; Problem posing; Referent whole; Pre-service teachers

Introduction

There are two main concepts underlying division (e.g., Bulgar, 2003): (1) the *measurement* concept (also known as quotation and grouping by quotient), i.e., to determine how many groups there are when the number of objects in each group is given; (2) the *sharing* concept (also known as partition, and grouping by divisor), i.e., to find how many objects are in a group when the given number (or fraction) of groups has been fixed. Bulgar (2003) noted that in real life there is no such thing as a fraction of a person, and thus using a fraction of a person as a divisor would be incorrect. For example, while a quarter of a pie can be used as a divisor, a quarter of a person cannot. In the first situation (a quarter of a pie), the *measurement* concept is appropriate, and in the second situation (a quarter of a person), it is not appropriate. Many studies, as well as this one, provide support for Bulgar's conclusion that students often employ the incorrect concept when they use fractions as a divisor (Kornilaki & Nunes, 2005; Liebeck, 1990; Smith, 2002; Squire & Bryant, 2002).

Mathematical situations that require division will involve only one of these two fundamental concepts. A competent teacher must understand both concepts and be able to identify which one to apply. This ability is an indication of the subject matter knowledge of the teachers. Substantial research has already been conducted in mathematics education regarding pre-service teachers' understanding of fraction division, where the divisor is a fraction. Several significant areas of fraction division related to the present study are the conceptual definition of division (Li & Huang, 2008), the mathematical interpretations for the underlying principles and meanings on fraction divisions (Ball, 1990; Cluff, 2005; Ma, 1999), and the importance of the changing role of the referent whole (Carbone, 2009a, 2009b, 2010; Carbone & Eaton, 2007, 2008; Cluff, 2005; Parker, 1996). For example, Graeber, Tirosh, and Glover (1986) found that pre-service teachers tended to treat division of fractions in terms of the measurement interpretation. Tirosh (2000) noted that pre-service teachers could not explain the reasoning behind the procedure of fraction division using the numerator as the dividend and the denominator as the divisor to obtain the quotient.

Even more studies appear to be warranted on this subject because division of fractions continues to be a very challenging topic for many pre-service teachers. Many pre-service teachers exhibit a difficulty interpreting every division problem, including those with fractions, as a direct grouping of completely countable and discrete objects. They seem to treat all kinds of division as a process separating some quantity into many smaller and equal portions. This unvarying use of division with a fractional divisor demonstrates a weakness in understanding of the concepts of fraction division. Pre-service teachers appear to habitually ignore the reality that in many situations a number of objects or people when used as the divisor in integral division (integers divided by integers) must be a whole number – a quantity that can be represented only by a natural number, which, by definition, is not the case for fraction division.

The cause may be partly identified by considering Piaget's theory (1972) of schemas of action. These are formally defined as "…actions that can be applied to a variety of objects which center on relations between objects and transformations…" (Nunes, 1999, p. 36). This concept is important because schemas of action could be the basis of students' first understanding of mathematical operations (Squire & Bryant, 2002). The grouping/sharing process of division should be treated as part of the schemas of action of repeated subtraction as students initially learn division. However, grouping/sharing is only applicable for the basic division problem when the divisor is a whole number. For fraction division, the learners need to reflect abstractly in order to transfer prior concepts (sharing as a division process with a whole number as a divisor) to new concepts where the divisor is a fraction rather than a whole number. In this case, the difficulty is to comprehend how to share a quantity into non-integral fraction groups.

Several writers (Austin, Carbone, & Webb, 2011; Carbone, 2009a, 2009b, 2010; Carbone & Eaton, 2007, 2008; Craig, 1999; Mack, 1990, 1995; Parker, 1996; Shin, 2010) suggest that the ability to pose problems, or to write problems for students to complete that are applicable to their lives, is an important measure of teachers' competence in terms of conceptual understanding. Analyzing the problems that they pose offers another way to assess the conceptual understanding of future teachers. The works of Brown and Walter (2005) suggest that problem posing offers new strategies for teaching. Other researchers tie problem posing with problem solving (Abu-elwan, 1999; Leung, 1997), thus encouraging teachers to actively pose problems to enhance mathematical thinking of their students (Silver, 1993). This capability to pose story problems involving division of fractions will indicate their pedagogical content knowledge (see for example the *MT21* Report, Schmidt, et. al. 2007). Thus, teachers must not only possess content knowledge about fraction division, but also the ability to deliver to their students the measurement concept of division using real life scenarios.

