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Preferred Representations of Middle School Algebra Students When Solving Problems

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Abstract: Preferences of middle school algebra students for certain representations were investigated in this research study. Previous studies reported that students from a wide range of ages and abilities have a preference for certain types of representation when solving algebraic problems. This study focused on representational preferences of middle school algebra students of different achievement levels when they solved problems involving linear relationships with one unknown. The study was conducted with a relatively large sample size (N = 443) of 7th and 8th grade students from one US urban district with low performance in mathematics. The students' achievement level was measured by the mathematics portion of the statewide Comprehensive Assessment System (sCAS). Analysis of the collected data generated explanations of the roots for their preferences and recommendations for teaching are provided.

Key words: Multiple representations; Linear equations with one unknown; Preferences; Middle school algebra

Introduction

In mathematics as a discipline, representations and symbol systems are fundamental since mathematics is "inherently representational in its intentions and methods" (Kaput, 1989, p. 169). More importantly, mathematical relationships, principles, and ideas can be expressed in multiple representations including visual representations (i.e., diagrams, pictures, or graphs), verbal representations (written and spoken language), and symbolic representations (numbers, letters). Each type of representation articulates different meanings of mathematical concepts. The recognition of multiple representational forms of mathematics concepts or problems and the implications of their use for educational outcomes have evolved considerably in recent years and stimulated the development of new approaches to mathematics education, especially at the elementary and middle school levels

(Boulton-Lewis & Tait, 1993; Boulton, et al., 1997; Goldin & Shteingold, 2001; Moritz, 2000; Outhred & Saradelich, 1997; Panasuk, 2010, 2011; Swafford & Langrall, 2000). It is now widely accepted that multiple representations are fundamental to mathematics education (Ainsworth, Bibby, & Wood, 2002; Moseley, 2005; NCTM, 2000; Niemi, 1996). There is broad agreement that students would benefit from concepts presented in a variety of forms, and that they should be encouraged to explore different aspects of each concept and solve problems through multiple representations (Goldin & Shteingold, 2001; NCTM, 2000).

Research on the use of alternative representations when learning mathematical concepts and solving problems has focused on students' ability to generate and use particular forms of representation (e.g., Brenner, 1995; Cifarelli, 1998; Diezmann, 1999; Diezmann & English, 2001; Hall, 2002; Moseley, 2005; Outhred & Saradelich, 1997). For example, previous studies (e.g., Boulton-Lewis & Tait, 1993; Dreyfus & Eisenberg, 1982; Keller & Hirsch, 1998; Ozgun-Koca, 2001) conclude that students have a preference for using certain types of representation when solving algebraic problems in general, and there is evidence that the preference is related to the students' ability level in mathematics.

As a contribution to this emerging research field, we report here on those aspects of a multiyear, large scale study (designed by Panasuk, 2006, 2010, 2011; Beyranevand, 2010) that focused on students' experiences and preferences towards different representations. The study was conducted with a relatively large sample size (N = 443) of 7th and 8th grade students from one of the New England (USA) states. The district is urban and is labeled as low performing in mathematics. The student achievement level was measured by the mathematics portion of the statewide Comprehensive Assessment System (sCAS), which has been proven to be a consistent and reliable assessment.

The research centered around two related ideas. First, and most important, is that multiple representations of structurally the same concept (Dreyfus & Eisenberg, 1982) or a problem situation are a powerful tool to develop and deepen mathematical reasoning. Interpretation, translation, and switching between representations contribute to building conceptual understanding. In fact, this research suggests that the emphasis in a single representation form can inhibit student's long term mathematical development.

Second, learning how to solve problems that involves linear relationships with one unknown, in a variety of representational modes, is the foundation for developing students' algebraic reasoning. Verbal, pictorial, and symbolic representations are commonly used to model mathematical concepts, in general and for linear relationships with one unknown, in particular. We examined the hypothesis that high-achieving students prefer symbolic representation, and low-achieving ones prefer working with pictures, and we found that there are important similarities and differences in their preferences.

In addition, this study generated explanations of the preferences to better understand how middle school algebra students, from a wide range of achievement levels, reason and use multiple representations when solving linear equations with one unknown.

Background: Representations

The following is an outline of the major current views that are essential to the concept of representations and a brief summary of the research on students' preferences for representations relevant to this study.

