

## **Mathematical Problem Solving for Everyone: A New Beginning**

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**Abstract:** Mathematical problem solving has been at the core of the Singapore mathematics curriculum framework since the 1990s. We report here the features of the Mathematical Problem Solving for Everyone (M-ProSE) project which was carried out in a Singapore school to realise the learning of mathematical problem solving and as described by Pólya and Schoenfeld. A mathematics problem solving package comprising “mathematics practical” lessons and assessment rubric was trialled in the school for Grade 8 in 2009. Responses from three students show mixed perceptions to the module, but an end-of-module assessment shows that the students were able to present their solutions along Pólya’s four stages. We also describe teacher preparation for teaching the module. After the trial period, the school adopted the module as part of the curriculum and it is now a compulsory course for all Grade 8 students in that school.

**Keywords:** Problem solving; Mathematics curriculum; Practical worksheet; Assessment rubric; Teacher development

### **Overview**

Mathematical problem solving has been the central theme of the Singapore mathematics curriculum since the 1990s. The focus has been on the four-step problem solving approach highlighted by Pólya (1957) and problem solving heuristics (Schoenfeld, 1985). Although all Singapore schools follow the national curriculum, issues with the teaching of problem solving have surfaced. For example, Hedberg et al. (2005) highlighted how Singapore teachers varied in their styles and approaches to problem solving, and others, such as Yeo and Zhu (2005) as well as Fan and Zhu (2007), have highlighted some rigid teaching approaches that are still prevalent in the Singapore mathematics classrooms. Attempts to infuse problem solving that typifies the kind of mathematical thinking used by mathematicians into the school mathematics curriculum have not been common practice.

We report here the features of the Mathematical Problem Solving for Everyone (M-ProSE) project to realise the learning of mathematical problem solving as practised by mathematicians and as described by Pólya. We have used a design experiment approach in a Singapore school to develop a problem solving package that involves positioning problem solving as “mathematics practical” lessons. Furthermore, we comment on our approach to teacher preparation for this problem solving curriculum and the fundamental issue of assessment of problem solving. Finally, we include some analysis of data from the end-of-module assessment for the first cohort of students who took the module; indeed, they were able to present their solutions along the lines of Pólya’s four stages of problem solving.

### Teaching Problem Solving: Some Issues

The publication of the *Agenda for Action* by the National Council of Teachers of Mathematics (NCTM) in the United States in 1980 and the *Cockcroft Report* in the United Kingdom in 1982 re-focussed the attention of the mathematics education community to problem solving. A main recommendation of these reports was that problem solving should be the focus of school mathematics. The *Agenda* recommended for action that: “[t]he mathematics curriculum should be organised around problem solving” (NCTM, 1980, p. 2), whereas the *Cockcroft Report* (1982), in *Paragraph 243*, reiterated that mathematics teaching at all levels should include opportunities for problem solving, including the application of mathematics to everyday situations.

In the United States, although the NCTM (2000) standards still emphasises the important role of problem solving in school mathematics, the situation about problem solving in schools has not improved much. There is little information in the standards document about how to implement problem solving in the school curriculum. There is now a mounting body of literature pointing to the fact that problem solving is still not implemented in mathematics classrooms in many parts of the world, or if implemented, then only certain routine approaches to heuristics are being adopted (English, Lesh, & Fennewald, 2008; Lesh & Zawojeski, 2007; Schoenfeld, 2007; Silver, Ghouseini, Gosen, Charalambous, & Strawhun, 2005).

Schoenfeld (2007) acknowledged that, in the 1980s, problem solving did become a fashionable term; however, its implementation in most classrooms was a travesty, probably due to a misinterpretation of problem solving in commercial textbooks. He also provided three reasons why a narrow view of problem solving was adopted in American classrooms:

1. Problem solving research was still in its infancy when the NCTM's *Agenda for Action* was published in 1980.
2. Teachers are part of a conservative force that resists change and as such it takes time to implement change.
3. The mechanics of the publishing industry militate against change by recycling traditional content in a lucrative market.

