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Attending to Oneself so as to Sensitise Oneself to Learners

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Abstract: The conjecture is proposed that by observing and paying close attention to the movements of attention and energy in oneself, it is possible to sensitise oneself to notice similar shifts in the attention and energies of learners. This enables teachers to make appropriate choices in the midst of a lesson. A phenomenological stance is taken, by offering some mathematical tasks to undertake through which it is possible to notice shifts in attention and energies. These possibilities are then described, and related to possible questions deserving of further research.

Keywords: Attention; Energy; Affect; Phenomenology; Noticing

Introduction

My approach to any question, issue or topic in mathematics education is to look for analogies in my own experience, in order to inform and sensitise me to what may be happening for others. My method uses the *discipline of noticing* (Mason, 2002), the products of which are task-exercises through which others can gain access to things that I have noticed and consider worthy of further attention. Thus the first section offers some mathematical tasks with some commentary: readers will get nothing from the rest of this note without undertaking the tasks themselves. One of the assumptions being made is that tasks generate activity which affords experience, but in order to learn effectively and efficiently from experience, it is necessary to withdraw from that experience and to consider (reflect upon) the mathematical and pedagogic actions being experienced with a view to identifying effective actions to use in the future. This is most effectively accomplished by recognising and labelling phenomena and associated actions so that they come to mind readily as possibilities in the future.

A phenomenological approach to mathematics education requires that everything that is said (in this case written) or observed by participants is considered to be a conjecture. Everything must be tested in your own experience, most especially the thesis of this note, namely that sensitising yourself through your own experience can sensitise you to the experience of learners. Conjectures are uttered with the intention of getting them “out” so that they can be considered dispassionately, and usually, modified. Mathematical thinking can only develop in a conjecturing atmosphere, in which those who are sure hold back, perhaps asking questions intended to assist others while those who are less sure to try to express and then modify their conjectures.

The themes that are particularly intended to come to the surface through the task-exercises presented here are the movements of attention when thinking mathematically, and the ebb and flow of different types of energies. An assumption being made is that if the full human psyche (enaction, affect, cognition and will/attention) is called upon, then learner experiences will be maximally rich and fruitful.

Tasks-Exercises

Calculations

376 + 459 = ?   438 + 526 – 438 = ?

679 + 847 – 677 = ?   (43 + 72) – (42 + 73) = ?

$$\frac{10\ 000 \times 10\ 004 - 10\ 002 \times 9998}{10\ 000 \times 10\ 001 - 10\ 001 \times 9999} =$$

Comment

The first few calculations are likely to have been met with a reaction of “why bother?”, “who cares?” or “do I really want to do this?” The second and third, if engaged in, might highlight the difference between *reacting* and *responding*: people who dive in tend to do the addition before realising that it is not necessary, whereas those who spend a moment gazing at the whole and then discerning details and recognising structural relationships don’t have to do any significant calculation. Some get stuck into calculations and then abort when they realise they didn’t need to calculate.

The last one looks completely off putting until you notice relationships between the numbers. Denoting the number 10,002 by a symbol converts the calculation into an exercise in algebra, leading to the simple answer of 4.

Learners who dive in to a task as soon as an action comes to mind often benefit from being held back to consider whether there might be a faster or more efficient approach, an example of what can be learned by reflecting on effective actions. Instead of seeing arithmetic as being about getting correct answers to calculations, arithmetic can be seen as the study of actions on objects and as the study of relationships between numbers. Considerable research has been done on recognising relationships in calculations (Mason, Stephens & Watson, 2009; Molina, Castro & Mason, 2007; Stephens, 2006).

No task of any quality or pedagogic effectiveness is an isolated island complete unto itself (apologies to John Donne (1572-1631) whose meditation "no man is an island" is much quoted). Any task of value can be extended and varied, including use of the mathematical theme of doing and undoing. Take for example the last calculation. The 4 in the answer is *not* the 4 of 10 004 in the question. This can be seen by replacing the original task by 10 000 + b or more generally by $a + b$ and expressing the other numbers in terms of these.