Seeger (1998) suggested that the process of representation or representing involves identification, selection, and presenting a concept through a device that is structurally similar and more easily understood. The alternative methods of representing algebra problems or concepts include images, diagrams, pictures, metaphors, analogies, symbols, and signs (language and notations). While using representations is necessary to student's understanding of mathematics concepts and the relationship among them, it is worth noting that each mode of representation provides only partial information and "stresses some aspects and hides others" (Dreyfus & Eisenberg, 1996, p. 268), thus is limited in certain, yet important ways.

Bruner (1973) distinguished three different systems of representation used to describe the process of cognitive development: enactive, iconic, and symbolic. Vergnaud (1997) suggested viewing representation as an attribute of mathematical concepts, which are defined by three variables: the situation that makes the concept useful and meaningful, the operation that can be used to deal with the situation, and the set of symbolic, linguistic, and graphic representation that can be used to represent situations and procedures.

Pirie (1998) associated representations with mathematical language, which she classified as ordinary language, mathematical verbal language, symbolic language, visual representation, unspoken but shared assumptions, and quasi-mathematical language. She asserted that the function of any type of representation is to communicate mathematical ideas, and that each type of representation adds to effective communication and helps to convey different meanings of a single mathematical concept.

External and Internal Representations

Distinguishing internal and external representations, Kaput (1999) used the term "fusion" to emphasize the actions surrounded by the experience of internalizing the external representation. Goldin and Shteingold (2001) asserted that the interaction between external and internal systems of representation is essential to mathematics teaching and learning.

Internal representations are usually associated with mental images that individuals create in their minds. Pape and Tchoshanov (2001) described mathematics representation as an internal abstraction of mathematical ideas or cognitive schemata, that, according to Hiebert and Carpenter (1992), the learner constructs to establish internal mental network or internal representational system. For the teachers of mathematics, the issues are how to facilitate in students the abilities to recognize, create, interpret, make connections, and translate among alternative representational modes. A difficult task in any case, it is made more complex by the students' pre-existing experience of potentially incorrect or inadequate internal representations and the use of different types of images.

External representations are usually associated with the "knowledge and structure in the environment, as physical symbols, objects, or dimensions" (Zhang, 1997, p.180). Goldin and Shteingold (2001) suggested that an external representation "is typically a sign or a configuration of signs, characters, or objects" and that external representation can symbolize "something other than itself" (p. 3). Most of the external representations in mathematics are conventional; they are "objectively determined, defined and accepted" (Goldin & Shteingold, 2001, p. 4).

Research on Multiple Representations and Students' Preferences

Pape and Tochanov (2001) observed that, when students generate representations and communicate their reasoning of a concept or problem situation, they reveal natural tendency to reduce the level of abstraction (given by the problem) to a level that is compatible with their existing cognitive structure. Similarly, Wilensky (1991) suggested that students tend to make the unfamiliar more familiar by "concretizing" the concepts they learn (p. 196). The concretizing is associated with the construction of an internal representation and presumably involves the process of reducing the level of abstraction while solving problems (Hazzan, 1999; Hazzan & Zaskis, 2005), including linear equations with one unknown (Panasuk, 2010).

Swafford and Langrall (2000) and Lowrie (2001) suggested that abstract representations of equations made little sense to early middle school students, and that they had certain representational preferences when solving equations. Ozgun-Koca (1998, 2001), Keller and Hirsch (1998), and Dreyfus and Eisenberg (1982) suggested that students of different ages and achievement levels have preferences in using certain mode of representation and found it easier to focus on one representation. Ozgun-Koca (1998) reported that low-achieving students, especially, have a preference for certain representations. Nevertheless, the knowledge about student preferences and the roots of the preferences is limited.

Method

Most of the previous studies related to multiple representations operated on a relatively small sample, did not include low-achieving students in their samples, and described their samples as selected students who were willing to participate. Previous research has convincingly shown that transition from arithmetic to algebra involves certain expected obstacles (e.g., Filloy & Rojano, 1989; van Ameron, 2002) due to the differences between arithmetic and algebraic forms of generalization and formalization. While the ability to solve linear equations with one unknown is the foundation for studying many concepts in algebra, there is evidence that student understanding and retention of the concept of linear equations is often minimal for certain subgroups of students (Hall, 2002; Vaiyavutjamai & Clements, 2006). Our study incorporated a non-experimental, mixed methodology design that involved multi-component survey, students' standardized test scores, and interviews with selected students.