Although it is agreed that problem solving is one of the fundamental goals of teaching mathematics, that goal remains one of the most elusive ones (Stacey, 2005). There are many issues confronting the teaching of problem solving in the school curriculum which make it hard to implement successfully. For example, Lovitt (2000, p. 9), proposed the following reasons in proclaiming that “problem solving has ‘failed’” in Australian schools:

1. Lack of clear and widely accepted criteria. All sorts of things, some diametrically opposite to each other are all dressed up as problem solving. The word has become so blurred that we have no common shared agreement on what it means.
2. Another reason is the unfortunate perception that one aspect of the problem solving picture is delivered through games and puzzles and therefore is relegated to the periphery or margins of mathematics. ‘I do these really interesting things on Friday afternoons,’ say many teachers to me. I am not sure if they are conscious that the act of doing so is to send a message to students that it is not really important — merely a bit of fun to be done after the ‘real’ stuff.

In Singapore, the picture is quite similar. While we acknowledge the efforts of teachers and schools to implement problem solving well, we are, however, dissatisfied with the “routinizing” manner that problem solving turns out to be when enacted in the classrooms, be it in the way it is taught or in the way it is learnt. The fault could well lie in the teaching and for that matter with the teachers who, to all good intentions, present problem solving as a series of steps to follow. It could also lie with the students who, despite the genuine efforts of the teachers, somehow perceive problem solving as a routinizing procedure.

A recent study conducted by Fang, Lioe, Ho, and Wong (2009) looked specifically into documenting and analysing actual classroom instructional practices. A total of 106 mathematics classroom lessons from three Primary schools and 53 lessons from three Secondary schools were observed, video-recorded and analysed. The authors summarised their findings as follows:

The teachers' approaches did not reflect an emphasis in the process of heuristics. The mostly closed routine word problems were not fully explicated in terms of the model proposed by Pólya (1957) and the range of heuristics used was limited. They generally read the problems, executed the solution and checked the answers. There was very little dwelling on the exploration or the planning aspect of the solutions. ... The emphasis appeared to be more to address the skills and procedures needed to solve problems than to tackle fresh problems anew where students have more chance of grappling with understanding and thinking about how to solve the problems. (p. 84)

The picture of classroom teaching of problem solving in Singapore portrayed in this study with local anecdotal observations — that of mathematical problem solving treated by teachers as supplementary rather than central to their instructional practices — coheres with the abovementioned reports about the situations in Australia and the United States. In a review of 60 projects carried out by mathematics departments of primary schools to improve students' mathematical proficiency, Foong (2004) found that 13 schools featured “problem solving” prominently in their listed projects. This indicates that a sizeable number of schools view problem solving as an important area of development for their students. However, these projects were carried out as enrichment activities. By “enrichment” in Singapore schools is meant add-ons to the core emphases of the school's mathematics instructional programme. Enrichment activities often take the form of mathematical games and competitions held outside the boundaries of regular curriculum time. In other words, even for schools that value its importance, mathematical problem solving is done as supplementary, not central, work in regular classroom teaching.

English, Lesh, and Fennewald (2008) attempted to shift the focus from “Pólya-style” problem solving to mathematical modelling, citing the lack of success in implementing the “Pólya's heuristics or ... Schoenfeld's metacognitive processes or beliefs” (p. 3). They cited the conclusion of Lesh and Zawojewski (2007) that when a field of research has experienced more than 50 years of failure using continuous embellishments of rule-governed conceptions of problem solving competence, perhaps the time has come to consider other options and to re-examine foundation-level assumptions about what it means to understand mathematics concepts and problem solving processes. This is a direct attack on the frameworks of Pólya and Schoenfeld so as to redirect efforts towards the theoretical perspectives and research methodologies of models and modelling perspectives (MMP) for teaching and learning mathematics problem solving. Other more generic models of thinking such as Understanding by Design (UbD) (Wiggins & McTighe, 2005) and Teaching for

Understanding (TfU) (Douglas, 2012) have been proposed. The fact remains that none of these alternatives have proven themselves any better in mathematics education than the approach of teaching Pólya-style problem solving. For example, when Hammerness, Jaramillo, Unger, and Wilson (1997) reported their analysis of the understanding of 38 students in four different subjects (History, Physics, English, and Mathematics) under the TfU programme, the mathematics class performed the worst of the four classes.