Learners often get given tasks to do without appreciating where the tasks came from or how they arose. Consequently it may be difficult to recognise the "type" of a task encountered on a test or examination. By inviting learners to construct their own variations they become aware not only of the origins of tasks and of mathematical objects, but also of the "type" or class being instantiated in the particular case. Thus learners who spend time expressing generality recognise algebraic expressions as having come from expressions of generality, and so may be more interested in and more willing to engage with the manipulations that form the techniques of algebra; and similarly with arithmetic.

Getting learners to construct examples of mathematical objects (Watson & Mason, 2005) and types of problems is more than a potentially interesting pedagogic strategy. It alters the ebb and flow of energies. Learners who construct their own examples on which to practice a technique, who develop techniques for tinkering with mathematical objects to create ones that meet special constraints not only enrich their personal example spaces (Goldenberg & Mason, 2008; Sinclair, Zazkis, Watson & Mason, in press; Watson & Mason, 2002a, 2002b) but also change their relationship with tasks. Instead of being oppressed by tasks being set by others, they can recognise what they are asked to do as a particular case of a general class of tasks which they are confident they can deal with.

Percentage Increase & Decrease

Consider (and check) the following facts

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)=1 \quad \left(1+\frac{2}{5}\right)\left(1-\frac{2}{7}\right)=1 \quad \left(1+\frac{3}{8}\right)\left(1-\frac{3}{11}\right)=1$$

Now make up some of your own, by attending to patterns of what is the same and what is different about these three "facts".

Comment
It is perfectly possible to make up examples like these that also "work", but the aim of the task is to express a generality, making use of relationships between the numerator and denominator of the fractions in each pair of brackets. Detecting and using the relationships to generate new instances is a form of *going with the grain* (Watson, 2000). Most people can detect such a pattern, though it is a sensitivity that can be developed, and is one of the hallmarks of a mathematician (Cuoco, Goldenberg & Mark, 1996).

Having detected and used relationships to express a general property, there is an orthogonal process, *going across the grain*. This involves interpreting the structure (Watson *op cit.*). Here, an increase of 50% is counteracted or undone by a reduction by 33%; a decrease by 2/7^{ths} is undone by an increase by 2/5^{ths}; and so on. Thus the mathematical theme of *doing and undoing* is manifested in and through detecting relationships.

Note however that going across the grain, seeking structural relationships, may not happen naturally. Learners may need to be helped to withdraw from the action of going with the grain, in order to become aware of the actions and the structural relations being used. Gattegno (1987) used the term *awareness* to indicate *that which enables actions*, whether conscious or unconscious. Here the pedagogic aim is to direct learner attention to the value and power of locating structural relationships.

At first the prospect of calculating with fractions may seem daunting or an obstacle (types of energy blockage), but once the learner is freed to construct their own, it is possible that they will experience a frisson of freedom, and if they detect and express a generality, a further frisson of pleasure. Using your own powers is pleasurable; having someone else (text-author or teacher) usurp those powers and do or try to do the work for you is blunting, dispiriting and ultimately unproductive. Detecting a pattern between the fractions in the bracket pairs, informed by what is the same and what different between the three instances offered, involves considerable movement of attention. Many learners won't at first "see" the detail of the fractions, but rather be aware of a "mess of stuff" or "a bunch of fractions and brackets". Gazing at the whole is how most people start. But then attention moves to discerning detail within what was previously treated as a whole. It is when there is a shift into seeking and recognising relationships between the fractions in a pair, between discerned details, that "pattern" emerges. When these are perceived as instances of a general property which is instantiated in three instances, it becomes possible to move reasoning on in order to justify the conjectured generality.

Circles in Circles

What is the ratio of the areas of overlap between the small circles, and the area of the large circle not in any of the small circles? See Figure 1.

