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Positional System: Pre-service Teachers' Understanding and Representations

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Abstract: This study reports 63 pre-service teachers' experiences of performing operations in bases other than ten. Identical, ten-item, pre- and post- group discussion questions were provided. Participants' written discussion logs and the instructor's observation notes were analyzed. The pre-discussions revealed the heavy dependency on base ten, which implied participants' limited understanding of the general concept of the positional system. In the post-discussions, participants demonstrated improved performance in several ways: increased correct response rate, increased discussion time, increased number of solution strategies, and decreased dependency on base ten. The availability of various modes of representation and the emphasis on the basic math concepts throughout the semester appear to be key factors of this improvement.

Key words: Positional system; Place value; Representations; Pre-service teachers' understanding

Introduction

Recent national mathematics education standards and policies in the US indicate that teachers' subject-matter competency and their ability to facilitate students' learning through effective pedagogical measures are becoming increasingly more valued in the learning and teaching mathematics (NCTM, 2000; No Child Left Behind Act of 2001). As Hill, Rowan, and Ball (2005) state, "teachers of mathematics not only need to calculate correctly but also need to know how to use pictures or diagrams to represent mathematics concepts and procedures to students, provide students with explanations for common rules and mathematical procedures, and analyze students' solutions and explanations" (p.372). This need for sound teachers' knowledge for teaching mathematics leads teacher educators to a question of how to design tasks for teacher candidates to be well-prepared in such a short time period. Previous research also indicates that many teacher educators often face a big gap between the understanding and beliefs that teacher candidates bring to teacher education and the current vision for effective mathematics teaching (Ball, 1990; Doerr & Lesh, 2003). Considering these issues of limited time and weak prior subject-matter knowledge, this study attempts to provide a group of pre-service teachers an opportunity to experience learning mathematics by sharing various modes of communications and representations. Utilizing teacher candidates' pre- and post-discussions on various positional systems, this study aims: (1) to investigate how teacher candidates' cooperative discussions can expand their knowledge on mathematics and pedagogical measures when meeting unfamiliar contexts, (2) to identify key factors contributing to the transformation, and (3) to reflect upon the implications of having quality tasks for the preparation of pre-service mathematics teachers. In other words, this study intends to focus on the process of communication through various representations among a group of pre-service teachers rather than highlighting their level of understanding.

Related Issues in the Literature

Through the review of related literature, this study recognized the importance of teachers' knowledge for mathematics teaching and found that there is a paucity of research on pre-service teachers' understanding of the concept of the positional system. This led us to develop a task that offers an opportunity for pre-service teachers to reflect upon their conceptual understanding and effective pedagogical measures. We intentionally included questions in different bases other than base ten for two reasons, partly, based on the implications from the previous studies: (1) The base ten system is just a specific case of the general concept of positional system. Thus, the understanding of overall positional system should precede understanding of the base ten system, and (2) The unfamiliar context, which is using different bases other than base ten, would be a quality task for the participants due to its challenging nature and uncertainty. A brief description of the related literature follows.

Teachers' Knowledge for Mathematics Teaching

From Shulman's (1986) *Pedagogical Content Knowledge* to Ball and her colleagues' (e.g., Ball, Hill, & Bass, 2005; Hill, Rowan, & Ball, 2005) *Mathematical Knowledge for Teaching,* for several decades, mathematics teacher educators have recognized the difference between mathematics subject-matter knowledge and knowledge for mathematics teaching. Although there was a unanimous consensus that teaching mathematics is a different entity from simply knowing mathematical concepts, there is an on-going investigation of how to define the level of knowledge and how to measure it.

This study employs the concept of "work of teaching" suggested by Ball and her colleagues (e.g., Ball, Hill, & Bass, 2005; Hill, Rowan, & Ball, 2005) to address teachers' knowledge for mathematics teaching. It includes all the interactions and the tasks that arise during teaching in order to support the instruction of students. In this study, teacher candidates interact with their peers, instead of their students, to complete a series of unfamiliar tasks. During this process, they still had to teach and learn from each other by clarifying their current understanding and skills, justifying and reasoning solution strategies, and providing feedback on each other's work. Thus, this study's context offers an opportunity for teacher candidates to experience the importance of knowledge for mathematics teaching before they actually interact with young students. It would also create a less stressful environment since the teaching responsibilities are shared within a discussion group.