Schoenfeld (2007), on the other hand, argued that the existing theoretical frameworks of Pólya-style problem solving remain valid:

That body of research — for details and summary, see Lester (1994) and Schoenfeld (1985, 1992) — was robust and has stood the test of time. It represented significant progress on issues of problem solving, but it also left some very important issues unresolved. ... The theory had been worked out; all that needed to be done was the (hard and unglamorous) work of following through in practical terms. (p. 539)

We agree with Schoenfeld and have decided to proceed on the “hard and unglamorous” work of following through the established theoretical framework by reasoning as follows.

The teaching of Pólya-style problem solving (which we will henceforth refer to as “problem solving”) has been successful under certain circumstances such as in Schoenfeld’s undergraduate classes (Schoenfeld, 1985). The processes, such as the cyclic nature of problem solving, are sound because these are the same processes professional mathematicians use (Carlson & Bloom, 2005). The methods of teaching are generally not complicated. A course on problem solving should be compact in terms of time and does not involve immense unsettling school changes such as the UbD and TfU programmes.

We are convinced that a root cause for the lack of success of learning problem solving is that problem solving is not assessed as part of summative assessment and national examination. Consequently, most students are not motivated to learn problem solving. Instead, they are more interested to learn the other components of the curriculum that will be assessed. For example, Holton, Anderson, and Thomas (1997) proposed a plan for teaching problem solving in New Zealand schools. The New Zealand Ministry of Education developed a national numeracy project which emphasised a problem solving approach and that has now been introduced to the majority of primary schools in the country. However, success is so far limited to the primary level (Ministry of Education New Zealand, 2006) because, similar to

Singapore, high stakes examinations have blunted the problem solving approach in mathematics classes at the secondary level. This is noted by Holton below:

First of all I think that you have to separate primary from secondary schools. There is a sense in which most primary schools are now using a problem solving approach and are being successful ... Moving further into the secondary school, there are certainly some good teachers who use problem solving especially to introduce new topics but many teachers at that level feel intimidated by the exam system (we have national exams in each of the last three years of school) and so teach at that level in a more 'didactic' manner. (D. Holton, personal communication, December 7, 2006)

One of the major issues in the implementation of problem solving in schools has been in the fourth stage of the Pólya model, namely "looking back". Silver et al. (2005) strongly pointed out that: "...instructional interventions intended to develop in students an inclination to 'look back' at their solution to a problem in order to generate alternate solutions have been largely unsuccessful" (p. 288). Students do not generalise and extend a problem and do not think that it is important to do so when solving problems.

The way out of this perennial quandary is by making a change in teaching approach. In a pilot project at an Integrated Programme school (Tay, Quek, Toh, Dong, & Ho, 2007), we constructed a worksheet similar to that used in science practical lessons and told the students to treat the problem solving class as a mathematics "practical" lesson. In this way, we hope to achieve a significant shift in the way students looked at these "difficult, unrelated" problems which had to be done in this "special" class. In this paper, we describe the Mathematical Problem Solving for Everyone (M-ProSE) project (2009-2011) that was a spinoff from the pilot study.

### **Implementing the Problem Solving Module in the Curriculum**

Stacey (2005) claimed that to get closer to the goal of problem solving requires research directed to understanding the problem solving process for mathematics (in all its aspects), developing effective curriculum processes, and designing excellent tasks. She added, "Moreover research needs to be closely intertwined with curriculum development and teacher development projects so that it can make an impact on practice" (p. 341). Thus, we are faced with the task of devising a problem solving curriculum, developing "excellent tasks," developing an assessment that focuses on the processes of problem solving and above all, providing teacher preparation for a problem solving curriculum.

