The Mathematics Educator
2010, Vol. 12, No.2, 103-124

Fraction Operations: Preservice Teachers' Misconceptions and Perceptions about Problem-Solving

Leah A. Nillas
Illinois Wesleyan University, USA

Abstract: This study investigated preservice teachers' content knowledge involving fractions operations and their beliefs about problem-solving. Preservice teachers took a semester of instruction on problem-solving that emphasized the use of multiple representations and solutions, communication of mathematical ideas, and working collaboratively. Common misconceptions were discussed and preservice teachers' beliefs about teaching mathematics were presented. Findings support initiatives to improve preservice teachers' content knowledge and beliefs in teaching and learning with understanding.

Key words: Fraction operations; Elementary Preservice teachers; Problem-solving, Beliefs; Teaching for understanding

Introduction

In their cross-cultural study, Zhou, Peverly and Xin (2006) found that U.S. teachers lag significantly behind Chinese teachers in subject matter knowledge and in areas of pedagogical content knowledge. They emphasized that the ability to understand mathematics conceptually is a significant factor in being able to teach with understanding. Similarly, in an earlier study, Ma (1999) found that Chinese teachers demonstrated a deeper understanding (both procedural and conceptual) of topics in elementary mathematics than U.S. teachers who exhibited mainly procedural understanding. She attributed these differences in ability and performance to differences in teacher preparation in China and United States. "Teachers must know the subject they teach" for the simple reason that "teachers who do not themselves know a subject well are not likely to have the knowledge they need to help students learn this content" (Ball, Thames, & Phelps, 2008, p. 404). In pursuit of understanding teachers' content knowledge, Ball, Thames, and Phelps (2008) developed a practice-based theory of content knowledge for teaching by studying actual mathematics teaching. They identified knowledge of content and students and knowledge of content and teaching as two sub-domains of pedagogical content knowledge. They, however, stressed that “just knowing the subject well may not be sufficient for teaching.

In support for policy initiatives, Hill, Rowan and Ball (2005) argued that improving teachers’ mathematical knowledge is an important component of improving students’ mathematics achievement. That is, teachers’ mathematical knowledge is significantly related to student achievement gains. How preservice elementary teachers drew their knowledge of mathematics and pedagogy in their teaching was what Rowland, Huckstep, and Thwaites (2005) investigated on using a grounded approach. They identified foundation, transformations, connection, and contingency as four dimensions that can be used as a framework for lesson observation where preservice teachers’ mathematics related knowledge could be observed in practice. In pursuit to understand more about what constituted teachers’ content knowledge, Hill, Schilling and Ball (2004) discussed their efforts to design and test measures of teachers’ content knowledge for teaching elementary mathematics. They found that knowledge for teaching elementary mathematics was multidimensional. It included knowledge of various mathematical topics such as numbers, operations, and algebra, knowledge of content, and knowledge of students.

There are studies that investigated the role of preservice teachers’ content knowledge and beliefs in relation to their preparation to teach and learn with understanding (Newton, 2008; Swars, Hart, Smith, Smith, & Tolar, 2007). Newton (2008), in particular, investigated extensively the nature of preservice teachers’ knowledge of fractions and examined it through multiple ways. Swars, Hart, Smith, Smith, and Tolar (2007) found that mathematics method courses that were taught with emphasis on understanding mathematical concepts and processes significantly influenced preservice teachers’ mathematical knowledge and pedagogical beliefs. With these contexts in mind, this current study is designed to investigate preservice teachers' content knowledge involving fraction operations and their beliefs about problem-solving.

The focus is on their misconceptions and beliefs about teaching mathematics via problem-solving, two areas that are not addressed all together in previous research studies. This current study integrates preservice teachers’ changes in beliefs about teaching mathematics as they learn mathematical content taught in a constructivist environment. For this current study, beliefs refer to attitudes and dispositions about teaching and learning. Beliefs play a critical role in defining preservice teachers’ behavior and organizing their knowledge. Problem-solving, on the other hand, refers to a process where an individual or group of individuals engages in finding solution to a mathematical problem that involves a conceptual task.

Studies support that teachers' knowledge has a huge impact on students' learning. Teachers' subject matter knowledge shapes the ways in which they teach mathematics (Leinhardt, et. al., 1991; Even, 1993; Langrall, Thornton, Jones, & Malone, 1996). In particular, Even (1993) showed how limited preservice teachers' concepts image are and how this limited knowledge affects students' vague mathematical conceptions. She then highlighted the need to provide better subject matter preparation for teachers. Also, a teacher's subject matter knowledge interacts with his or her assumptions and explicit beliefs about teaching, learning, students, and contexts (Ball, 1991). Sufficient depth of understanding of the subject matter will enable teachers to show the connection of mathematical ideas to students (Ball, 1990b). Thus, since knowledge often develops based on the teacher's pedagogical knowledge and through classroom interactions with the subject matter and the students, the teacher's knowledge has a critical role in student's learning (Fennema & Franke, 1992). As a consequence, transformation of knowledge should be viewed as an important goal in teaching. In this context, teaching should aim for both conceptual and procedural understanding.

The choice of mathematics content for this study is supported by research findings that rational numbers are still a difficult topic for students (Mullis, et. al., 1999; Wearne & Kouba, 2000). In the Third International Mathematics and Science Study (TIMSS) and the mathematics assessment of the National Assessment of Educational Progress (NAEP), rational numbers, in particular fractions, continue to be a problematic topic in eight-grade mathematics. Students have a vague conceptual understanding as well as a very procedural understanding about fractions. In most cases, classroom instructions focus on algorithmic or rule-based approaches in teaching the concept of fractions. There is universal call to promote teaching and learning mathematics with understanding. Most research studies highlight the need to develop and promote the use of assessment tasks that represent a balanced approach to assessment on the understanding of fractions. Grouws & Smith (2000) argued that improving students' learning of mathematics depends on knowledgeable teachers who conduct high-quality lessons, which focus on important mathematics that support students' opportunity to learn.