Mathematical Tasks

The importance of quality mathematical tasks has been widely noted. A common feature of quality tasks has been expressed in several ways, such as the demand for mathematical challenge (Jaworski, 1994) and uncertainty (Zaslavsky, 2005). In other words, students' meaningful mathematical learning and understanding would occur while they experience difficulties in solving their tasks and resolving their difficulties through various measures. At the same time, quality tasks can enhance teachers' mathematical and pedagogical knowledge via reflecting upon their students' needs and difficulties (Cooney, 1994, 2001; Zaslavsky, 2005). In the study presented in this paper, teacher candidates engaged in such a task. Teacher candidates are in the transition period from being not only students, but also teachers of mathematics. It was expected that this opportunity would help teacher candidates see this specific mathematical learning experience from both student and teacher perspectives. Thus, the task used in this study was chosen due to its mathematical importance and unfamiliarity to participants, since the majority of participants have not had experiences using and operating numbers in bases other than base ten throughout their K-12 education.

Research on Positional System

The current study chose the task of solving operations in various base systems other than base ten. Current elementary mathematics content in the US does not specifically suggest teaching multiple bases (NCTM, 2000, 2006), and consequently, this concept is not sufficiently addressed in mathematics methods course textbooks or mathematics manipulatives for classroom use. In short, the current elementary mathematics curriculum and methods courses heavily depend on the base ten to explain the concept of positional system and to perform operations. However, as Vygotsky (1962) claims, we cannot claim that we understand the base ten system without mastering the more general concept of positional system, covering other bases. Rather, we are bound by it.

Studies on children's computation errors indicate that the lack of understanding of positional system causes systemic error patterns (e.g., Ashlock, 2002). Various instructional methods were proposed to enhance students' understanding of positional system (e.g., Bove, 1995; Fuson & Briars, 1990; Hiebert & Wearne, 1992; Nagel & Swingen, 1998; Varelas & Becker, 1997), but most of them explain the structure of positional system within the base ten system. A notable exception was a study of implementing a Russian experimental curriculum, which was developed based on Vygotsky's theory (Lee, 2002, 2007; Schmittau, 2004). Unlike other conventional curricula, this Russian curriculum introduced various positional systems prior to the base ten system, which is considered a specific case of the general system of numeration. Other studies utilized patterns in foreign languages or cultural artifacts to explain the numeration system and place value (e.g., Alsawaie, 2001; Cotter, 2002; Uy, 2003; Zaslavsky & Crespo, 2000).

While many research studies have investigated children's understanding of place value or proposed instructional strategies, there has been a paucity of research on pre-service teachers' understanding of the positional system. McClain and Bowers (2000) and McClain (2003) report on their teaching experiment with the Candy Factory context where eight pieces comprised a roll and eight rolls comprised a box. They deliberately use the base eight system instead of base ten to focus more on the development of conceptual understanding rather than mere proficiency with rote algorithms. Their teaching experiment reports that pre-service teachers' understanding of place value was superficially grounded in rules for manipulating algorithms.

Schmittau and Vagliardo (2006) report on a pre-service teacher's case of understanding and explaining the concept of positional system using concept mapping. The concept map presented in Schmittau and Vagliardo's study reveals the pedagogical content knowledge required to successfully teach the concept of positional system and other related concepts beyond the base ten system. Pre-service teachers' reasoning and arguments on the questions which are non-integer rational numbers, i.e., decimals and fractions, represented in numeration systems in bases other than ten were also investigated (Khoury & Zazkis, 1994; Zazkis & Khoury, 1993). Khoury and Zazkis found that the majority of participants believe that fractions change their numerical values in different bases, and that participants' knowledge of place value and rational numbers is more syntactical than conceptual. Overall, few researchers have undertaken research on pre-service teachers' understanding of the positional system and their pedagogical approach to teach the positional system.

Method

Participants

Sixty-three teacher candidates enrolled in one of three sections of a four-credit required K-8 mathematics methods course over one semester in 2007 in a Midwestern university in the US participated in this study. All of them had successfully completed mathematics content courses prior to this methods course. This four-credit course is required for elementary educations majors and is usually taken prior to student teaching. The instructor for all sections was the first author of this study. Participants consisted of 55 female and 7 male teacher candidates. The majority of participants (about 67%) were non-traditional students in the sense that there was a gap of time between their enrollment in college and graduation from high school.

Context

Throughout the semester, participants engaged in various modes of instruction. Based on the current national and state curriculum documents, teacher candidates were encouraged to utilize multiple ways of representation, including hands-on manipulation, pictorial representation, and explanation of the meaning of mathematics processes in cooperative groups.