Sample and Process

Four hundred and forty three (N = 443) 7th and 8th grade students participated in the survey. The students in the district had regularly performed in the bottom third on the annual state standardized mathematics test. In the year prior to data collection, 8th grade failure rate was 43% as compared to 24% in the state. In our study, three demographically compatible middle schools in the district (total 9) administered the survey to all 7th and 8th grade students from 21 classes. The district is well known for its low socio-economic status and very diverse and transient population. Some of the participating students had not been in the school the previous year, thus the data of the corresponding sCAS score was available for only three hundred ninety students (N = 390). The state Comprehensive Assessment System (sCAS) reports the scores in six categories, which include Low Warning, High Warning, Low Needs Improvement, High Needs Improvement, Proficient, and Advanced. Table 1 shows the scores of the participating students in each category of the sCAS.

Although there are other ways to classify students' achievement levels (e.g., report cards and benchmark tests) and single standardized test high- or low-achieving categorization limits the study's conclusions and its generalizability, given the relatively large sample size, the sCAS scores were considered the most attainable source of reliable and consistent data.

Table 1
Participating Students' Standardized Test Scores

SCAS Classification	Frequency	Percent
Low Warning	4	1.0
High Warning	89	22.8
Low Needs Improvement	60	15.4
High Needs Improvement	73	18.7
Proficient	119	30.5
Advanced	45	11.5

The district had consistently used the mathematics curriculum (Connected Mathematics Project [CMP], 2009) that was recognized by the state for its central goals to promote multiple representations, critical thinking, and problem solving. Since the curriculum introduces the concept of linear relationship with one unknown in the 7th grade and further develops it in the 8th grade, these two grade levels were selected for the study. The survey was administered after the completion of the units on linear relationship with one unknown. Students' identification number was available in order to distinguish each student's standardized test score. We do recognize that there are other ways to classify students' achievement levels (e.g., report cards and benchmark tests), and that single standardized test high- or low-achieving categorization limits the study's conclusions and its generalizability. However, given the relatively large sample size, the sCAS scores were the most attainable source of reliable and consistent data.

Instrument

In this paper we report on only three parts of the comprehensive survey aimed to investigate several hypotheses. Originally, Part I combined 12 items with five Likert scale response choices (always, often, sometimes, rarely, never). We report on 9 items (see Table 2) that are relevant to this paper. All items were broken into three constructs shown in Table 2 and scoring codes were clustered around students' preferred mode of representation. The items were coded from 5 ("always") to 1 ("never"), except negatively worded items that were reverse coded.

Table 2
Survey Part I Items, Means and Standard Deviations

Construct	Survey items	Mean	SD
Preference Towards Pictures	1. I need to draw a picture when I solve problems.	2.41	.798
	5. When I solve problems, pictures help me.	3.10	.964
	6. Pictures confuse me when I solve problems.	4.10	.853
Preference Towards Words	2. I am able to solve word problems.	4.02	.792
	9. Word problems confuse me.	3.68	.885
	12. I am able to solve a word problem without drawing a picture.	3.44	1.011
Preference Towards Symbols	4. Solving problems with symbols confuses me.	3.67	.915
	7. I am comfortable with math symbols and can easily work with them.	3.95	.960
	10. I am good at memorizing rules.	3.60	1.011

Note: Items 4, 6, and 9 were reversed coded.

Part II contained three sets of statements with three choices, and asked the students to select either *all that appropriate* options which would best reflect their current learning practices or one option that would reflect the *most preferable* mode of thinking (words, pictures, numbers/symbols) when solving linear equations with one unknown. The students were asked whether they prefer to use trial and error analysis, or general rules, or pictures, or whether they recall the steps shown in class to find the unknown number.

Part III had three sets of problems: Set W (words), Set P (pictures), and Set S (symbols). Each set consisted of three problems that involved linear relationships with one unknown, and each problem had its counterparts on the other two sets. The students were asked to solve each problem. Set W had three word problems that can be solved by using both arithmetical procedures and algebraic model (e.g., Mark is 10 years older than his sister. He is now 28. How old is his sister now?). The students were particularly encouraged to exercise algebraic reasoning and to apply their algebraic skills when solving the problems. Set P posed three linear relationships presented in visual form via pictures. One of the Set P problems is shown on the Figure 1.