Research studies suggest that teachers' beliefs are important in teaching mathematics for understanding (Collier, 1972; Cooney, et. al., 1998; Ernest, 1991; Gregg, 1995; Skemp, 1976; Swars, Hart, Smith, Smith, & Tolar, 2007; Thompson, 1992). Teachers' conceptions entail more than just knowledge of specific mathematical content and pedagogical skills but include beliefs about teaching and learning mathematics. Skemp (1976) characterized teachers' understanding or beliefs as "instrumental" or "relational." In particular, teachers whose beliefs are characterized as “instrumental” depend on teaching by telling, using memorization of rules, and direct instruction in teaching mathematics. Those whose beliefs are characterized as “relational” are inclined to provide opportunities for students to explore, investigate, and use multiple strategies in solving problems. Ernest (1991), on the other hand, identified five categories of educational ideologies of mathematics education: ‘industrial trainer’; ‘technological pragmatist’; ‘old humanist’; ‘progressive educator’; and ‘public educator’. Specifically, teachers having these ideologies perceive teaching as passing a body of knowledge (industrial trainer), imparting knowledge through practical experience (technological pragmatist), lecturing and communicating about mathematics (old humanist); problem-solving, investigating and exploring (progressive educator); and a socially constructed process (public educator). Regardless of what philosophies teachers have, their beliefs about mathematics, teaching, and learning, significantly influence the ‘modelling’ of their pedagogies (Thompson, 1992). A teacher’s conception of the nature of mathematics refers to “teacher’s conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics” (Thompson, 1992, p. 132).

Research studies suggested the findings that student teaching had very little impact on the student teacher’s views about teaching (Tabachnick & Zeichner, 1984) and that student teachers' knowledge of teaching gained from earlier experiences was highly influential in their views on teaching and learning (Bramald, Hardman, & Leat, 1994; Calderhead & Robson, 1991; Carter & Doyle, 1995). Preservice teachers’ beliefs about teaching mathematics as well as their pedagogical content knowledge could influence their instructional decisions (Pajares, 1992). It is important to study the beliefs of preservice mathematics teachers because beliefs guide their instructional choices and decisions (Ernest, 1991; Skemp, 1976; Thompson, 1992). One important implication of these research studies is the need to recognize the power and importance of preservice teachers' preconceptions about teaching (Carter & Doyle, 1995). I agree that it is significant that teacher educators should help preservice teachers to conceptually shift from their personal view to a more professional view of teaching (Feiman-Nemser & Buchmann, 1986). Details about the study are presented in the methodology section.

Methodology

This research study addresses the following research questions: What are preservice teachers’ misconceptions about fraction operations? How do preservice teachers’ describe their problem-solving experiences and plans to teach problem-solving?

Participants

Thirty preservice elementary teachers from a large Midwestern university in the United States participated in the study. They took a general education mathematics course which emphasized conceptual understanding of fractions and problem solving involving fractions using the four basic operations: addition, subtraction, multiplication, and division. The course was designed for non-mathematics specialists (i.e., preservice elementary teachers and special education majors) and with a focus on the use of multiple ways of solving problems (e.g., using modeling approach, manipulatives, arithmetic, and algebra). Preservice teachers were expected to present multiple solutions in solving a problem and explain, justify and make sense of different ways of solving problems involving fractions.

Procedures

At the beginning of the semester (pre-intervention), preservice teachers' definitions of problem-solving were obtained using a brief questionnaire. For the entire semester, they were given instruction on problem-solving with fractions for each of the four basic operations. At the end of each operation, they were asked to complete a short assessment. Four questions, each representing the four basic operations, were chosen to be analyzed in this study. All questions were similar to the types of problem-solving task given in class and were collected as part of class assessment. Also, all questions were designed to assess preservice teachers' conceptual and procedural understanding of the mathematical concepts and skills involved in problem-solving with fractions. These problems included questions on comparing fractions as well as those using the four operations. Responses to these problems were analyzed in terms of what were preservice teachers' common misconceptions about fraction operations.

At the end of the semester, preservice elementary teachers were asked to write a reflection (guided by reflection prompts) of their learning experiences. Reflection prompts consisted of questions pertaining to their experiences as well as changes in their beliefs about mathematics and the teaching of mathematics. These written reflections were used to determine their perceived conceptions or beliefs about problem-solving after taking the course.

Instruments

During the semester, preservice teachers completed several problem-solving questions (as part of their in-class assessment). For this article, four questions were chosen, one for each of the four basic operations. Open-ended questions were used and collected through questionnaire to provide additional information on preservice teachers' mathematical understanding. Foong (2002) argued that the use of short, open-ended questions in the classroom enabled teachers to see students' thinking rather than the teacher's own thinking. Questions were designed to ask for explanations to generate preservice teachers' thinking (Szetela, 1993) and emphasize contexts and knowledge of mathematics and pedagogy (Joyner & Bright, 1998). Using these open-ended questions, the teacher teaching the course not only helped preservice teachers generate creative strategies but also enabled her to gain a better understanding of what they were able to do.

Preservice teachers were asked to complete written reflections on their (a) definition of problem-solving and about teaching problem-solving, (b) course learning experiences, and (c) beliefs about teaching mathematics. These reflections aimed to provide a wider range of their understanding by including assessment of attitudes and perceptions (Swan, 1993). Attitudes influence mathematics learning and beliefs about mathematics competence are positively related to students' achievement in mathematics (McLeod, 1992). Discussion on common misconceptions about fraction operations is presented in the following section.