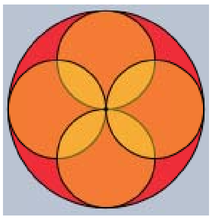

*Figure 1.*Areas of Overlapping Circles

Comment

At first, there is the whole diagram. Then the four smaller circles come into focus. The task itself directs attention to two areas, so these have to be discerned, and then held in relation to each other. The task may seem mysterious (a chance to experience a chasm or obstacle between current state and goal state) until the need to work out some relative measurements comes to mind. Once the area of each of the small circles is related to the area of the large circle, the required ratio becomes evident, even though the eye finds it hard to believe.

This is an instance of a much more general result known as the "carpet theorem," which has two versions.

> Imagine a bare room with two carpets exactly covering the floor without overlap. If the carpets are moved so as to overlap, then the area of floor uncovered equals the area of overlap.
> Imagine two carpets in a room, perhaps overlapping. The carpets are moved slightly so as to change the amount of overlap. Then the change in overlap is equal to the change in uncovered floor.

There are several opportunities here. First, there is the realisation that the circles result was obtained by using the first carpet theorem without, possibly, recognising it as a general principle. One of the important features of my way of working is to try to provoke participants into experiencing something spontaneously, intuitively, naturally, and then to draw their attention to it as an action that might be useful elsewhere. Second, there is the overcoming of the initial obstacle of how to compare the areas, resolved by focusing attention on one of the small circles and finding its area, an action that can be carried out, even if there is some resistance because of the possible involvement of π. Third, the shift from a static version of the carpet theorem to a dynamic version requires a shift in thinking, which involves a different way of thinking (change instead of steady state). Fourth, there is the possibility of invoking the theme of doing & undoing: instead of being asked the area ratio, start with a number of circles uniformly distributed around a larger circle. If it is asserted that the overlap and the uncovered areas are the same, what must be the radius of the smaller circles?

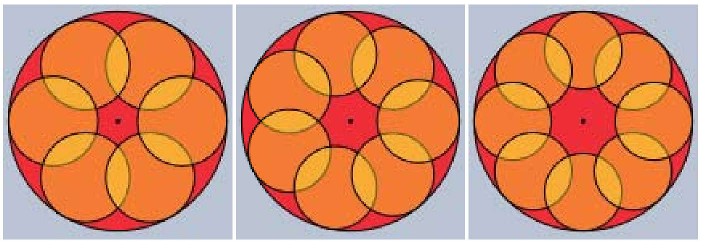

*Figure 2.*Areas of Different Number of Overlapping Circles

This raises the further question of how many such circles are required so that there is room for one more circle in the middle? Is there an instance when it fits exactly? More generally, what sort of carpet theorem would apply to three or more carpets in a room?

The posing of my own question changes the flow of energies, engages my interest, and directs my attention. When I take the initiative, more actions are possible than if I am always acted upon by tasks set by others. This shift is manifested in changes of awareness (cognition), emotion (affect), behaviour (enaction) and will (attention), and thus involves all aspects of the psyche as promulgated in the Upanishads for example (Rhadakrishnan, 1953, p. 623).

Attention

Reflecting on these and many other tasks suggests the conjecture that attention works on three levels. First there is the macro level which includes such things as attending to two things at once (conversation while driving; listening to a lecture while doodling, …) and whether my focus is broad or narrow in scope, and fuzzy or clear. Then there is a meta level illustrated here by the shift from a static version of the carpet theorem to a dynamic version, and this is paralleled by shifts from discrete to include the continuous, from accepting things to be as they seem ("it just is") to seeking logical connections ("it must be") among several others (Watson, 2010). Finally there is a micro level in which there are rapid shifts between:

Holding wholes (gazing at some aspect without significantly discerning details)
Discerning Details
Recognising Relationships
Perceiving Properties
Reasoning on the basis of agreed properties.