Regarding the understanding of positional system, chip trading materials were briefly introduced to all participants in the third class session, which occurred after pre-discussion in the second class session. Instruction and practice time lasted about an hour. To maintain the unfamiliarity of the task, participants were exposed to whole number addition and subtraction contexts in various bases in this session. Two sections, totaling 35 participants, also had an opportunity to use the Prairie Rainbow Blocks for about an hour in the fourth class session in addition to the chip trading materials. The instruction was offered by the developer of the material, Dr. George Gagnon. Again, only whole number operations were addressed. The other parts of the course instruction addressed models and algorithms for four fundamental operations within the base ten system. Thus, the multiplication and division of whole numbers and fractional number comparisons in non-base ten systems were not directly instructed in class.

Data Source

A list of ten discussion questions was created by the researcher (see Table 1 below).

Table 1

Discusion questions

Part 1: Solve. Explain your solution process	Part 2: Compare. Explain your solution process
1) $23_{(4)} + 11_{(4)}$	7) $30_{(4)}$ ____ $30_{(5)}$
2) $21_{(4)} - 13_{(4)}$	8) $3_{(4)}$ ____ $3_{(5)}$
3) $1.3_{(4)} + 1.1_{(4)}$	9) $\frac{1}{2}_{(4)}$ ____ $\frac{1}{2}_{(5)}$
4) $13_{(4)} \times 11_{(4)}$	10) $0.3_{(4)}$ ____ $0.3_{(5)}$
5) $12_{(4)} \div 2_{(4)}$	
6) $12_{(4)} \div 0.2_{(4)}$	

Part I asks students to discuss and solve whole number and fractional number operations in bases 4 and 5. The first two whole number addition and subtraction questions were similar to what we discussed in the third class session. The other operations were not introduced in different bases. Also the fractional numbers in different bases were not directly covered in the course instruction. Part II contains several comparison questions of whole number forms and fractional number forms in different bases. The last two fractional number operation questions were adapted from Khoury and Zazkis's (1994) study of examining the reasoning strategies and arguments given by pre-service school teachers through individual clinical interviews and written responses. The identical pre- and post-discussion questions were given to participants at the beginning of semester and at the end of semester respectively. Partner or group work was encouraged to increase the visibility of their sense-making process through communications, as well as to reduce anxiety level. For both discussions, participants were asked to record their discussion in words, pictures, or symbolic notations. The instructor also observed participants' work during the pre- and post-discussions and wrote observation notes.

Data Analysis

Participants' pre- and post-discussion results along with detailed discussion logs and the instructor's observation notes were analyzed. The analysis included both correct and incorrect types of strategies and focused on identifying key factors contributing to each reasoning process and common pedagogical measures used during the peer group discussion. Authors individually reviewed the data collected and identified frequently employed modes of representations and solution strategies by the participants. Later, authors jointly synthesized their individual findings.

This study adopts an exploratory character offering plausible explanations for further investigation of quality tasks in mathematics teacher education, rather than providing conclusive evidence regarding pre-service teachers' understanding of positional systems (Yin, 1994, 2006).

Results

Pre-Discussion

Pre-discussion was conducted on the first day of class. Participants had the choice of completing the tasks alone or with a partner. Most participants completed pre-discussion questions with their partners (29 pairs and a group of 3 students). Only two participants chose to complete the tasks alone. Table 2 shows the summary of pre-discussion results.

Table 2
Summary of Pre-discussion

Response / Question	Correct answer	Correct #/no base notation	Incorrect answer	No response	**Total**
1	11 (34%)	12 (38%)	8 (25%)	1 (3%)	32 (100%)
2	11 (34%)	5 (16%)	13 (41%)	3 (9%)	32 (100%)
3	8 (25%)	13 (41%)	7 (22%)	3 (9%)	32 (100%)
4	9 (28%)	11 (34%)	8 (25%)	4 (13%)	32 (100%)
5	7 (22%)	7 (22%)	8 (25%)	10 (31%)	32 (100%)
6	4 (13%)	2 (6%)	10 (31%)	16 (50%)	32 (100%)
7	18 (56%)		4 (13%)	10 (31%)	32 (100%)
8	10 (31%)		12 (38%)	10 (31%)	32 (100%)
9	1 (3%)		19 (59%)	12 (38%)	32 (100%)
10	7 (22%)		13 (41%)	12 (38%)	32 (100%)

N=32 Sets (2 individuals/1 group of 3 participants/29 pairs) = 63 students

Even though unlimited time was given, most of them finished their discussion in 20 minutes and not much writing was included in the explanations. In particular, many questions in Part 2 remained unanswered. Participants were informed that manipulatives were available upon request, but none of them actually requested physical materials. It was also noted that there was a big gap among participants' prior knowledge regarding positional systems. In particular, their experience in the mathematics content course, which was the prerequisite for this methods course, could be the factor, since all of them stated that they encountered the different bases other than ten in that course for the first time. The following sections provide more information regarding participants' pre-discussion results.