Findings from previous research (Li & Huang, 2008; Li & Kulm, 2008; Li, Ma, & Pang, 2008) show that there is a discrepancy between what many pre-service teachers perceive as their level of knowledge about fraction division and their actual knowledge. Thus, they believe that they already know almost everything about division of fractions, including its definition, meanings (sharing and measurement concepts), representation, and related properties. Research, however, has shown that pre-service teachers do not understand why the reciprocal of the fraction divisor is necessary when completing a fraction division problem (Li & Huang, 2008), and thus they tend to employ simple algorithms (Kamii & Dominick, 1998; Maurer, 1998). They also have difficulty explaining and representing the meanings of fraction division (Rizvi & Lawson 2007), in particular, the measurement concept (Ball, 1990).

Aim

This study investigated pre-service teachers' understanding of the meanings of division of a mixed number by a fraction divisor (called fraction division below). The research questions were:

1. Was there a significant difference in the ability of the pre-service teachers in this study to pose a real life story problem for their future elementary

students that reflects the underlying meaning of fraction division after they had undergone a teaching intervention involving the use of a referent whole for addition and multiplication of fractions?

2. Did these pre-service teachers show an improvement in their conceptual knowledge of the measurement concept of fraction division through the problem posed?

Methodology

Participants

The participants included 72 pre-service primary level mathematics teachers in the College of Education in a university in Hong Kong. They were from four different groups in their first, second, or third year of studying toward the Bachelor of Education degree in primary mathematics. In addition to prescribed undergraduate level mathematics, such as calculus, linear algebra, and number theory, these pre-service teachers were required to study courses in principles and methods of mathematics teaching, assessment methodology, and mathematics problem solving. Their instructor verified that all the participants already knew the procedure to compute fraction division by taking the reciprocal of the divisor and performing the fraction multiplication.

Procedure

The pre-service teachers were asked to complete the following task:

Write a story problem that shows the meaning of $2\frac{1}{2} \div \frac{1}{2}$.

They responded to this task twice, as a pre- and post-test, with two weeks apart. The responses were collected for two years from November 2008 through November 2010 from several different groups.

For each group, between the pre- and post-tests, their instructor clarified the concept of the referent whole in the context of fraction addition and multiplication only. No instruction and hints were given to the pre-service teachers before the pre-test. During the intervention between the two tests, the participants were reminded of the changing role of the “referent whole” in the fraction operations of addition and multiplication using the examples below. There was no discussion about fraction division. The assumption of this study was that discussing the meanings of a referent whole for fraction addition and multiplication might help pre-service teachers obtain a conceptual understanding fraction division.

The intervention included the following examples.

Addition: $\frac{2}{3}+\frac{1}{4}$, both fractions represent the values that refer to the same referent whole, i.e., "1".

Multiplication: $\frac{2}{3}\times\frac{1}{4}$, the multiplier $\frac{1}{4}$ refers to the value of $\frac{2}{3}$, which acts as a different whole. See Figure 1(b). In this case, "1" is the original referent whole when multiplication begins. The referent whole changes in the middle stage of the multiplication of fractions, where the new whole, $\frac{2}{3}$, is separated into 4 equal portions, and 1 out of the 4 portions is taken.

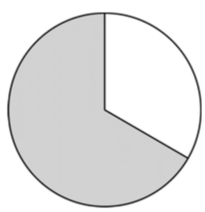	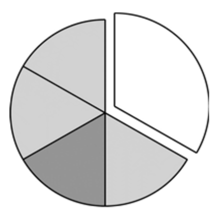	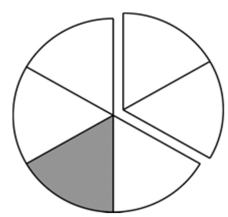
$\frac{2}{3}$, the original whole is "1".	$\frac{1}{4}$ of the $\frac{2}{3}$, $\frac{2}{3}$ is temporarily treated as the new whole. The $\frac{2}{3}$ portion is separated into 4 equal parts.	$\frac{2}{3}\times\frac{1}{4}$ equals $\frac{1}{6}$, $\frac{1}{6}$ refers to the original whole "1" again.