We used “design experiments” (Brown, 1992; Collins, 1999; Wood & Berry, 2003) as the methodological backbone of our project. Design experiments arose from attempts by the education research community to address the demands of conducting research in real-life school settings in all their complexity. It argues for the application of multiple techniques to study a complex phenomenon such as mathematical problem solving. This approach permits the use of several methods, such as participant observation, interview, video-taping, and paper-and-pencil testing to provide corroborative evidence for findings. The envisaged outcome of M-ProSE was to produce a workable “design” (an initiative, artefact, or intervention, for instance) that could be adapted to other schools. In Gorard’s (2004) words, “The emphasis [in design experiments], therefore, is on a general solution that can be ‘transported’ to any working environment where others might determine the final product *within their particular context* [italics added]” (p. 101).

### **The Problem Solving Module**

The research school is a special school with students of high ability, and it has a flexible curriculum that can accommodate a new module (or course). Our aim was to show that this new problem solving module would work in this school. It consisted of 10 lessons, each of 55 minutes. With the exception of Lesson 1, every other lesson was divided into two parts. In the first part, the teacher reviewed homework of the previous lesson and explained one aspect of problem solving, such as a particular stage in the Pólya’s model. The second part focused on one problem called the “Problem of the Day”. Table 1 shows the scheme of work of the module.

The problem solving lessons were accompanied by a mathematics practical worksheet and an assessment rubric. Full copies of these two documents can be found in Toh, Quek, Leong, Dindyal, and Tay (2011) and so will be described only briefly below.

Table 1  
*The 10 lesson problem-solving module*

Lesson	Activity
1	<ul style="list-style-type: none"> <li>• Distinguish between a problem and exercise</li> <li>• Model successful problem solving</li> </ul>
2	<ul style="list-style-type: none"> <li>• Introduce Pólya's problem solving model</li> <li>• Introduce Stage I of Pólya.</li> </ul>
3	<ul style="list-style-type: none"> <li>• Introduce the meaning of the word <i>heuristics</i> and provide a list of the common heuristics for mathematical problem solving</li> <li>• Introduce Stages I to III of Pólya</li> </ul>
4	<ul style="list-style-type: none"> <li>• More on heuristics</li> <li>• Practice on using Stages I to III of Pólya</li> </ul>
5	<ul style="list-style-type: none"> <li>• Introduce to the practical paradigm of mathematical problem solving</li> <li>• Formal use of the Practical Worksheet to solve Problem of the Day and Homework Problem</li> </ul>
6	<ul style="list-style-type: none"> <li>• Focus on Check and Extend, i.e., Stage IV of Pólya</li> <li>• Emphasis on adapt, extend and generalize a mathematical problem</li> <li>• Introduce the assessment rubric</li> </ul>
7	<ul style="list-style-type: none"> <li>• Identify the features of Check and Extend</li> </ul>
8	<ul style="list-style-type: none"> <li>• Introduce the importance and use of Control (Schoenfeld, 1982) in mathematical problem</li> </ul>
9	<ul style="list-style-type: none"> <li>• Introduce the use of the Control Column in Stage III of Pólya</li> </ul>
10	<ul style="list-style-type: none"> <li>• Revision on the key features and processes of mathematical problem solving</li> </ul>

### The Practical Worksheet and Assessment Rubric

The practical worksheet, as shown in Figure 1, is a scaffold in the form of a template with the moniker “practical worksheet” to tie in with the practical paradigm. It was developed based on earlier work by Tay et al. (2007). The worksheet contains sections explicitly guiding the students to use the Pólya's stages and problem solving heuristics to solve mathematics problems. It has four sections corresponding to the four Pólya stages. Each section takes up a page. Consistent with Kantowski's (1977) caution, the worksheet allows for looping back to previous stages. Thus, students may use one or more of each of the sections I, II and III, and indicate their different attempts at each stage accordingly as Attempt 1, Attempt 2, etc. The worksheet is gradually introduced across a few lessons to induct students into the Pólya's stages and Schoenfeld's framework of problem solving.

Walvoord and Anderson (1998) claimed that effective assessment practice begins with and enacts a vision of the kinds of learning that we most value for students and that we help them achieve. An assessment rubric was designed based on the Pólya's model and Schoenfeld's framework, and it was used to assess the students' problem-solving processes, which we value. The rubric is shown below.