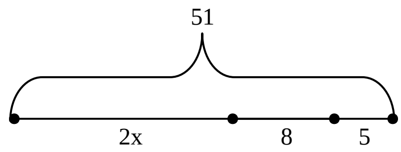

Figure 1. Find the Length of the Unknown Segment

Set S contained three linear equations with one unknown presented in symbols (e.g., $3x + 2 + 4 = 27$). For each Problem Set in Part III, a coding system was created, which included 4 codes from 0 to 3. The code 0 was assigned if the students made no attempt (i.e., left it blank). The code 1 was assigned to the answers which indicated that the students did not recognize (answered "no") the same relationship presented via three different representations. The code 2 was assigned if the students recognized the relationship presented via three different representations (answered "yes"), but did not explicitly verbalize their thinking in a clear and coherent way. The code 3 was assigned if the students recognized the same relationship (answered "yes") and explicitly described the relationship presented via three modes.

The choice of this particular level of linear equations with one unknown was based on the following reasoning. Given the sample of the study, if we had chosen to work with more complex linear equations with one unknown, we would had risked getting a greater number of students who would be unable to solve the equations, and this would not be helpful for testing our hypotheses. These problems were relatively straightforward so that, given the sample of the study, it was possible to have sufficient number of students who could solve the problems for us to test our hypotheses.

Followed by the analysis of the data, fifteen most representative surveys were selected and the students were invited for interviews. The selection was based on the responses that had a potential to provide information related to the different achievement level students' preferences. Of those fifteen, nine agreed to be interviewed. It was determined, after the fact, that there were two students in each of the High Warning, Low Needs Improvement, High Needs Improvement, and Proficient category, and one student in the advanced category.

Results

Quantitative Component

All collected surveys were coded on a nominal or ordinal scale, and frequencies, means, standard deviations, regression/correlations analysis, and chi-square tests of independence were calculated for each part of the instrument as appropriate. The nine survey items reported in this paper were organized in three constructs related to the students' individual preferences for representational modes. Table 2 illustrates the data for the Part I for all students from both grades.

The analysis of the data showed that low-achieving students had a preference for working with pictures (iconic representation) and numbers, while high-achieving students had a preference for working with symbols, with $\chi^2 = 43.7$ ($df = 10$) and p-value < .001 (Table 3).

Table 3
Chi-square Test for Part II

All participants	Pictures		Words		Numbers/Symbols	
	Frequency	Percent	Frequency	Percent	Frequency	Percent
	81	19	90	21	252	60
Low Warning	2 (0.692)	50.0	1 (0.947)	25.0	1 (2.36)	25.0
High Warning	25 (17)	28.4	36 (23.2)	29.5	37 (57.9)	42.0
Low Needs Improvement	12 (10)	20.7	17 (13.7)	29.3	29 (34.2)	50.0
High Needs Improvement	10 (12.6)	13.7	14 (17.3)	19.2	49 (43.1)	67.1
Proficient	18 (19.9)	15.7	20 (27.2)	17.4	77 (67.9)	67.0
Advanced	1 (7.79)	2.2	5 (10.6)	11.1	39 (26.6)	86.7

* Expected contingency value in parentheses

Table 4 shows the mean, standard deviation and frequency of the correct solutions for each of the nine problems in the Part III. The majority of the students were most successful in solving equations presented symbolically, which was in line with their reported preferences for thinking in numbers/symbols, as compared to words and/or pictures. In addition, symbolic equations do not involve deeper processing required for word problems and problems presented in pictures.

Table 4
Part III: Mean, Standard Deviation and Frequencies

Part III Problem Sets	Mean of All Three Problems	SD	Problem: Percent Correct #1	#2	#3
W: Verbal representation	2.34	.634	86.7%	98.2%	49.2%
P: Pictorial representation	2.36	.879	88.0%	87.7%	60.6%
S: Symbolic representations	2.62	.731	92.9%	88.7%	80.7%

Qualitative Component

The interviews followed a semi-structured format and provided another layer of evidence. Watching how the students solved the problems posed in different representations informed our understanding of the students' representational preferences. Students in the Low-Warning category articulated a preference to work with pictures, which was consistent with the data obtained from the survey. A High Warning category student (coded as Lisa) stated that working with pictures was "easier for me so I can visualize" the problem. Lisa indicated that she struggled to understand word problems and symbols. She did not attempt problem #3 from the Problem Set W when she took the survey. During the interview, Lisa was encouraged to draw a diagram that reflected the information given by that same problem, and she was able to do so; see Figure 2.