Figure 1. Changing Roles of "the referent whole" in Multiplication of Fractions.

Data Analysis and Results

In general, the problems posed followed a simple sentence format:

Given a background real life situation (with prescribed condition)
→ *Stem question (scenario leads to the division of fraction)*

In posing the problem, pre-service teachers would initially describe some background scenarios. These background scenarios were expressed in phrases such as “cutting the pizza,” “drinking juices,” or “eating few pieces of cake,” and similar activities. The key was to investigate how the actions of cutting, eating, and drinking were expressed in the prescribed conditions (“eating a piece every day”). If this action was executed by some people (for example, “my father and I”), it was classified and counted as a *sharing* concept. In this case, the size of the smaller pieces was compared with the size of the whole object, and the pre-service teachers usually used the words “share with” and “give”. If the action was to measure, to calculate, or to estimate how many “parts” are in the “whole,” then it was classified and counted as a *measurement* concept (“How many plates are needed if one serves a half a piece of cake on each plate?”). In the latter case, the whole, represented by a big piece, was divided into smaller pieces with known size and the smaller pieces were compared with the large one, and the words “separate,” “divide,” and “cut and compare” were used to express this idea. The use of suitable units also helped to identify which concept, *sharing* or *measurement* was being employed. A third category (Other) was created for responses that were mathematically wrong or irrelevant to primary pupils. For each category, the problem was counted as correct, incorrect, or inappropriate. “Inappropriate” refers to the use of an inappropriate real life situation even though the mathematics may be correct.

Figure 2 shows an example of proper posing of the measurement concept. Considering that the participant explained the story problem in a primary class, the processing was clear and easy to follow, as it was equivalent to repeated subtraction for division (Lakoff & Nunez, 2000).

You get $2\frac{1}{2}$ L orange juice. Now, you have to divid $\frac{1}{2}$ L in each bottle. So, you can divid

1st bottle, and remain $2\frac{1}{2} - \frac{1}{2} = 2$ L
2nd bottle, and remain $2 - \frac{1}{2} = 1\frac{1}{2}$ L.
3rd bottle, and remain $1\frac{1}{2} - \frac{1}{2} = 1$ L.
4th bottle, and remain $1 - \frac{1}{2} = \frac{1}{2}$ L.
5th bottle

Totally 5 bottles of orang juice

Figure 2. Proper Posing: Measurement (Script: 027).

In Figure 3, an inappropriate posing of measurement concept was attempted. It was not a valid story, as the participant just rephrased the operation of division into words correct story problem. (Script: 033).

How much is a 1/2 of the fraction $2\frac{1}{2}$?

Figure 3. Inappropriate Posing: Measurement (Script: 033).

Figures 4 and 5 show examples of inappropriate posing of sharing. For Figure 4, the participant initially used 5 cakes shared by 2 people (an integral divisor) to get 2 and ½, which is incorrect. He suddenly grouped such 2 and ½ by ½ (measurement) by asking how many halves, which would be classified as "correct" mathematically, but it does not relate to a real life situation.

5 cakes is shared by 2 people, how much cake(s) can each person share?
Hence, for the cake shared, how many half of cake can be counted?

Figure 4. Inappropriate Posing: Sharing (Script: 024).

In Figure 5, "…sharing the piece of apple with son and father" refers to a division by 2, not by ½. This participant confused measurement with sharing division.

You share 2 and half apple with your father.
How many apple do you have?

Figure 5. Inappropriate Posing: Sharing (Script: 014).

The data were independently classified and counted by two different mathematics educators. There were only two (out of the 72 response sets with a pre- and a post-test in each set) where disagreements were found; the 98.6% reliability rating shows the validity of the results. For these two cases, a third person reviewed them to arrive at a majority decision for the classification. Table 1 summarizes the results for both tests.