- *Pólya's Stages* [0-10 marks] — this criterion looks for evidence of the use of cycles of Pólya's stages (Understand Problem, etc).
- *Heuristics* [0-4 marks] — this criterion looks for evidence of the application of heuristics to understand the problem, and devise/carry out plans.
- *Checking and Expanding* [0-6 marks] — this criterion is further divided into three sub-criteria:
  - Evidence of checking of correctness of solution [1 mark]
  - Providing for alternative solutions [2 marks]
  - Extending and generalizing the problem [3 marks]; full marks for this are awarded for one who is able to provide (a) two or more generalizations of the given problem with solutions or suggestions to solution, or (b) one significant extension with comments on its solvability.

**I Understand the problem (UP)**

(You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.)

- Write down your feelings about the problem. Does it bore you? scare you? challenge you?
- Write down the parts you do not understand now or that you misunderstood in your previous attempt.
- Write down the heuristics you used to understand the problem.

**II Devise a plan (DP)**

(You may have to return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.)

- Write down the key concepts that might be involved in solving the question.
- Do you think you have the required resources to implement the plan?
- Write out each plan concisely and clearly.

**III Carry out the plan (CP)**

(You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.)

- Write down in the *Control* column, the key points where you make a decision or observation, for e.g., go back to check, try something else, look for resources, or totally abandon the plan.
- Write out each implementation in detail under the *Detailed Mathematical Steps* column.

**IV Check and Expand (C/E)**

- Write down how you checked your solution.
- Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of.
- Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

*Figure 1.* Instructions on a practical worksheet

The rubric encourages students to go through the Pólya stages when they are faced with a problem. They should return to one of the first three Pólya's stages upon failure to realise a plan or solution. Students who show control (Schoenfeld's framework) over the problem solving process will gain marks. A student who does not obtain a completely correct solution will still be able to score up to 11 out of 20 marks, if they show evidence of cycling through the stages, using heuristics, and exercising control. The maximum mark for getting a correct solution is 14, and the remaining 6 marks come from *Checking and Expanding*. We hope to push students to check and expand the problem (Pólya's Stage IV), an area of instruction that has not been successful so far (Silver et al., 2005). A distinction is for marks above 75%.

### Teacher Development

Teachers are not mere implementers of revised curriculum. They bring with them their beliefs, knowledge, goals, and experiences in mediating the intended curriculum and the enacted curriculum. Thus, our approach had to start with and include the changes in teachers' mindset about the centrality of problem solving in the teaching of mathematics. Teacher development is therefore an important component in our overall effort to raise the prominence of problem solving in the classroom. This paper will briefly mention this aspect, and the readers are directed to Leong, Dindyal, Tay, Toh, Quek, and Lou (2011) and Leong, Tay, Toh, Quek, and Dindyal (2011). Such a teacher development programme should contain the following features:

- Feature 1. Re-designed curriculum and structures in place to necessitate a change in approach from existing instructional practices that are not congenial to the carrying out of regular problem solving in the classrooms.
- Feature 2. A substantial amount of time given to teachers to experience and reflect on mathematics problem solving, to the point that teachers "buy-in" to the processes and heuristics and can confidently use them in their own problem solving attempts.
- Feature 3. Opportunities to observe positive enactments of problem solving in actual classrooms and participation in discussion about how essential elements can be incorporated into their own practices.

Three phases of teacher development were designed to roughly correspond to the three features above. Phase I involved the re-design of curriculum and structures as described. This was completed in 2009. During Phase II, the participating mathematics teachers attended five sessions, each of 90 minutes, in 2009. During the sessions, Tay, the trainer, covered several heuristics, using problems as

examples. The primary goals were to provide teachers with time to experience problem solving themselves and to help them develop problem solving habits such as use of the Pólya's stages. For the last two sessions, the trainer introduced the Practical Paradigm to help teachers re-conceptualise mathematics problems solving as integral to mathematics work in the classroom. They were also asked to use the practical worksheet as a way to record their attempts at solving the problems given to them. Through this exercise, the teachers experienced firsthand a structural tool that they could use to guide the students' problem solving efforts along the lines of the Pólya's stages and heuristics.