Examples of correct solution strategies

There was a lack of variety of solution strategies in the pre-discussion. Mainly, the primary strategy used was to convert the given numeration into base ten, and then reconvert to the original base to answer the question. For example, the following strategies resulted in producing correct answers.

e.g.) $23_{(4)} = 2x4^1 + 3x4^0 = 11$, $11_{(4)} = 1x4^1 + 1x4^0 = 5$
$23_{(4)} + 11_{(4)} = 11 + 5 = 16 = 1x4^2 = 100_{(4)}$

Examples of incorrect solution strategies

The most frequently appearing incorrect solution strategy was to conduct operations as if the numbers were all in base ten. An entry stated, "If they [the numbers] are the same base, you can just add them together," revealing misconception.

e.g.) $23_{(4)} + 11_{(4)} = 34$

In some entries, participants attempted to apply the typical conversion process, but they did not recognize that they were using an incorrect unit system:

e.g.) $23_{(4)} = 2x\,4^2 + 3x4^1 = 32+12 = 44$, $11_{(4)} = 1x\,4^2 + 1x4^1 = 16+4 = 20$
$44+20 = 64$

Some entries demonstrated the incomplete regrouping process. For example, in the example below, one more regrouping should have occurred.

e.g.) $23_{(4)} + 11_{(4)} = 40_{(4)}$

Modes of Representation
Participants heavily depended on symbolic representations to discuss the solution strategies. A few additional representations other than symbolic forms, whether or not they led to the correct solutions, were used. Those included number lines and place value charts. However, those representations did not explicitly show the quantities involved.

Some Reactions from the Pre-discussion Session
Some verbal or written statements indicated that participants had learned the different base systems but had a hard time in recalling the procedures:

- I knew how to do this before, but I completely forgot…. Very frustrating….
- We don't remember how

The other responses indicated that the questions were new or very unfamiliar to them.

- Do all the questions have answers?
- We haven't learned how to do this with fractions
- I don't even know where to begin
- We are not familiar with base 4 and 5 in this form of a problem

Post-Discussion
A total of 16 groups, 11 groups of four participants and 5 groups of three participants, were formed for the post-discussion. Once again, unlimited time was given, and groups took two to three hours to complete the tasks. Various manipulatives were available upon request. The requested materials include the chip trading materials, unifix cubes, counters, fraction circles, fraction bars, and base 10 blocks.

Table 3 shows the summary of post-discussion. Overall, participants provided more correct answers than they did on the pre-discussion. Questions similar to the first two were discussed in class early in the semester. However, the formats of the rest of questions were not directly introduced in class. Students tried to connect several concepts and representations they learned during the semester to answer these questions. Table 4 (See Appendix) briefly reports on participants' solution strategies and modes of representation for each question.

Table 3
Summary of Post-discussion

Response / Question	Correct answer	Correct #/no base notation	Incorrect answer	No response	**Total**
1	14 (88%)		2 (13%)		16 (100%)
2	15 (94%)		1 (6%)		16 (100%)
3	15 (94%)			1 (6%)	16 (100%)
4	15 (94%)	1 (6%)			16 (100%)
5	16 (100%)				16 (100%)
6	12 (75%)		3 (19%)	1 (6%)	16 (100%)
7	15 (94%)		1 (6%)		16 (100%)
8	14 (88%)		2 (13%)		16 (100%)
9	4 (25%)		12 (75%)		16 (100%)
10	10 (63%)		6 (38%)		16 (100%)

N=16 Sets (11 groups of 4 / 5 groups of 3) = 63 participants

Discussion and Implications

This study aimed to investigate a group of pre-service teachers' construction of knowledge in unfamiliar contexts, using positional systems other than base ten, to identify key factors of knowledge construction, and to reflect upon the implications on quality tasks in the preparation for mathematics teachers.