Research Findings

Preservice Teachers' Misconceptions about Fraction Operations

In the discussion that follows, preservice teachers' common misconceptions about four basic operations involving fractions are presented. These misconceptions include concepts on applying the fraction addition rule in adding fractions with different units, interpreting fraction subtraction like subtraction of whole numbers, multiplying fractions out of the context of the word problem, and that dividing by ½ is like dividing by 2. Fractional concepts included in these assessment questions were: identification of the unit, partition of units into same-sized parts, representation of fractions as a relationship between two continuous or discrete quantities, making sense of the manipulations on fractions, equivalence, comparison, the four basic operations, and the recognition that there are different but related ways of thinking about a rational number written as fraction.

Applying the Fraction Addition Rule in Adding Fractions with Different Units

This question was designed not to ask preservice teachers to perform a procedural solution for the expression 3/4 + 5/8 but to critique a hypothetical student's response. It was interesting that the questions elicited issues pertaining to preservice teachers' understanding of adding fractional parts of different units and determining a pictorial representation of the sum. In particular, preservice teachers were asked to respond to the following question: *The two pictures below [see Figure 1] show a* *representation of 3/4 and a representation of 5/8, respectively. (All small squares in the pictures are assumed to be of the same size partitions.) When asked what fraction is represented by the shaded portion of the combined pictures, Masako added 3/4 and 5/8 to get 11/8 and explained that 11/8 represents the shaded portion of the combined pictures. Comment on the correctness of Masako's solution. Justify whether or not her work deserves full points*.

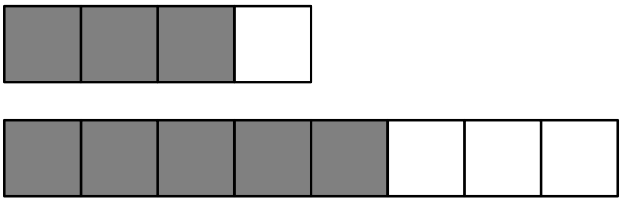


Figure 1. Two Pictorial Representations

Arnie drew and explained Figure 2. In her explanation, Arnie validated that Masako's solution was correct and explained that Masako converted the fractions into similar terms. In this case, changing 3/4 to 6/8. Arnie concluded that once the terms were alike (i.e., with the same denominators) the fractions could then be added.

Arnie failed to recognize the difference between adding fractional numbers 3/4 and 5/8 from the combined pictorial representations of shaded portions with adding 3/4 and 5/8. It made sense to her that the sum of the two fractions is 11/8, as validated by her application of the addition rule. It is emphasized that students having fraction sense should be able to estimate a reasonable answer to the fraction addition problem (Cramer & Henry, 2002). However, in the case of Arnie, she did not pay attention that these two fractions represented parts of different units (i.e., four squares and 8 squares). For her, adding 3/4 and 5/8 was the same as generating the fraction that represented the sum of the combined figures' shaded parts. In other words, the mathematical task was asking for the sum of 3/4 and 5/8.

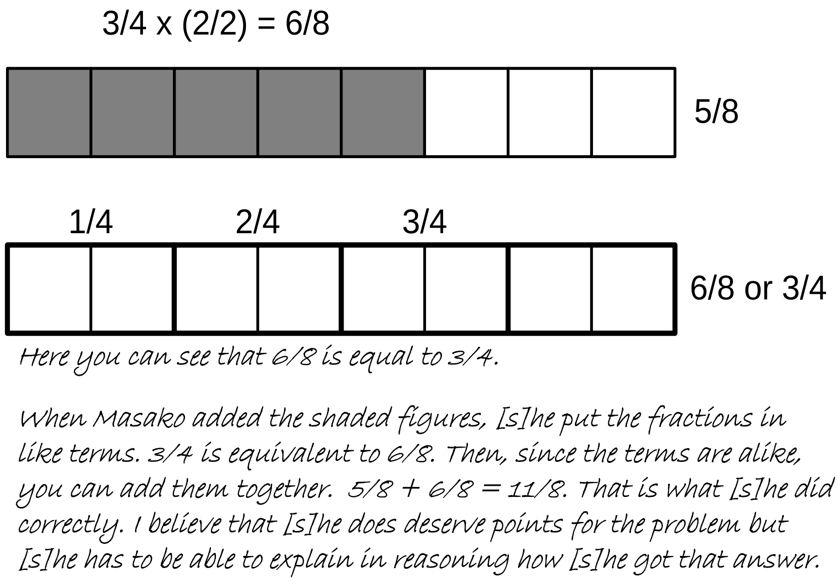


Figure 2. Arnie's Solution

Similarly, Helen explained that Masako's solution was correct by trying to reenact what Masako could have done to get the answer 11/8 (see Figure 3). She argued that Masako was right but failed to recognize that adding four more squares to the given picture changed the condition of the initial problem.