In Phase III, the teachers were provided with the opportunities to observe and discuss the enactments of problem solving in actual classroom teaching. We used some approaches from *Lesson Study* to re-direct the focus of teachers from the experience of problem solving to teaching it to students in an actual classroom. One important feature of *Lesson Study* is the observation and revision of a lesson by a team of teachers. This practice is premised upon the idea that "teachers can best learn from and improve their practice by seeing others teach" (Isoda, Stephens, Ohara, & Miyakawa, 2007, p. xvi). This characteristic of *Lesson Study* is congruent to Feature 3 of the teacher development design used in this study, rendering *Lesson Study* useful for our purpose.

In 2009, the trainer taught an elective Year 9 module over ten one-hour lessons in the same school. Twenty one students took the module. This module was similar to the teacher module carried out during Phase II, but the pace, tone, and issues raised for discussion were adjusted to suit the needs and abilities of the students. After Lessons 1, 3, 5, 6, 8, and 10, post-lesson meetings were conducted with the teachers to discuss the lessons. We focused on the suitability of the problems, student responses to those problems, and the adjustments needed for Cycle 2, when the teachers had to teach the module in their own classes later on in 2009. These meetings helped the team and teachers gather ideas for improvement and clarify the instructional practices that were demonstrated.

### **School Implementation**

After completion of the above three phases, the next stage of the project was for the mathematics department of the school to conduct the problem solving lessons in 2009. At this point, we let the teachers decide and modify the resources and materials to suit the needs of their classroom practices. It turned out that the school had used almost all the instructional resources given to them in Phase III, with very slight modifications. That the school adhered so closely to our materials and intent of the course can be seen as an indication of success of the teacher development process in persuading the mathematics department as a whole to “buy-in” to the enterprise.

The school decided to offer the problem solving course as a compulsory module for the entire Grade 8 cohort of the school, totalling 159 students. Three teachers were selected to teach the module. Our research focused on only two of these teachers as one of the teachers was new to the school and the project.

### **Student Learning and Responses to the Module**

We next report on the semi-structured interviews (Bogdan & Biklen, 1998) with three students who completed the problem solving module. The purpose is not targeted at verification or generalisation, but to discovery, a description that is not necessarily typical, but unique and individual. The prompts for the interviews include:

- Name one thing that you learnt from the course.
- How does the practical worksheet help you in solving problems?
- What do you think about the assessment of the course?

The three interviewees were Way Nam, Jen Non, and Zen Kon (all pseudonyms), and they were chosen to represent bands of mathematical ability. Way Nam was among the highest mathematical ability group. He was a member in the school’s Mathematical Olympiad team. Jen Non was of middle ability. She was a conscientious student who took pride in her work. Zen Kon was from the lower mathematical ability group. He showed a lack of interest in the problem solving module and would not try hard in the class work.

A detailed discussion of Way Nam’s interview is given in Leong, Dindyal, Toh, Quek, and Tay (2011). He found the module complementary to his Mathematical

Olympiad training. He had learnt to think along the sequences of the Pólya's stages, in contrast to his Olympiad Training, where his approach was to search for the appropriate "formulae" to apply. He found the practical worksheet useful for the difficult questions but a hindrance for the easy ones.

**Researcher:** Did it [practical worksheet] help ... page one, two, three, section one, two, three help you in the practical work. Were there any problems?

**Way Nam:** For the more difficult ones... yah. The practical worksheet will help. For the more basic ones...

**Researcher:** Means you already got the solutions, you want to quickly run to the...

**Way Nam:** Extension... For simple questions sort of hinders me.

Way Nam had obviously got the message that he still needed to go through Stages I, II, and III even though he could solve the problem. In this respect, he found the practical worksheet had hindered his progress for the simple problems.

Way Nam felt that marks should be awarded for students who had gone through the problem solving processes but were still unable to obtain the correct answer.

**Researcher:** The other big component in this course is the assessment rubric ... how you feel about the assessment rubric? How you're being assessed. Do you like it? You don't like it? Does it help you?

**Way Nam:** Ya but I feel that if I'm only going to get marks because of correct solutions, I don't think it's like Pólya's method is emphasised a lot. Because if I'm supposed to use Pólya method and I get it wrong, I should be at least given some marks... 'cause correct solution you don't show your thinking process, you don't know how can you get your answer. You can just copy from someone else.