The results of pre-discussion reveal that the majority of participants' understanding of positional system is limited to base ten. In addition, their choices of solution strategies or representations lack variety. This outcome was not much different from the previous studies (e.g., Khoury & Zazkis, 1994; Zazkis & Khoury, 1993).

The results of post-discussion indicate several distinctive features: increased discussion time, increased correct response rate, increased number of solution strategies, and decreased dependency on base ten. We consider a couple of factors may have affected these changes. First, the availability of various modes of representation appears to be one of the key factors. While the participants' explanations in the pre-discussion were brief, mostly using symbolic forms of representation only, the post-discussion reports contain various pictorial representations and verbal (written) explanations in addition to the symbolic or abstract form of representations. For example, for Question 6, several groups justified their solution using different modes of representation (see Figure 1).

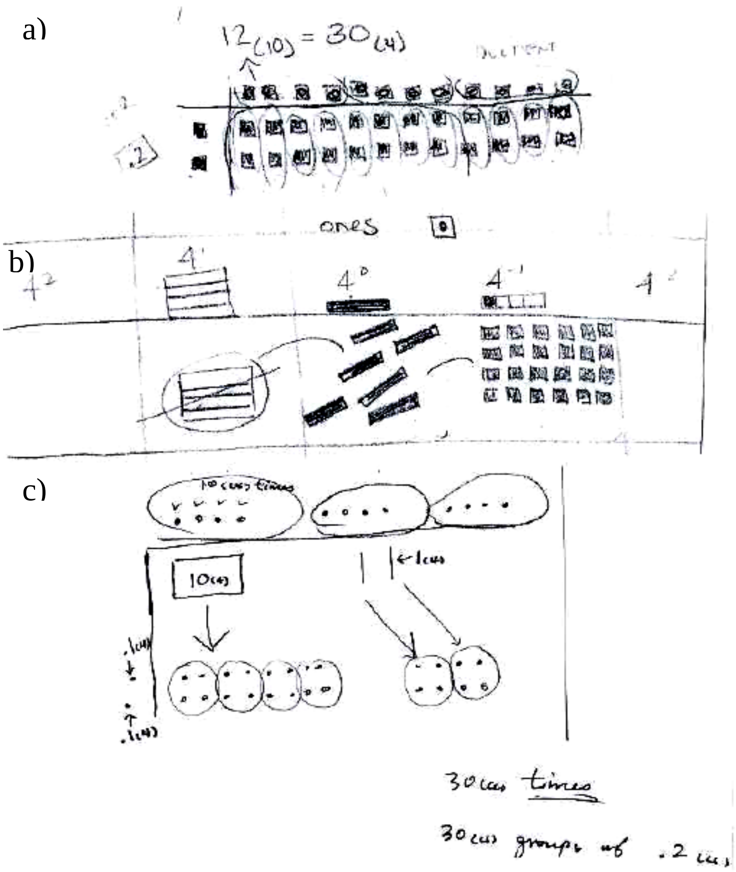

Figure 1. Different Representations for Question 6

Also, within a discussion group, participants kept using different materials and representations to verify their solutions. For example, one group had a very long discussion on Question 9. Initially, they concluded that ½ $_{(4)}$ is bigger than ½ $_{(5)}$. When one person asked for verification, the members tried to model their solutions. They used fraction bars to compare two quantities and re-checked using fraction circles. They recorded their manipulation in pictorial form along with verbal explanation and symbolic forms. During this process, they realized that their original answer did not make sense (see Figure 2 for their discussion notes).