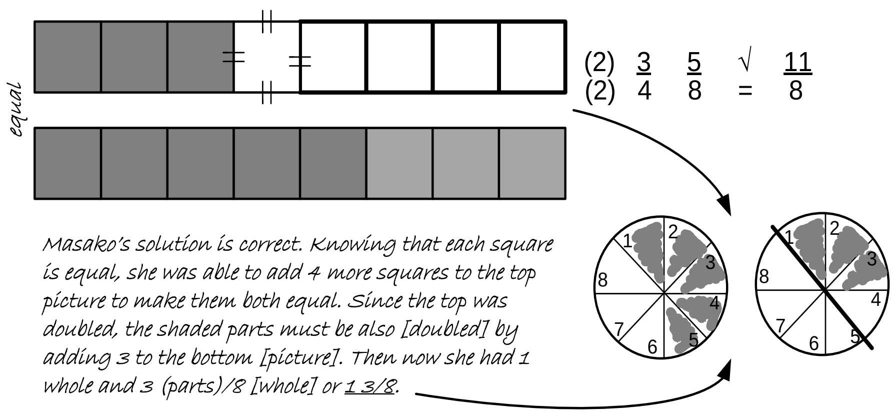


Figure 3. Helen's Solution

However, like Arnie, Helen did not simply add numerators and denominators, a common addition error pattern (15 out of 30) among preservice teachers (see Newton, 2008). Arnie and Helen both recognized that 3/4 needed to be of the same denominator as 5/8 before adding them. Their addition error was characterized by applying the addition algorithm for fractions even if the two fractions represented different units. Arnie and Helen generalized that the addition rule holds true even if the fractions represent different units.

Interpreting Fraction Subtraction like Subtraction of Whole Numbers
One problematic aspect of subtraction of fractions for preservice teachers is clearly shown in the following mathematical task.

Jean's Word Problem: *Daisy has 3/4 of a pie and Jonathan ate 2/3 of what Daisy has. What part of the pie was left?*

First, when asked to present an argument to justify whether or not the given word problem (Jean's word problem) truly asks for the answer to the expression 3/4 - 2/3, Emy wrote the following response (see Figure 4).

I feel that Jean's question [word problem] is correct. She is telling us that Daisy has 3/4 of a pie, which would represent the 3/4 in the problem. Then she says that Jonathan ate 2/3 of Daisy's 3/4. This means he took away 2/3 of Daisy's 3/4. This impl[ies] that Jean is asking you to subtract 2/3 from 3/4.

3/4 - 2/3 = ? ← will be how much is left

how much pie Daisy has   how much Jonathan ate of Daisy's total pie

Overall, Jean's worb problem representation for the [expression] 3/4 – 2/3 is correct.

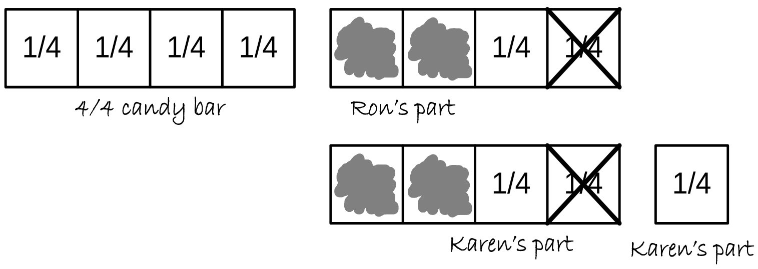
Figure 4. Emy's Justification

Emy misinterpreted when she stated that Jonathan ate 2/3 of Daisy's 3/4 of a pie. This means Jonathan took away 2/3 of Jean's 3/4 of a pie (i.e., $3/4 - 2/3(3/4)$). For Emy, this statement was the same as $3/4 - 2/3$. In this case, Emy translated the phrase "took away" as an operation of subtraction without considering whether the fractional parts given are from the same unit.

Subtraction operation usually means separating an amount into subgroups. Emy knew that "taking away" implies subtraction and so 2/3 should be subtracted from 3/4. In this case, Emy did not understand the effect of subtraction to a pair of fractions. She was correct in describing that "Jonathan took away 2/3 of Jean's 3/4 of a pie" but failed to represent it correctly in a form of mathematical expression. "Fundamental to operation sense is an understanding of the meanings and models of operations" (Huinker, 2002, p. 72). Clearly, Emy did not understand the meaning of fraction subtraction. When asked to write a word problem (see Figure 5) for the arithmetic expression 3/4 -1/2, Emy maintained that misconception and failed to recognize the *unit* or *referent* for the first fraction. She did not understand that the algorithm in subtraction of fractions is applicable only if the *unit* or *referent* is the same.

In Emy's written word problem for the arithmetic expression 3/4 -1/2, she was clear that Ron took "1/2 of the 3/4 of a candy bar". However, her pictorial representation (i.e., two shaded 1/4 parts in Figure 5) of that statement mathematically means 1/2 of a candy bar. Emy was not able to interpret correctly the change in the unit from the first fraction to the other. This misconception, like in fraction addition, was common (18 out of 30) among preservice teachers in the study. Also, problem posing was not an easy task for them, and their responses to the problem posing task implied that they do not have a clear understanding of the meaning of operations of fractions.

Karen and Ron bought a giant candy bar together. They broke the candy bar into four parts. They decided they were only going to eat 3/4 of the candy bar. Ron took 1/2 of the 3/4. What fractional part does Karen get?



Karen will get 1/4 of the candy bar. I drew four pieces of chocolate and crossed one piece out because they were not going to eat it. Ron eats 1/2 of the 3 pieces of the four whole. This leaves Karen with 1 piece of the whole 1/4.

Figure 5. Emy's Word Problem

Multiplying Fractions out of the Context of the Word Problem

Most preservice teachers had the tendency to perform an operation involving fractions without actually validating whether the problem asks for it. In this *Car Gas* problem, the majority of the preservice teachers (20 out of 30) multiplied the two given fractions without carefully considering the context of the problem. This suggested that they tended to perform procedural solutions even in a non-routine problem-solving context like the one given in this problem: *Noel has 3 ¼ gallons of gas left in the tank of his car. This is 2/5 of the amount of gas left in Debra's tank. How much gas does Debra have in that tank?*

Melie interpreted the first two sentences of the problem correctly (see Figure 6). However, she simply multiplied the two given fractions (i.e., 13/4 ● 2/5) without recognizing that it violated her first two written statements. Melie's solution was a representative of the common misinterpretation and incorrect solution of most preservice teachers who simply jumped into multiplying the two given fractions without realizing a conflict in its context. True indeed, Melie and the rest could perform multiplication of fractions but failed to show understanding of what was asked in a contextualized problem like this.