Jen Non took up the problem solving module as she had always been interested in problem solving. She found that the mathematics practical lessons were very different from other mathematics courses, but she appreciated the importance of the skills taught through this course.

**Researcher:** So if I ask you one thing you learn from this course.... What would it be?

**Jen Non:** Yeah. Cos initially I thought generalisation is like a very professional thing... Then I am like wah actually I also can do it. Initially I cannot. When I start[ed] the course of course I cannot do anything. I am like how to. So every time you know the back part, the fourth page I always leave it blank. I only can do the checking part. Then ... I learn

generalisation is not that difficult after all. It can be accomplished. So I gained more confidence on the way. Yeah.

She was impressed by her own ability to generalise a mathematical problem. The practical worksheet had helped her to stay focussed while solving mathematical problems:

**Researcher:** How has this practical worksheet helped you in solving the problem?

**Jen Non:** Initially I thought that the understand the problem is a bit redundant. Okay. Cos for almost every worksheet I read almost the same thing... Because it can help me to... for example when I get lost then I get back to see what I really want to do. Yeah. Then I can stay focus. Then if I want to change plan I also... So I thought stage two was quite important as in the devise of plan. Then of course the carrying out of plan... I will say that the worksheet helps me to stay more focus... Cos in our problem solving it's very often very frequent that you actually go off track... But then this worksheet actually help me to go back and think about my plan whether it is right or wrong and then analyse and how can I further develop it.

Jen Non was satisfied with the assessment rubric, since the skills of problem solving were taught during the practical lessons.

**Researcher:** How has this practical worksheet helped you in solving the problem?

**Jen Non:** Yeah. I think it's alright... 'Cos we are assessing with the things that we have actually learned and the skills that we have learned.

As mentioned earlier, Zen Kon was a low achieving student who was sceptical about the problem solving module as he felt that it was "not worth of his time".

**Zen Kon:** It's not that worth it of my time cause I didn't really learn much other than solving problems and getting exposed to new problem. I didn't learn much... I expected something more, my expectation was higher...

**Researcher:** What did you expect, then?

**Zen Kon:** Something like real-life application of mathematics, not just solving problems again and again.

He was unhappy with using the practical worksheet because the process of filling it up was redundant, as the same process holds for solving all the problems.

**Researcher:** You know this thing that you said you won't write in right, the practical worksheet?

**Zen Kon:** No. I won't. I will just give the solution immediately... You know because for this is like especially the first part it's really... every time I solve a problem it's the same thing. How do you find it? I either write boring or challenging. Then write out the part you don't understand, none. Then see also none. So it's like very redundant. I always write the same thing. Boring, none, none.

Interestingly, as the interview continued, Zen Kon changed his tone:

**Researcher:** Do you think students will go through the Pólya's stages in solving the problem?

**Zen Kon:** I think if they don't like understand the problem, then in the first place they won't be able to solve it. So it's like if you can solve it then you have gone through thinking. So it's like one come the other...

**Researcher:** you mean devising a plan?

**Zen Kon:** Devising a plan might not be that effective but part A as in writing down the... knowing the key concepts it will help a bit la.

He reflected that the use of the practical worksheet for Stage I was redundant, but for Stage II (devising a plan), the process of writing down the key concepts could be helpful. However, he expressed his dissatisfaction that he was required to *Check the problem* (Stage IV) and it was to allocate marks to the checking process.

**Zen Kon:** during the exam, the final test, then there was this thing about how do you check your answer about the timer one? I mean it's like when you think of it it's already so obvious, so how else can you check it? It just works that way. So how can you check it? If like 2 plus 2 equals to 4, how do you check your answer? It's like you cannot check it. There's no way to check. So I just wrote like I look at the timer of the clock then after that I tap my finger every 4 minute and it just works and ya it's the best way of checking I can think. But then I still got 0 marks because there's still no other way to check it, like it's so obvious already. So the checking part, I don't think it's very fair to say that you have 1 method of checking you get 1 mark, 2 methods 2 marks. I don't think it's very fair that way.