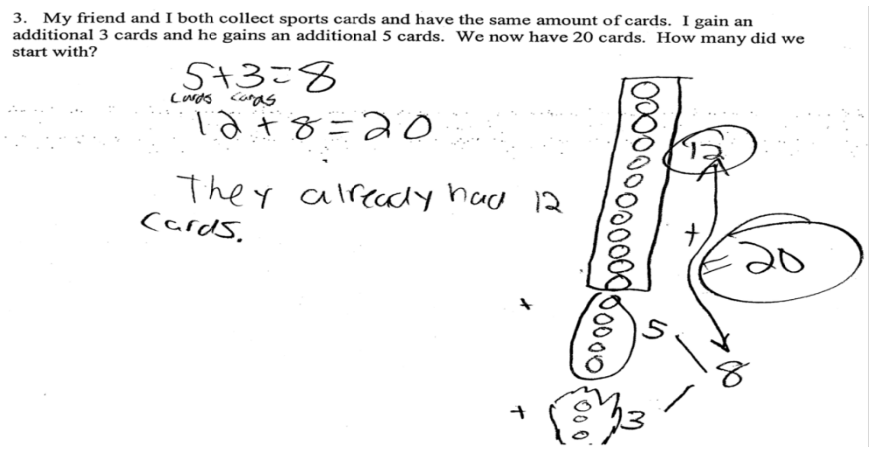

Figure 2. Lisa's Translation Word Representation into Pictorial Representation

When asked to write an *algebraic* equation that would describe the relationship posed by the problem, Lisa produced two numerical equations shown on the left side of Figure 2. She did not like the picture where the numbers and unknown were presented as line segments, and apparently was quite confused with such number/unknown image representation. We speculate that Lisa had difficulty recognizing the relationship between the lengths of the segments because she was thinking of numbers that can be represented only as discrete objects. The sophisticated type of number/unknown image such as line segment representation requires more advanced number sense and "structural awareness" (Cifarelli, 1998) level of reasoning.

A Low Needs Improvement student (coded as Ann) stated that she preferred her own drawing because "pictures created by others confused" her. When asked to draw her own picture to model the *relationship* described in words (Part III, Problem Set W, problem #2), Ann produced the picture shown in Figure 3.

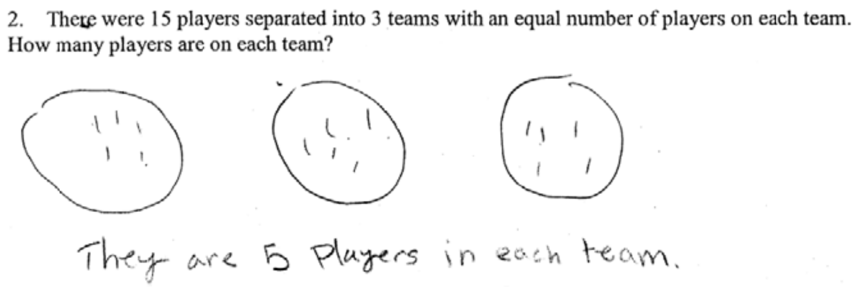

Figure 3. Ann's Picture

Such visualization provided some organizational structure and possibly helped the student process the relationship between numbers and unknown. It must be noted that while the picture on the Figure 3 seemingly assists calculations (i.e., process), it does not exactly represent *the linear relationship* the word problem describes. When asked to write an *algebraic* equation that would describe the relationship posed by the problem, Ann, being determined to solve the problem and to get the correct answer, was avoiding thinking in abstract terms, and preferred thinking in numerical mode, 3×5. One of the goals of the interviews was to gain insight into the students' ability to think in algebraic terms. They were asked to produce algebraic sentences and drawings that would reflect their *algebraic* reasoning. The algebraic sentence that would describe the relationship stated in words is either $15 \div 3 = n$, or $3 \times n = 15$. The pictorial representation that would adequately describe the relationship is shown in Figure 4.