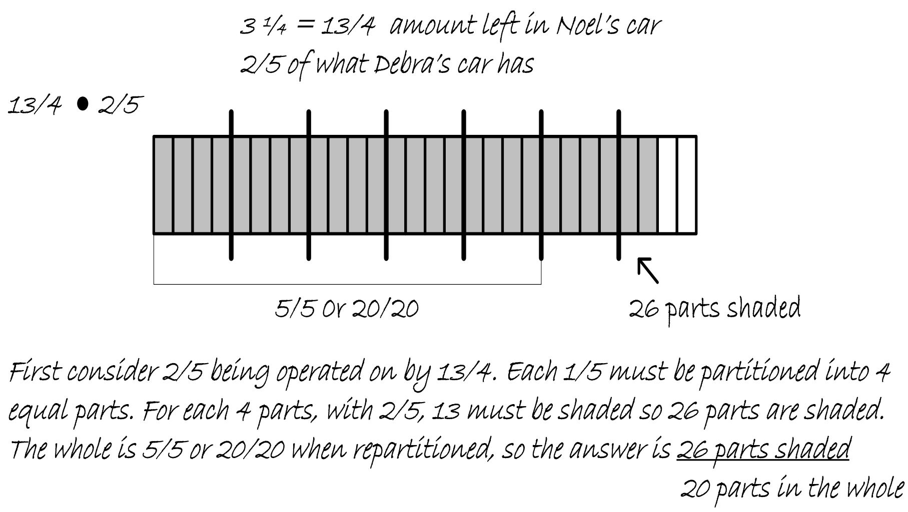


Figure 6. Melie's Solution

Melie's misconception was not unique to fraction multiplication alone. This thinking could be a product of traditional and procedural teaching of problem-solving in earlier grade levels. That is, focus on finding the two numbers in the problem and identify the keywords that represent the operations (i.e., addition – sum, total, subtraction – take away, multiplication – of, division - divide) then proceed with the algorithm. Melie, in this case, did not understand that the problem was of the form 3¼ = 2/5x and finding x means dividing 3¼ by 2/5. In her study, Newton (2008) posited that preservice teachers did not see that the division algorithm is contained in multiplication algorithm. This is the same case as in this current study. Preservice teachers, like Melie, viewed multiplication and division as unconnected.

Dividing by ½ is like Dividing by 2

When asked to solve the mathematical expression 1¾ ÷ ½ in two different ways, the majority of preservice teachers (25 out of 30) used the invert and multiply rule. As for the other solution, a pictorial solution (i.e., modeling approach) was a second choice. Even with a semester of lessons, most preservice teachers (20 out of 30) failed to apply the definition of division in coming up with distinct solutions for this expression. Also, when asked to write a word problem for the expression 1¾ ÷ ½, Emy's word problem was an example of a common misconception among preservice teachers (20 out of 30). Emy wrote:

> *Karen has 2 boxes of cookies. She asks Jason if he wants a box and he says no. So Karen eats 1/4 of a box of cookies. Then Jason decides he wants 1/2 of Karen's new whole of 1 ¾ of cookies. How much of the boxes of cookies does Jason get?*

Clearly, Emy interpreted that the phrase 1/2 of 1¾ means "a half of 1¾". In this case, it meant "1¾ divided by 2" which is clearly not "1¾ divided by 1/2". Most preservice teachers like her (20 out of 30) failed to use the appropriate mathematical term to refer to the divisor 1/2. This is consistent with Ball's (1990a) study where some preservice teachers perceived the expression 1¾ divided by 1/2 as division by *two* instead of division by *one half*. Also, on division by fractions, Ball found that most preservice teachers perceived the problems on division by fraction in terms of fractions and not as a division problem. They also had difficulty relating the fractional expression to real life situations.

This claim is supported by the results of the study I conducted (Nillas, 2003) on preservice teachers' understanding and use of strategies in solving different types of division of fractions problems. I found that preservice teachers were able to use different strategies: repeated subtraction, repeated addition, pictorial solution, multiplicative partitioning, finding parts of a whole, setting up an equation, direct division, and use of area formula. In most cases, preservice teachers in my previous study used these strategies as a result of modeling the given division of fractions problems. They used the context of the problems as a means to find solutions. Regardless of their ability to generate solutions in different ways, it was evident that preservice teachers still lack a strong conceptual understanding of division of fractions. It is imperative that they understand and comprehend what division of fractions means. With a semester of problem-solving involving fractions, preservice teachers in this current study reflected on their experiences as problem-solvers. Details are discussed in the following section.

Preservice Teachers' Reflections on their Problem-Solving Experiences

At the beginning of the semester, preservice teachers defined problem-solving as (a) looking at a problem in depth, (b) figuring out a solution, (c) solving mathematics problems and equations the best way, (d) applying a certain set of rules, (e) making sense of a problem, (f) answering a math problem, and (g) looking at problem and breaking it down to simpler parts. Most of these definitions changed accordingly (25 out of 30) throughout the semester with the inclusion of how they described a successful problem solver (i.e., someone who does not get the correct answer every time, one who thinks critically, and looks at a problem logically and doesn't necessarily know the answer right away). They described problem-solving as a process that involves: reasoning beyond the rule (30 out of 30), solving in distinct multiple ways (25 out of 30), and patience (20 out of 30).