**Researcher:** Oh ok. You mean like if you look at the marks here it tells you if you go and write down more cycles [of Pólya's stages in the worksheet], you get more marks. That didn't make you write down many cycles?

Zen Kon thought that *Check and expand* was about learning and it should not be classified under problem solving.

**Zen Kon:** This thing is more about ah more of the generalising part. If you want to generalise, it's more of learning, not problem solving. Cause when they said problem solving, they only wanted you to solve the problem. So checking if you go into check the extend it's like not so much of a problem solving but learning about the problem.

He would write down the steps using the practical worksheet purely to obtain the marks for the assignment. His behaviour is further evidence that assessment is necessary for learning to take place, especially for students who have not bought-in the idea of problem solving via these practical lessons.

**Zen Kon:** I just normally I just write down then normally I just get the marks la... I don't really look at the rubrics. I just solve the problem, get the correct problem and add it up... Without the marks, I no wait ya I wouldn't have written those.

#### ***End-of-module assessment***

This assessment consisted of the following problem that all the 159 Grade 8 students were given 55 minutes to solve using the Practical Worksheet.

There are two timers: one for 5 minutes and one for 9 minutes. I want to heat up a beaker of water for 11 minutes. How can we do this using only these two timers?

We thought that this problem was at a difficulty level suitable for almost all the students. They should be able to proceed to the *Check and Expand* stage. This problem was similar to the "Jugs Problem" (to measure out a given volume of liquid using two jugs of given capacities using Diophantine equations) that was given to the students in the first lesson. Using a related problem was in line with the emphasis on the Pólya's heuristics, but the contextual specifics of the problem would still present a substantial challenge, rendering it a genuine problem to the students. In response to the prompt "Write down your feelings about the problem," 78 students found the problem "challenging" or "very challenging," 12 thought it was "boring," and the remaining 69 were non-committal. Their performance is given in Table 2 below.

Table 2  
*Results of end-of-module test (n = 159)*

Components	Max mark	% scoring max	Mean	SD
First 3 stages of Pólya	10	69.8	9.41	1.05
Use of heuristics	4	74.8	3.72	0.52
Stage IV	6	7.5	2.97	1.61
Overall	20	7.5	16.06	2.71

A high percentage of the students could complete the first three stages and apply the heuristics in solving the problem. Many of them had also demonstrated Pólya stage IV to some extent: checked the reasonableness of their solution (74.2%), provided alternative solutions (35%), or generalised the given problem by offering at least one related problem, thus obtaining at least 1 out of 3 marks (89.8%).

### Conclusion

Is it possible, within a school mathematics curriculum, for every student to experience mathematical problem solving as practised by mathematicians? In this paper, we describe the backdrop of four decades of teaching and learning of problem solving in school mathematics against which we have designed an approach to enable students in a Singapore school to learn Pólya-style problem solving. The effort to expose students to Pólya-style problem solving is daunting for two main reasons: it is being abandoned by a lack of definitive success by some of its early proponents for a mathematical modelling approach or it has become “routinized” as applying a set of heuristics to problems. Novel approaches are needed if we wish to realise the central aim of the Singapore school mathematics curriculum.

To this end, we designed a problem solving package for trial in a selected school of high ability students and provided support to its teachers to implement the module. A crucial aspect of our approach is the Practical Worksheet, which serves as a pedagogical and assessment tool. We developed a rubric to assess the process as well as the product of problem solving. We insisted that students be assessed for the module so as to convey to them what is valued in Pólya-style mathematical problem solving. The findings from the in-depth interviews with three students and the results of the end-of-module assessment are encouraging. They illustrate the potential of the approach we are recommending to enable every student to experience genuine problem solving. There still remains the hard arduous work of refining and accommodating the problem solving design for students in mainstream schools. To be realistic, the approach will have to withstand the uncompromising and real threat of it being routinized by teachers and students because of the

pressure of high-stakes national examinations. Will the M-ProSE approach to Pólya-style problem solving help students in these examinations? We believe that, at the least, it is not detrimental to student performance in national examinations but there are benefits to students learning it.

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