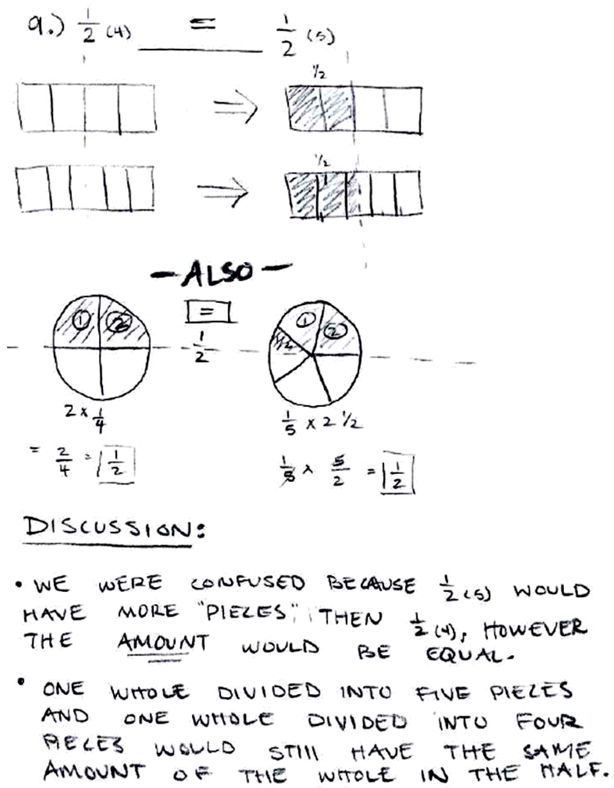

Figure 2. A Group's Discussion Notes for Question 9

Also, the availability of various representations enhances the level of discussion focusing on the quantities involved beyond symbolic-only manipulation, which resulted in less dependency on base ten.

Another factor that may contribute to the change is the emphasis on the basic concepts involved. In the pre-discussion, participants tended to approach the task by asking questions, such as "Do you know how to do it?," and "Did you learn this?" These questions easily led to giving up on further discussion with the unfamiliar contexts. For example, for the majority of participants, the multiplication of two numbers in base 4 was new, meaning that they have not been directly instructed to solve this type of problem. However, there is no doubt that participants already had learned the meaning of multiplication and the concept of place value in their K-12 experiences. The problem was that their prior knowledge was not expanded to the new situation, remaining isolated knowledge. In the post-discussion, we observed some changes. It was visible that participants' initial focus was placed on the meaning of involved operations or concepts. For example, the measurement division concept was frequently used for Questions 5 and 6 regardless of whether the given divisor was a whole number or a fractional number. In Question 8, the clear meanings of denominator and numerator played an important role.

Spontaneously during the post-discussion, participants defined the denominator as "the number of equal-sized pieces my whole is cut into" and the numerator as "the number of equal-sized pieces in my whole I am discussing," and were able to apply these basic meanings to various bases. In other words, participants tried to expand their extant prior knowledge to this unfamiliar situation. Throughout the semester, the instructor emphasized the meaning of basic concepts and participants practiced defining basic mathematical operations and concepts in their own words. This instruction, partly, may influence the post-discussion results.

The results of this study imply that the improvement of participants' performance depends on how to connect and expand their extant prior knowledge, rather than acquiring new rules for new cases. The question items included in the current study may be directly taught over a longer period of time, which could produce higher correct response rates. However, the intention was not to teach these items in an isolated manner. It was our hope that participants had an opportunity to engage in the refinement and reconstruction of their prior knowledge, which in turn may work as a turning point for many pre-service teachers. The challenge for mathematics teacher educators is to develop such tasks that support pre-service teachers combining their fragmented prior knowledge.

Limitations of the Study

As designed, this study provides a snapshot of a group of pre-service teachers' experiences regarding the understanding of positional system in order to see overall changes as a group. In this sense, each individual pre-service teacher's thinking was not clearly described in this paper. Although it was beyond the scope of this study, it would be meaningful to probe individual participants' reasoning for some question items, such as Questions 6, 9, and 10, which produced lower correct response rates. Also, in the pre-discussion, the size of discussion groups ranged between one and three participants. In the post-discussion it ranged between three and four. The size of discussion groups may have influenced the dynamics of group discussion.

Concluding Remarks

In terms of answering all questions correctly with sound understanding of the positional system, we could not say that the participants' performance was perfect. However, in terms of refining prior knowledge and choosing appropriate measures to justify their solutions, this study provides participants with a valuable opportunity. With the scarcity of teaching the general positional system covering bases other than ten, one might argue that it is not necessary to include this task in a mathematics methods course for pre-service teachers because it is not included in the current school curriculum. However, we remain convinced that teachers' understanding of the general positional system, which can be demonstrated in their ability to interpret various bases, is critical to teach the place value concept to students and to remedy students' misconceptions or errors even if they teach everything in base ten. Teachers' sound understanding of the general positional system may not be visible. However, we believe that, in the end, students will make it visible with increased success in learning mathematics.