Reasoning beyond the rule

Sally identified the importance of justifying solutions beyond the rule. She said:

> *So far this semester of math has taught me a lot of different learning/teaching techniques. I have also come to grips with not allowing the response "it's just a rule and that's why". This should never be an answer to someone's question.*

Arnie reiterated what Sally expressed. She wrote:

> *Math has been my strong point in school, but this class is different for me. In high school there was never a time where you needed to explain <u>why</u> something equals something. Equations were just given and "that's how it is".*

Both Sally and Arnie stressed learning the importance of being able to explain beyond the rule. They recognized the difference of the traditional conception of what mathematics is, which for most cases is rule-based. They learned and had seen the importance of explaining beyond the rules. They recognized the significance of explaining *why* as an indication of learning beyond the procedures. Tirosh (2000) found that even though most preservice teachers knew how to divide fractions they could not explain the procedure. She recommended using children's common errors to familiarize preservice teachers on children's ways of thinking.

Just like in traditional high school classrooms, rule-based methods are used mainly in solving problems. In Yea-ling's (2005) study, low ability students tended to use rule-based methods more often that the high ability students. They relied on standard written algorithms more than reflecting number-sense-based methods. To encourage the use of higher order thinking skills in problem solving, Crespo and Nicol (2006) recommended the use of mathematical tasks that situate inquiry in teaching practice. Preservice teachers need to experience inquiry in order to teach mathematics with understanding.

Solving using distinct multiple ways
Annie wrote:

> *In math there has usually always been one right answer. With this class, that's not the case. I've also learned how to explain my answers at the overhead in front of the whole class so that my point is made clear.* Lara wrote: *I think for me it's a lot of the initial understanding that I struggle with. I've found that I see things a little different than the rest of the class but in the end come out with the same thing.* Likewise, Jane remarked: *I also learned that may people have different ways of doing things. When we meet in small groups to discuss or answers it always amazes me how everyone can think of different representations to doing the same problem.*

All three preservice teachers experienced that there are multiple ways of solving a particular problem. Finding distinct ways of representations and solutions was difficult at first. However, having the opportunity to discover, share, and discuss distinct solutions helped them think beyond one way and be familiar with different perspectives. Bischoff and Golden (2003) found that working in groups was also found to be helpful in generating novel solutions to a problem. This was consistent with what Lara observed (i.e., multiple solutions were evidence of creative outcomes of the group who engaged in the process of negotiating meaning-making).

Blessie, another student, added that the process of solving problems required patience. She wrote:

> *Trained to memorize formulas and other ideas, it was difficult for me to come into and basically be told to disregard all the rules we*

> *had previously been taught and learned. Now I am finally starting to get accustomed to the restrictions that have been placed on us and I am beginning to see another side of math. I have learned that there is not only one solution to each problem and even if there was only one solution, there are several ways to arrive at that answer. Also, sometimes the final answer isn't as difficult to reach as we think it is. I have discovered that this course demands patience and if you don't have it, then it is more difficult to be successful in the end. At the start of the semester, I only relied on my work to get through the majority of the problems. After class presentations and just by talking with other girls in the class I've learned that it is helpful to see how other people come to a conclusion - it just gives you another insightful perspective to rely on.*

For preservice teachers, their learning experiences made them understand that problem-solving is a process that requires patience and time, solving multiple ways, and reasoning beyond the rule. Blessie recognized these elements as essential in being a successful problem solver. After their experiences as problem-solvers, I also asked them to describe how they plan to teach mathematics in their future classes. Details are discussed in the next section.

Preservice Teachers' Plans on How to Teach Problem-Solving
The question of whether preservice teachers' beliefs changed after experiencing a constructivist-based (i.e., characterized by building on prior knowledge, discussing multiple ways of solving a problem, using multiple representations, and explaining and communicating mathematical reasoning) problem-solving class was one of the objectives of this study. The question is how then these preservice teachers plan to teach problem-solving.

Using multiple methods (25 out of 30)
Melie said:

> *Teach each basic step one by one until they [students] have an understanding of each problem. Problem-solving is like building blocks. Without the base you can make no tower. Teach them [students] multiple methods of problem-solving because not all students look at problems the same way and therefore have a different way of learning.* Lisa described her future plans in teaching: *In the future I will teach problem-solving an entirely different way th[a]n I was taught in grade school and high school. I would rather have my students know how to read the problem,*

> *solve the problem, and then explain the problem. If a student cannot perform all those steps he or she then does not fully understand the concept of the problem. I will teach my students how to use pattern blocks, circles, pictures, and algebraic solving methods. Instead of focusing on only using one way to solve a problem I will show them different ways with the pictures and algebra. By showing them different ways to solve a problem they may be able to find what works best for them. Hopefully this will help them to understand and like math instead of dreading the next problem or even coming to class.*

The use of alternative methods and multiple solutions beyond the traditional algorithm has proven to enrich preservice teachers' ability to pose problems (Rizvi, 2004). Mathematics teachers need to provide opportunities for their students to learn and experience solving problems using multiple methods and tools. Melie's plans for how she would teach problem-solving resonated with the instructional and pedagogical goals of the problem-solving course she took. Similar to Swars, Hart, Smith, Smith, and Tolar's (2007) study, preservice teachers became cognitively aligned with their curriculum program. They believed that skills should be taught through problem-solving and with understanding. They found that curriculum programs which emphasized reform –based ways of teaching for understanding developed stronger beliefs in preservice teachers' mathematics skills and abilities to teach effectively.