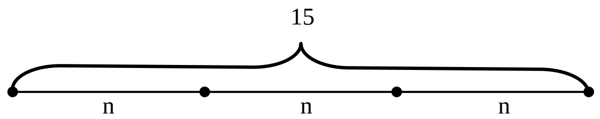

Figure 4. Pictorial Representation of the Equation: 3× *n* = 15

The interviews showed that Lisa and Ann preferred to solve word problems by arithmetic reasoning rather than first representing the problem by an algebraic equation and then applying algebraic transformations to that equation. This supports the large body of research related to the representation of word problems by equations. Stacey and MacGregor (1999) found that many students preferred arithmetic approaches to the problems in spite of the fact that they were specifically encouraged to use algebraic methods. Striving to solve the problems, Lisa and Ann tended to reduce the level of abstraction (Hazzan & Zaskis, 2005) embedded in the problems to arithmetic calculations showing a persuasive dependence on numbers, which was consistent with their survey responses. They seemed to lack the ability to recognize equivalent and identical structures (Dreyfus & Hoch, 2004) that constitute the nature of the linear equations with one unknown; thus they preferred pictorial based thinking.

High-achieving students voiced their preference for solving problems with symbols. A Proficient student (coded as Alex) stated, “I don’t really like pictures, sometimes they confuse me.” Another high-achieving student (coded as Noel) stated that she was “not comfortable drawing pictures to solve problems and [she] did not like drawing them down.” The most telling comment was written on the survey by the Advanced student (coded as Michael) shown in Figure 5. He articulated a strong aversion to pictures saying that the pictures confuse him and that he strongly preferred using the symbolic mode of representation.

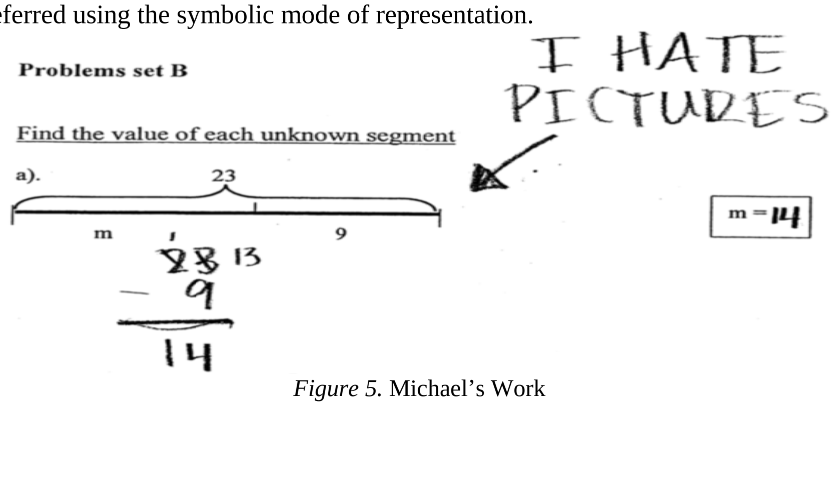

Figure 5. Michael’s Work

Another Proficient student (coded as Sonya) said that "equations make it easier because you can just put in the numbers and solve for it." Such response was alarming and when probed for conceptual understanding, Sonya revealed well-trained procedural skills and minimal flexibility in thinking while dealing with the linear relationship. All high-achieving interviewees justified consistently their preference for symbolic representation saying that "it is easier for me to look at." These students explained that they "got confused with the pictures" because they did not know "what to do" and "how to do it." "How to do it" would involve the extraction of and analysis of the structure of the relationship which is not immediately obvious and explicit when presented in words and/or pictures.

Interviewees of all levels lacked solid understanding of the deep structure of the linear relationship and the nature of unknown, and they mechanically used the rules and followed the steps, showing their preference for procedure. In other words, they were procedure oriented and did not achieve abstract thinking.

Discussion

Understanding Students' Preferences

This study supports previous research (e.g., Dreyfus & Eisenberg, 1982; Keller & Hirsch, 1998; Neria & Amit, 2004; Ozgun-Koca, 1998, 2001; Swafford & Langrall, 2000) and suggests that low-achieving students who participated in this study were more likely to prefer pictorial representations when solving problems involving linear relationships with one unknown, while high-achieving students preferred working with symbolic representations. However, the students preferred different representations for the same reason, namely procedure rather than conceptual.