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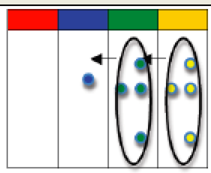
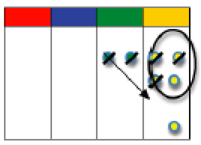
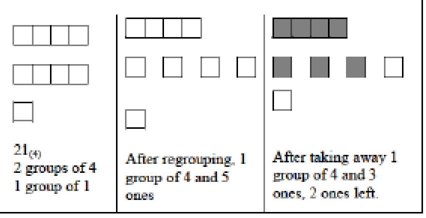
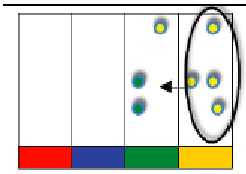
Appendix

Table 4
Strategies used in the post-discussion

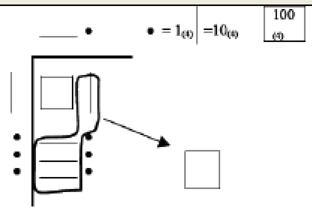
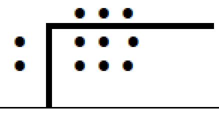
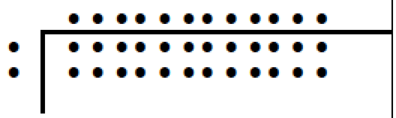
Qn	Strategies/Modes of Representations	Frequency [N (%)]*	Examples of students' work**
1	Chip trading materials/drawing	7 (44%)	
	Algorithm	4 (25%)	1 1 $23_{(4)}$ $+11_{(4)}$ $100_{(4)}$
	Convert to base 10 (with/without place value chart)	2 (13%)	$23_{(4)} = 8 + 3 = 11$ $11_{(4)} = 4 + 1 = 5$ $11 + 5 = 16 = 100_{(4)}$
	Written explanation of the regrouping process	3 (19%)	"It is like monopoly: 3 houses + 1 = hotel. Whenever we reach 4, we have to regroup."
2	Chip trading materials/drawing	11 (69%)	
	Covert to base 10	2 (13%)	$21_{(4)} = (2x4) + (1x1) = 9$ $13_{(4)} = (1x4) + (3x1) = 7$ $9 - 7 = 2,\ \ 2_{(10)} = 2_{(4)}$
	unifix cubes/drawing	1 (6%)	

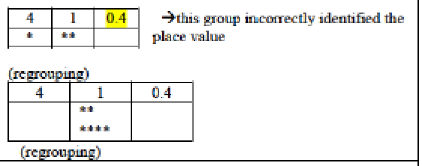
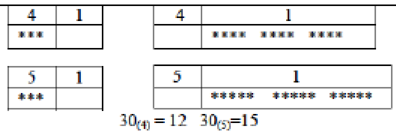
<table>
<tr><th>Qn</th><th>Strategies/Modes of Representations</th><th>Frequency [N (%)]*</th><th>Examples of students' work**</th></tr>
<tr><td rowspan="2">2 (cont.)</td><td>Number line</td><td>1 (6%)</td><td>$1_{(4)}$ $2_{(4)}$ $3_{(4)}$ $10_{(4)}$ $11_{(4)}$ $12_{(4)}$ $13_{(4)}$ $20_{(4)}$ $21_{(4)}$

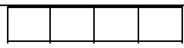
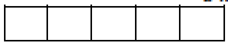
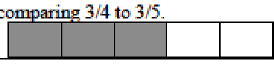
From $21_{(4)}$ count backward $13_{(4)}$</td></tr>
<tr><td>Algorithm</td><td>1 (6%)</td><td>1 4
$21_{(4)}$
$-13_{(4)}$
$2_{(4)}$</td></tr>
<tr><td rowspan="5">3</td><td>Chip trading materials/drawing</td><td>10 (63%)</td><td></td></tr>
<tr><td>Algorithm only</td><td>3 (19%)</td><td>1
$1.3_{(4)}$
$+1.1_{(4)}$
$3.0_{(4)}$</td></tr>
<tr><td>Number line</td><td>1 (6%)</td><td>$1.3_{(4)}$ $2.0_{(4)}$ $2.1_{(4)}$ $2.2_{(4)}$ $2.3_{(4)}$ $3.0_{(4)}$

From $1.3_{(4)}$ count forward $1.1_{(4)}$</td></tr>
<tr><td>Place value chart</td><td>1 (6%)</td><td><table><tr><td>4^1</td><td>4^0</td><td>4^{-1}</td></tr><tr><td></td><td>1^1</td><td>3</td></tr><tr><td></td><td>1</td><td>1</td></tr><tr><td></td><td>3</td><td>0</td></tr></table></td></tr>
<tr><td>No explanation</td><td>1 (6%)</td><td>"This really confuses us. We don't really know where to begin. We suspect that one of the places on our chart would be the decimal, but how would we change them"</td></tr>
<tr><td>4</td><td>Partial product algorithm (written)</td><td>11 (69%)</td><td>$13_{(4)}$
$\times\ 11_{(4)}$
$13_{(4)}$
$130_{(4)}$
$203_{(4)}$