It seems that the shift from recognising relationships in the specific or particular to perceiving these as instantiations of a more general principle is rather subtle (Mason, 2006). An example is the shift between using the carpet theorem principle (a form of "theorem in action" as described by Vergnaud, 1994) to perceiving the carpet theorem as a general result that could be used in many situations, as indeed it can. If teachers are talking to learners about instantiated properties when learners are struggling to recognise the relationship, or if the teacher is talking about relationships while the learners are struggling to discern the relevant details, or if the teacher is discerning and referring to details while learners are still gazing at the whole, communication is likely to break down. Thus different forms of attention can provide an explanation for why it is that learners do not appear to "take in" what they are told, or make the sense that the teacher imagines them to be making during exposition, exemplification of procedures, or in mathematical discussion.

Energies

The term *energies* refers here to the ebb and flow, the flux of what might be called drive or motivation. Where does the initiative lie and how much force is there connected with it? In considering the three tasks presented above, there may have been moments of reluctance (for example to engage in purposeless calculation, or uncertainty as to what action to initiate), moments of more intense resistance, moments of sudden engagement or welling up of interest (particularly if you posed your own related questions). Your sense of agency, of what you felt empowered to do is strongly influenced by how you perceive the situation (whether the answers matter, for example) and your developing habits, which turn into dispositions.

People experience both rapid changes and stabilities in the nature and flow of energy they experience. Stabilities are associated with learned habits linked to dispositions. Once learners have become used to thinking that they can't do something, they slide down the slippery slope of "won't" and "don't" to become non-participants. Agency informs identity. Working on the language they use so that “can’t” is turned into “could” (in the sense of try harder) can be very effective (Dweck, 2000).

For example, it is sometimes useful to distinguish between learners who *assent* to what they are told, sitting back, as it were, and receiving without apparently acting upon what they receive, whether it is exposition, explanation or invitation to engage in activity. Thus when a task is proposed, some learners dive in immediately, others wait until they are told specifically what they are to do, and others consider things and then act. Others, or the same person at other times, may be more *assertive*, in the sense of taking initiative, making conjectures, modifying those conjectures and apparently working on and with what they are offered. The *assent–assert* distinction between learner stances towards mathematics, or towards particular mathematical topics, cuts across another classic distinction between *reacting* and *responding*. The former is usually automatic and based on habit whereas the latter involves participating in some choice, acting freshly in response to the situation and taking possible consequences into account. Responding includes actions such as:

> re-constructing rather than relying on remembering, though the original meaning of re-membering is much more to do with constructing-again from the constituent “members”;
> re-generating rather than relying on re-call;
> re-presenting rather than re-gurgitating (pouring out what was poured in).

The notion of human beings as composed of multiple selves is suggested in a metaphor for human beings used by Plato in *The Republic* (Hamilton & Cairns, 1961, p. 353-384). It is based on the image of a mansion whose owner has gone away and in which the various servants (selves or aspects of the self) vie with each other for ascendancy. So too at different times human beings seem to act differently. Each self is constructed to deal with specific situations (being a mathematics learner in an educational institution, a sports person or a computer expert, a son or daughter, grandson or granddaughter, sister or brother and so on). Each self is characterised by different flows of energy activating different aspects of the psyche (Bennett, 1964; De Geest, 2006). One way to chart these is to use *gunagrams*, based on the notion of the three *gunas* or tendencies that comprise *prakriti* and block access to “reality” (Mason, 2002):

> Rajas: tendency towards creation and hence action or initiation
> Tamas: tendency towards resistance and hence destruction or acceptance
> Sattva: tendency towards preservation and hence order, purity, balance

Placing these at the corners of a triangle offers the possibility of charting energy flows through placing a point in the triangle presenting the perceived balance at that time between the three gunas, as manifested in particular types of situation such as being presented with an unfamiliar mathematical task.