When asked to create *algebraic equations* to represent the relationships posed in words and pictures from the Part III, Problem Set W and P, interviewees of all achievement levels generated numerical types of equations (e.g., 28 – 10 = 18) rather than algebraic type ($x + 10 = 28$, or $28 - 10 = n$). Setting up an algebraic equation requires an analytic mode of thinking that is quite different from that used when solving a problem arithmetically (Kieran & Chalouh, 1993). We agreed with Nathan and Koedinger's (2000) assertion that when permitted to choose their own methods, students prefer problems presented in verbal form and found them easier to solve than comparable questions presented in other forms, such as equations, or "word-equations," or diagrams. This explains why students of all achievement levels chose to present their solutions via non-algebraic methods. The participating *algebra* students were likely to think in terms of "process conception" (Sfard, 1991) by undoing the chain of operations and following this string of reasoning: 20 cards minus 5 cards and minus 3 cards would bring to 12 cards. Or, 15 players divided by three gives 5 players in each group. The algebraic method for solving the problems from the Problem Set W, #2 and #3, which we encouraged and expected the students to demonstrate, is to represent the unknown number by a letter, e.g., "*n*", construct the equation $3\times n = 15$ for the problem #2 (Figure 3) and $n + 3 + 5 = 20$ for the problem #3 (Figure 2), and then solve the equations for *n*. Apparently, because these *algebra* students showed the preference of operating over "process conception" of the linear relationship with one unknown, they functioned as "arithmetic" students and "concretized" (Wilensky, 1991) the concept by reducing the level of abstraction (given by the problem) to a level that was compatible with their existing cognitive structure. They preferred to think in numbers and operations and used established techniques learned in elementary school (Kieran, 1992). When encouraged to create a visual representation of the linear relationship with one unknown, the interviewees generated pictures that supported their calculations (i.e., process oriented arithmetic mode of thinking) and focused on the "surface structure" rather than the "deep structure" (Skemp, 1982, p. 286) of the problem. We can speculate that since these students had difficulties conceptualizing linear relationship with one unknown, they preferred memorizing the rules and steps and dealing with numbers and symbols.

High-achieving Students

High-achieving students preferred to think in a symbolic mode rather than in an iconic one. First, it is likely that they have already transitioned to the symbolic mode of thinking (Bruner, 1973) and have probably developed conceptual understanding of the linear relationship with one unknown (Panasuk, 2010b). If this were the case, then they did not need to reduce the level of abstraction (Hazzan, 1991; Hazzan & Zaskis, 2005) of the problem to a lower level, i.e., pictorial (iconic) representation. This supports Diezmann's (1999, 2002) assertion that once students have developed an understanding of the structural relationship of the concept or problem, they may no longer have the need for pictorial representation.

The second assumption is that the high-achieving students have not developed solid conceptual understanding, yet have well trained procedural skills and could efficiently manipulate symbols (as indicated by Michael who "hated pictures"). It is credible to believe that these students might have preferred symbolic representation due to its utility as a mnemonic device for recollecting and re-enacting steps and procedures.

Low-achieving Students

There are several reasons that can explain why low-achieving students reported preference towards pictorial representations. First of all, it is likely that they have not transitioned from the iconic to the symbolic mode of thinking. When faced with algebraic problems, they experience the need to reduce the level of abstraction to a level that is compatible with their existing cognitive structure (Pape & Tochanov, 2001), which tends to use imagery representation.

However, we observed a gap between the students' *perception* about their preference for representational mode (as shown in the survey) and their *factual* preference (during interviews). These students intuitively believed that pictures (concrete metaphor) would help them to solve a problem, but some of the pictures they created were more confusing than helpful. When asked to illustrate (not to solve) the equation, $x + 10 = 28$, they were not thinking of how to present the relationship (conceptual skills), but rather how to solve the equation (procedural skills). For example, some drew 28 discrete objects, then 10 more of the same objects, and suggested to take away 10 objects. Then, they realized that this sequence of actions did not produce plausible results. They were puzzled and reported that they disliked pictures because their pictures neither helped them to represent the mathematical relationship nor to solve the problem. They were more successful with the routine procedures of subtracting 10 from 28, using mostly vertical column format (i.e., arithmetic mode of thinking). The pictures they created were even more confusing than just manipulating numbers and symbols. Our findings support previous studies (Garderen, 2006; Koedinger, Alibali & Nathan, 2008) that students with learning difficulties have trouble working with certain representations, such as diagrams. As Pimm (1995) posited, "…because diagrams seem so iconic, so transparent, it is easy to forget that they too need to be read rather than merely beheld" (p. 41). It is likely that these students were rather focused on "surface details" (Diezmann, 1999, p. 8) than on the mathematical relationship posed by the problems to the detriment of the representation of the problem structure. Thus, we suggest that an overemphasis on the notion that pictures always help seems somehow unwarranted and over-enthusiastic.