We used a standard multiplication algorithm while paying attention to the bases. We make sure the answers did not go over 4 when adding.</td></tr>
</table>

Qn	Strategies/Modes of Representations	Frequency [N (%)]*	Examples of students' work**
4 (cont.)	Array/area model	6 (38%)	
	Convert to base 10	2 (13%)	$13_{(4)} = 7$, $11_{(4)} = 5$ 7x5=35 $35 = 2x4^2 + 0x4^1 + 3x4^0 = 203_{(4)}$
	Repeated addition	1 (6%)	$11_{(4)}$ groups of $13_{(4)}$ $13_{(4)} + 13_{(4)} + 13_{(4)} + 13_{(4)} + 13_{(4)} = 203_{(4)}$
5	Find the number of groups of 2 using proportional or non proportional materials (chip trading/unifix cubes)	9 (56%)	**** ** → ** ** ** $12_{(4)}$ (regrouping) $3_{(4)}$ groups of $2_{(4)}$
	Array/area model	4 (25%)	
	Convert to base 10	3 (19%)	$12_{(4)} = 6$, $2_{(4)} = 2$ $6 \div 2 = 3$, $3 = 3_{(4)}$
6	Array/area model	5 (31%)	 ● = $.1_{(4)}$ interpretation of the quotient: There are $30_{(4)}$ groups of $0.2_{(4)}$ in $12_{(4)}$.

Qn	Strategies/Modes of Representations	Frequency [N (%)]*	Examples of students' work**
6 (cont.)	Chip trading materials/drawing	3 (19%)	
	Induce from Question #5/Conventional rules	4 (25%)	"We simply moved the decimal point one place to left from Question #5."
	Convert to base 10	1 (6%)	$12_{(4)} = 6$, $0.2_{(4)} = 0.5$ $6 \div 0.5 = 12 = 30_{(4)}$
	Partial quotient	1 (6%)	$.2_{(4)}$ goes into $2_{(4)}$, $10_{(4)}$ times $.2_{(4)}$ goes into $10_{(4)}$, $20_{(4)}$ times So, $.2_{(4)}$ goes into $12_{(4)}$, $30_{(4)}$ times.
	No explanation	2 (13%)	
7	Convert to base 10 using chip trading drawing or expanded form)	12 (75%)	
	Rules	2 (13%)	"You can *always* assume that the same number in a higher base will be greater than that number in the lower base."
	Convert to base 4	1 (6%)	"If we count one base five stick into base four; we will get one stick and one single unit: $10_{(5)} = 11_{(4)}$. So, $30_{(5)} = 33_{(4)}$."
	Use the nearest round number	1 (6%)	"30(4) is closer to being 100 than 30 in base 5."
8	Convert to base 10 (with or without drawing the quantities)	11 (69%)	$3_{(4)}$: *** $3_{(5)}$: ***
	Rules	1 (6%)	"A given number in a higher base will *always* be larger than that same number in a smaller base."

Qn	Strategies/Modes of Representations	Frequency [N (%)]*	Examples of students' work**
8 (cont.)	Rules	1 (6%)	"A given number in a higher base will *always* be larger than that same number in a smaller base."
	Use the nearest round number	1 (6%)	"3 in base 4 is closer to being 10 than 3 in base 5."
	No explanation	3 (19%)	
9	Convert to base 10	9 (56%) 1 correct 8 incorrect	$\frac{1}{2}_{(4)} = .2$ $\frac{1}{2}_{(5)} = .25$
	Drawing/concept of fractions	4 (25%)	 1 whole unit, $\frac{1}{2}_{(4)}$: half of a whole  1 whole unit, $\frac{1}{2}_{(5)}$: half of a whole
	Rule (incorrect over-generalization)	2 (13%)	"Half of a larger base would be a larger number."
	No explanation	1 (6%)	
10	Convert to base 10	9 (56%)	 We are comparing 3/4 to 3/5. 
	Use the nearest round number	4 (25%)	$.3_{(4)}$ is closer to a whole unit.
	Rule (incorrect over-generalization)	2 (13%)	"They are equal because they do not need to be regrouped." [using the rationale from Question #8].
	No explanation	1 (6%)	

*There were multiple strategies found for some group's work.
** Some solution strategies did not produce correct reasoning and answers.