Emphasizing WHY (25 out of 30)

Lara expressed her changed beliefs about teaching a mathematics course. She wrote:

> *My views of teaching young children mathematics have changed considerably over the past semester. I had not thought about the things that I took for granted that I knew. There are a lot of basic concepts that I have to understand in order to explain things in terms that little children will understand. I can't say that something happens because it is a rule and that's just how it is. I have to understand why things happen so that I can teach them and they will then be able to apply their knowledge in a better way.*

Judie, on the other hand, expressed her realizations of the importance of her understanding to be able to teach competently. Judie wrote:

> *I think that my understanding has expanded whether my grades have shown it or not. I think that this is more important when teaching a class. I have to be having the bigger understanding of why things work, th[a]n the students because I have to be able to*

> *answer their questions that they have without just saying "because that's just the rule!"*

Lara and Judie's reflections were similar to Ms. Moseley's (a middle school teacher) reflections as she learned to connect her understanding of her students' work to her teaching rather than limiting her discussion to her students and their understanding (Kwon & Orrill, 2007). It is apparent from these reflections the importance of teaching with understanding. Judie called it "bigger understanding" and Lara described it as understanding "why things happen." For these two preservice teachers and all others who learned the value of explaining why, teaching would be more conceptual.

Although most preservice teachers expressed change in their views about teaching problem-solving and mathematics in general, a few of them, like Jennifer, were still adamantly opposed to the different style in which the problem-solving course was taught. Jennifer expressed a strong view about the nature of mathematics. She wrote:

> *Math in my mind is still something that is entirely rule-based and must be taught in this manner.*

Unlike Jennifer, Jane's views changed even though she was unsuccessful and struggled in completing the course. She wrote:

> *I guess I had to think beyond the way that I was thinking, and so I ended up not getting any points for my work. I was unsuccessful with these problems because I did not understand the correct approach needed to solve them. The thing that was lacking in my approach to solve these problems was the ability to see the big picture. I saw everything is such a narrow perspective that it was too hard to solve the problem.*

Preservice teachers like Jennifer and Jane's experiences were examples of those conflicted by constructivist way of teaching problem-solving. Their previous, more comfortable learning experiences made it a struggle to be more accepting and open-minded about new ways of teaching. In a similar study, Raymond and Santos (1995) found how a reform-based course challenged preservice teachers to "reassess their relationships with mathematics in terms of their disposition toward, their confidence in, and their views of the usefulness of mathematics" (p. 67). Challenging preservice teachers' strongly-held beliefs about mathematics and how to teach it can be problematic yet rewarding in pursuit of helping them learn how to teach with understanding.

Conclusion

This article discusses common misconceptions of preservice teachers in solving problems involving addition, subtraction, multiplication, and division of fractions. Misconceptions identified are: applying the fraction addition rule in adding fractions with different units, interpreting fraction subtraction like subtraction of whole numbers, multiplying fractions out of the context of the word problem, and dividing by ½ is like dividing by 2. Knowing these misconceptions is important in understanding preservice teachers' notions about the meaning of operations of fractions, ability to problem-pose, and limitations of the algorithm of operations.

Preservice teachers' reflections about their learning experiences in solving problems described their renewed understanding and changing beliefs about teaching and learning problem-solving and mathematics in general. They described problem-solving as a process that requires reasoning beyond the rule and solving using distinct multiple ways as well as time and patience.
It is hoped that through their experiences they will be able to facilitate a mathematics class that promotes use of higher level thinking, reasoning and communicating ideas, and use of multiple representations. Despite the fact that preservice teachers' beliefs about teaching problem-solving have changed, it was evident that they struggled in transferring what they have learned to new situations (as shown by the misconceptions about fraction operations discussed in this article). A longer period of instructional time is needed to achieve better retention. This element of the study should be considered for future research investigation.

Most preservice teachers' views about teaching problem-solving changed from rule-based to a constructivist-based teaching. They understand the value of using multiple methods and explaining why the procedures or solutions work. Although there were some students who remained traditionalist in thinking about the nature and teaching of mathematics, others who struggled understood that changing their ways of thinking could change their ability to see the big picture when solving problems. Although findings on preservice teachers' beliefs are interesting, the narrative is less compelling since there is no evidence that their teaching reflects their changed beliefs. Whether these preservice teachers actually implement what they have learned in their own classrooms is a good topic for follow-up research.

Findings of this study support initiatives promoting teaching for understanding and changing preservice teachers' beliefs about teaching and learning mathematics.

References

Ball, D. L. (1990a). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*, 132-144.

Ball, D. L. (1990b). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90, 4,* 449-466.

Ball, D. L. (1991). Research on teaching mathematics: Making subject-matter knowledge part of the equation. *Advances in Research on Teaching, 2,* 1-48. JAI Press.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59, 5,* 389-407.

Bischoff, P. J., & Golden, C. F. (2003). Exploring the role of individual and socially constructed knowledge mobilization tasks in revealing preservice elementary teachers' understandings of a triangle fraction task. *School Science and Mathematics, 103*(6), 266-273.

Bramald, R., Hardman, F., & Leat, D. (1994). Initial teacher trainees and their views of teaching and learning. *Teaching & Teacher Education, 11,* 23-31.

Calderhead, J., & Robson, M. (1991). Images of teaching: student teachers' early conceptions of classroom practice. *Teaching and Teacher Education, 7,* 1-8.

Carpenter, T. P., & Lehrer, R. (1999). Teaching and learning mathematics with understanding. In E. Fennema, & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 19-32). Mahwah, NJ: Lawrence Erlbaum Associates.

Carter, K., & Doyle, W. (1995). Preconceptions in Learning to Teach. *The Educational Forum, 59,* 186-195.

Collier, P. C. (1972). Prospective elementary teachers' intensity and ambivalence of beliefs about mathematics and mathematics instruction. *Journal for Research in Mathematics Education.*

Conney, T. J., Shealy, B. E., & Arvold, B. (1998). Conceptualizing beliefs structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education, 29,* 306-333.