External and Internal Representations

This study revealed that both low- and high-achieving students preferred their own internal mental network and did not favor someone else's pictorial representations. While a line segment image is an adequate and algebra-looking illustration of the linear relationship with on unknown, apparently this external representation was not intuitive and created "cognitive obstacles to be overcome" (Goldin & Shteingold, 2001, p.10) because they have "some ambiguity" (p.2). A picture, metaphorical by nature, can either help or confuse the formation of students' mental images of mathematics concepts. If the students have no previous training and lack diagrammatic literacy skills (Diezmann, 2002), the challenge of deducing meanings from unfamiliar external representation of mathematics concepts may trigger aversion to the pictorial representation and to favor "mechanized" manipulation of symbols.

It seems unjustifiable to wait for the students to discover how to adequately represent mathematical concepts diagrammatically. Diezmann and English (2001) suggested that the students must be exposed to systematic instruction which would help them to move beyond the surface or literal representation of information to a more structural and "sophisticated representation" of the problem information (p. 83). We support the idea that algebra students, particularly low-achieving ones, should be explicitly and consistently taught certain models of visual thinking (e.g., line segment) to represent numbers and/or unknowns.

Conclusion

The results of this study highlight several points of interests and implications for teaching and research.

First, it seems that the middle school algebra students who are in transition from arithmetic to algebra are highly procedural oriented. These students tend to spontaneously apply arithmetic algorithms to solving problems without making sense of the relationship between the knowns and unknowns presented by the problem situation. Because they possess a tool to process (i.e., routine steps for solving equations), they have little need to make sense of the situation. The high-achieving students with symbol sense can carry out transformations with relative fluency, thus they do not like pictures which require them to interpret the relationship and present it in different modes. Some students (most likely low-achieving) who lack symbol sense prefer pictures. When the pictures they produce do not help, they turn to operate on algebraic entities using familiar and routine arithmetic procedures.

While pictorial representations are essential for building an understanding of the mathematical concepts, it is hard to create a picture that all students would understand. Thus, the teachers need to systematically incorporate into classroom instruction different types of pictorial illustrations (networks, matrices, hierarchies, line diagrams, etc.) of the same concept.

Second, it is critically important to focus classroom instruction on "diagram literacy" (Diezmann & English, 2001, p.77) and develop in students of all achievement levels the ability to think and to learn in terms of images. It is highly likely that students, who have not been instructed in the general purpose of pictures and in possible variations of imagery representations of the concepts and processes, would produce structurally inadequate picture of the problem. Thus, they need explicit and organized instruction on how to generate appropriate pictorial images of the concept and to reason with the images. Teachers need systematic professional development focused on diagrammatic literacy, including (a) training in pictorial representations as an effective tool for thinking, (b) testing representations for adequacy, and (c) observing, monitoring, and responding to correct and incorrect representations provided by students of structural information in a given problem.

While we stress the importance of diagrammatic representations, one must keep in mind that they do not demonstrate the *process* of problem solving; rather they are fixed in one "frozen" instant. Most of the pictures used in the classroom are not dynamic and show no transformation of the images. Since both high- and low-achieving students are procedural oriented (but for different reasons), they might significantly benefit from instruction that incorporates animated pictorial representations with technology. Interactive simulations and animations can provide a rich environment to all students regardless of their preferences of representational modes, can help clarify abstract relationship, and thus can positively motivate the students to learn. Whenever it is difficult to explain and grasp complex connections in words or with one type of picture, a dynamic simulation can assist in displaying such connections. Further research is needed to study how different groups of students learn about linear relationship using videos with animated transformations.

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