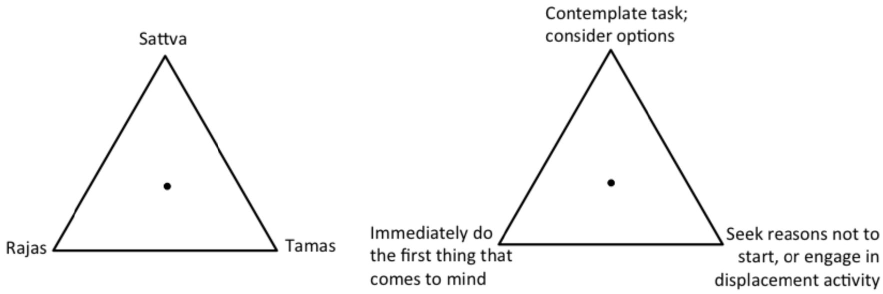

*Figure 3.*Energy Flows through Three Gunas

The centre of gravity representing complete balance is marked for easy reference. Using the three gunas to discern details in different situations leads to multiple gunagrams according to the circumstances as a way to chart the ebb and flow of energies or dispositions.

Further Research

There is much to explore concerning the ways in which attention flits from one form to another, and ways in which a teacher can recognise the phenomenon of mismatch and change actions accordingly. What are the verbal and behavioural clues that attention is in one form or another? What helps a teacher be aware of the state of their attention while in the midst of attending in that way?

In order to study energies coherently, it will be essential to develop a vocabulary which expresses what people can notice. Terms such as excited, bored, hopeful, anticipating, fearful, and doubtful all point to various ebbs and flows of energy, but it would be really helpful to capture specific experiences with more detailed observation-based description. Most terms that point to energies are themselves part of an evaluation, explanation or justification, and these get in the way of clarity obtained from brief-but-vivid descriptions (Mason, 2002). A good place to start would be in identifying different "selves" acting at different times. For example, the self that plans or prepares for a lesson, the self that initiates the lesson and the self that finishes the lesson may not always be the same, or if you prefer, the states experienced during preparation, initiation and completion may be quite different. Finding descriptions of these selves or states that colleagues recognise would be a valuable contribution to educating awareness.

One of the current foci in mathematics education has to do with *identity* (as a mathematics learner). There is a wide variety of interpretation as to what this means, but a good starting place is with Dweck (2000) who looks at self-theories, which are epistemological stances overlaid with affective assumptions. It would be really useful to know what learners at different ages think their role is in school, which involves the didactic contract (Brousseau, 1984, 1997). For example, some learners act as if they see their task as simply to attempt the tasks they are assigned, with an implicit assumption that somehow this will cause the "learning process" to take place. Most learners will not have thought about this issue, so what really matters is developing ways of working on this issue with learners without taking attention away from mathematics, so that they develop positive dispositions towards developing mathematical thinking rather than negative dispositions towards mastering mathematical procedures.

Some learners are likely to favour "working hard," but what does it mean to "work hard" at mathematics? How do learners at different ages set about studying for an examination? Plausible conjectures include correlation between success and those who go beyond "doing lots of questions" to reflecting on what is the same and different about different questions, what it would take to do another task "like this one" and asking themselves which of their actions proved successful and which in need of improvement. To what extent are learners exposed to and aware of, these extra dimensions to their activity? What strategies might learners encounter in lessons that could inform their revision for examinations?

It would be a major contribution to capture and distinguish various tamasic-dominated tendencies to reluctance, resistance or obstruction, and both their rajasic-dominated and sattvic-dominated analogues in descriptions of specific experiences that others can recognise and build upon. This could help both teachers and learners become aware of obstructions and associated actions that might reduce those obstacles. A first step is to learn to recognise subtle differences in oneself, and then test whether others recognise something similar. Then see if it sensitises you to what learners are experiencing. The proof is in the practice.

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