Cramer, K., Henry, A. (2002). Using manipulative models to build number sense for addition of fractions. In B. Litwiller & G. Bright (Eds.). *Making sense of fractions, ratios, and proportions: NCTM Yearbook* (pp. 41-48). Reston, VA: National Council of Teachers of Mathematics.

Crespo, S., & Nicol, C. (2006). Challenging preservice teachers' mathematical understanding: The case of division by zero. *School Science and Mathematics, 106*(2), 84-97.

Ernest, P. (1991) *The philosophy of mathematics education*. Basingstoke, UK: Falmer Press.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education, 24,* 94-116.

Feiman-Nemser, S., & Buchamann, M. (1986). The first year of teacher preparation: Transition to pedagogical thinking. *Journal of Curriculum Studies*, *18*, 239-256.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: Macmillan.

Foong, P. Y. (2002). The role of problems to enhance pedagogical practices in the Singapore mathematics classroom. *The Mathematics Educator, 6*(2), 15-31.

Gregg, J. (1995). The tensions and contradictions of the school mathematics tradition. *Journal for Research in Mathematics Education, 26*.

Grouws, D. A., & Smith, M. S. (2000). NAEP findings on the preparation and practices of mathematics teachers. In E. A. Silver and P. A. Kenney (Eds.), *Results from the Seventh Mathematics Assessment of the National Assessment for Educational Progress* (pp. 107-139). Reston, VA: National Council of Teachers of Mathematics.

Hill, H. C., Schilling, S. G., Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for learning. *The Elementary School Journal, 105*(1), 11-30.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.

Huinker, D. (2002). Examining dimensions of fraction operation sense. In B. Litwiller, & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions: NCTM Yearbook* (pp. 72-78). Reston, VA: National Council of Teachers of Mathematics.

Joyner, J. M., & Bright, G. W. (1998). Recommendations and starting points. In G. W. Bright & J. M. Joyner (Eds.), *Classroom Assessment in Mathematics: Views from a National Science Foundation Working Conference* (pp. 59-77). New York: University Press of America.

Kwon, N., & Orrill, C. H. (2007). Understanding a teacher's reflections: A case study of a middle school mathematics teacher. *School Science and Mathematics, 107*(6), 246-257.

Langrall, C. W., Thornton, C. A., Jones, G. A., & Malone, J. A. (1996). Enhanced pedagogical knowledge and reflective analysis in elementary mathematics teacher education. *Journal of Teacher Education, 47*(4), 271- 281.

Leinhardt, G., Putnam, R. T., Stein, M. K. & Baxter, J. (1991). Where subject knowledge matters. *Advances in Research on Teaching, 2,* 87-113.

Ma, L. (1999). *Knowing and teaching elementary mathematics : teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ.: Lawrence Erlbaum Associates.

McLeod, D. B. (1992). Research on affect in mathematics education: Reconceptualization. In D. A. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 575-596). New York: Macmillan.

Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., Gregory, K. D., Garden, R. A. O'Connor, K. M., Chrostowski, S. J., & Smith, T. A. (1999). TIMSS 1999 International Mathematics report: Findings from IEA's Repeat of the Third International Mathematics and Science Study at the eighth grade. Retrieved Dec. 3, 2002 from http://isc.bc.edu/timss1999i/math_achievement_report.html

Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal, 45*(4), 1080-1110.

Nillas, L. A. (2003). Division of fractions: Preservice teachers' understanding and use of problem solving strategies. *The Mathematics Educator*, *7*(2), 96-113.

Pajares, F. (1992). Teachers' beliefs and educational research. Cleaning up a messy construct. *Review of Educational Research*, *62*, 307-332.

Raymond, A. M., & Santos, V. (1995). Preservice elementary teachers and self-reflection: How innovation in mathematics teacher preparation challenges mathematics beliefs. *Journal of Teacher Education, 46*(1), 58-70.

Rizvi, N. F. (2004) Prospective teachers' ability to pose word problems, *International Journal for Mathematics Teaching and Learning*, *12*, 1- 21.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi, *Journal of Mathematics Teacher Education, 8* (3), 255–281.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

Swan, M. (1993). Assessing wider range of student's abilities. In N. L. Webb & A. F. Coxford, *Assessment in the Mathematics Classroom* (pp. 26-39). Reston, VA: National Council of Teachers of Mathematics.

Swars, S., Hart, L. C., Smith, S. Z., Smith, M. E., & Tolar, T. (2007). A longitudinal study of elementary pre-service teachers' mathematics beliefs and content knowledge. *School Science and Mathematics*, *107*(9), 325-335.

Szetela, W. (1993). Facilitating communication for assessing critical thinking in problem-solving. In N. L. Webb, & A. F. Coxford, *Assessment in the Mathematics Classroom* (pp. 143-151). Reston, VA: National Council of Teachers of Mathematics.

Tabachnick, B. R., & Zeichner, K. M. (1984). The impact of the student teaching experience on the development of teacher perspectives. *Journal of Teacher Education, 11*(1), 23-31.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 127-146). New York: Macmillan.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5-25.

Wearne, D., & Kouba. V. L. (2000). Rational Numbers. In E. A. Silver and P. A. Kenney (Eds.), *Results from the Seventh Mathematics Assessment of the National Assessment for Educational Progress* (pp. 163-191). Reston, VA: National Council of Teachers of Mathematics.

Yea-ling, T. (2005). The number sense of preservice elementary school teachers. *College Student Journal, 39*(4), 647-679.

Zhou, Z., Peverly, S. T., & Xin, T. (2006). Knowing and teaching fractions: A cross-cultural study of American and Chinese mathematics teachers. *Contemporary Educational Psychology, 31*, 438-457.

Author:

Leah A. Nillas, Center for Liberal Arts 114, Illinois Wesleyan University Bloomington, IL 61701, USA; lnillas@iwu.edu