Table 1
Frequency of Types of Problems Posed: Pre-test (Post-test)

Classification	**Correct**	**Incorrect**	**Total**
Sharing (*partition*)	0 (0)	26 (15)	26 (15)
Measurement (*quotition*)	27 (36)	0 (0)	27 (36)
Others	0 (0)	19 (21)	19 (21)
Totals	27 (36)	45 (36)	72 (72)

Of those pre-service teachers who employed the sharing concept (e.g., in sharing apples with father or sharing some pieces of cake), all failed to produce a proper story problem on both tests; see Figures 4 and 5 above. The main difficulty was expressing the fraction divisor with a number representing the number of groups (or people).

While those employing the measurement concept showed a 33% increase in correct problem posing (from 27 in the pre-test to 36 in the post-test), for those who posed the incorrect problems (45 in the pre-test and 36 in the post-test), many of them rephrased the expression of division by describing a situation where one must cut a cake (or pizza) into smaller pieces. Typically, it is an approach to employ the concept of sharing. It is not measurement because the measuring unit (the "half") is fixed and is already there; thus, we do not need to cut anything. One could say that "If a half of a cake is put into each plate, how many plates do we need for 2 and ½ cakes?" Such an example is typically a measurement approach, with 2 and ½ measured normally by ½ producing 5 units.

A relatively high percentage (36% on the pre-test and 50% on the post-test) of the participants rephrased the expression by employing the concept of measurement. To be scored as a correctly posed problem, the given real life situation must include the proper meaning of *two and a half divided by a half.* We can see that the value of one is the original whole, however, 2½ becomes the new referent whole when the division operation is performed, and ½ is the unit that we count for the measurement approach. The new referent whole (2½) consists of 5 units of such halves. If the pre-service teachers had interpreted the problem in this manner, a correct story problem would be produced.

The problems were scored based on correctness in using the measurement concept. A correctly posed problem received one point and an improperly or wrongly posed problem (including those under "Others") received zero points. The McNemar test (Pett, 1997) was used to determine if the discussion about fraction addition and multiplication during the teaching intervention had changed their ability to use the measurement concept. The result ($\chi^2 = 2.083$, $p < 0.05$) shows a significant improvement; see Table 2. The increase in correct responses in Table 1 (from the correct-incorrect ratio 27:45 in the pre-test to 36:36 in the post-test**)** shows that understanding the meaning of the referent whole for the operations of addition and multiplication of fractions improved the pre-service teachers' understanding of fractional division because they became familiar with the concept of the referent whole. The increases in the number of correct responses show that the short intervention regarding the referent whole allowed them to transfer this important concept and apply it to the division of fractions.

Table 2
Cross-tabulation of Pre-test vs. Post-test Scores on Measurement Concept

	Post-test		
Pre-test	0	1	**Total**
0	35	11	46
1	1	25	26
Total	36	36	72

Conclusion

All the 72 pre-service teachers in the study were able to solve the division problem procedurally, but some did not demonstrate a conceptual understanding of the division of a mixed number by a fraction. Some participants originally did not understand that the divisor must be a positive integer (natural number) when the sharing interpretation is applied. However, after the teaching intervention regarding the referent whole with addition and multiplication of fractions, the number of incorrect responses involving sharing decreased from 26 to 15, with more participants giving correct measurement story problems. The Hong Kong pre-service teachers' initial knowledge is similar to the initial knowledge of subjects in other studies (Cluff, 2005; Li & Huang, 2008; Li & Kulm, 2008). Only one-third of the pre-service Hong Kong primary teachers in our sample were initially able to correctly pose a problem of a mixed number divided by a fraction. The data supports that the brief teaching intervention shows an improvement in the conceptual understanding of the pre-service teachers in the study. Additional studies should be conducted using different teaching interventions to enhance conceptual understanding of fraction division. Problem posing is found to be a useful method to measure conceptual understanding, and it may be helpful in future research.

This study shows that many pre-service teachers do not have a thorough understanding of the meaning of "dividing by a fraction." Being unaware of the importance of the fundamental concepts of fraction division, they may skip explanations of these concepts with their future students and just lead them to memorize the rule and procedure of computation. Making a connection to the referent whole in studying all operations with fractions will support more conceptual understanding rather than an undesirable method of merely teaching procedures.

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