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Analyzing Errors Made by Eighth-Grade Students in Solving Geometrical Problems in China

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Abstract: In mathematical problem solving, students may make various errors. In order to draw useful lessons from the errors, and then correct them, we surveyed 24 eighth-grade students' performances in geometrical problem solving according to Casey's hierarchy of errors. The paper also discussed that (1) students' motivations and beliefs can lead to errors at the stage of comprehension, strategy selection, and skills manipulation; and (2) students' geometric schemas also influenced their strategy selection.

Key words: Geometrical problem solving; Motivations; Beliefs; Schemas

Background

Many researchers have expressed interest in analyzing students' errors from different perspectives (e. g., Cox, 1975; Dai, 1996; De Bock, Van Dooren, Janssens, & Verschaffel, 2002; Fiori & Zuccheri, 2005; Knifong & Holtan, 1976; Lannin, Barker, & Townsend, 2007; Newman, 1977). After literature review, we found many studies have used Newman procedure (Newman, 1977) for analyzing students' errors in problem solving since 1977 (Casey, 1978; Clarkson, 1991; Clements, 1980, 1982; Pan & Wu, 2008; Watson, 1980). It seems that this analysis procedure attracted considerable attention from mathematics education researchers.

Newman (1977) claimed that if a student wished to obtain a correct solution to a one-step word problem, he or she should ultimately proceed according to the following hierarchy: (1) read the problem (reading); (2) comprehend what is read (comprehension); (3) carry out a mental transformation from the words of the question to the selection of an appropriate mathematical strategy (transformation); (4) apply the process skills demanded by the selected strategy (process skills); and (5) encode the answer in an acceptable written form (encoding). Casey (1978), by modifying and extending Newman's hierarchy of errors, produced a more general hierarchy which could be applied to the analysis of errors made on multiple-step word problems in mathematics: (1) question form，which is the first point of interaction between the written task and the person attempting it; (2) question reading; (3) question comprehension; (4) strategy selection; (5) skills selection; and (6) skills manipulation. In his hierarchy, Newman’s transformation category was redefined in terms of strategy selection and skills selection. This was necessitated by the fact that he was concerned with multiple-step problems but Newman was concerned with one-step problems only. Newman and Casey claimed that carelessness and motivation were the two error causes which could lead to various errors at any stage of the problem solving process (Clements, 1980).

In the above-mentioned papers, the analysis technique founded on the Newman procedure was limited to probing students’ errors in arithmetical or algebraic word problems. There are very few empirical studies on this issue in Chinese context in literature (Gao & Xue, 2009). In this study, we used the analysis technique based on Casey’s hierarchy to explore secondary students’ errors in solving multiple-step geometrical problems.

Methodology

Participants

The participants were 24 eight-grade students from a middle-class suburban junior school. The school is located in Nantong, a city near Shanghai, China. It has seven Grade 8 classes. The classes were ranked on the basis of their students’ performance (the total score of Chinese, mathematics, and English tests) in the final exam in the previous academic year. The 24 students were from the first author’s two classes (at that time, the first author was a mathematics teacher in this school.), which were the mid-ranked classes in this grade. They were selected to participate in the study. All the students had high geometric achievements in the midterm exam. This study was conducted in November 2002 after the midterm exam.

Materials

The mathematics subject at this school included two parts: algebra course and geometry course. Grade 8 students at this school learned algebra course during the previous Autumn term (the first term, from Sep. 2001 to Jan. 2002). Until the following Spring term (the second term, from Mar. 2002 to Jun. 2002), their mathematics subject didn’t include geometry course. The geometric series textbook used by students was *Junior Geometry*, which was published by the People’s Education Press in 2002. Before the midterm of Autumn 2002, Grade 8 students had learned some geometric topics, such as parallel lines, construction with straightedge and compass, congruent triangles, and isosceles triangle.

The purpose of the study was to investigate students' errors in solving geometric problems. In order to reduce some influence on students' errors (e.g. teacher's instruction (Frank, etc., 2009), and student' practice (Newell & Rosenbloom, 1981)), we would rather choose a geometric topic which hadn't been learned by these students, but also could be understood based on their prior knowledge. So that it will be propitious to help us explore students' learning difficulty. Compared with the series textbook *Junior Mathematics,* published by Shanghai Education Press in 2002, we found that a geometric topic on the theorem - "the median to the hypotenuse of a right-angled triangle equals to the half of the hypotenuse" (MHRT theorem) followed close behind "isosceles triangle" in *Junior Mathematics*. According to *Junior Mathematics*, the MHRT theorem would be taught in the autumn term of Grade 8 (the first term). However, in *Junior Geometry,* this theorem would be taught as a corollary of a property of rectangle in the spring term of Grade 8 (the second term). It was assumed that the high-achieving participants had necessary knowledge and sufficient competence to understand the MHRT theorem, because in *Junior Geometry* they had learned contents of midpoint connector of triangle, congruent triangles, and isosceles triangle, which were used to prove the MHRT theorem in *Junior Mathematics*. Hence, it is reasonable that we selected the geometric topic on the MHRT theorem to investigate students' errors made in solving problems on this particular topic.

Instrument

In fact, students usually don't immediately know what to do when a novel problem is encountered for the first time. Therefore, they might attempt to recall a similar problem encountered in the past and try to use that to solve the current one. This means that students' problem solving is based on their past experience in a problem solving episode (Robertson, 2001). Anderson (1993) argued that all skill learning occurs through analogical problem solving, in which examples have an important rule. He claimed that even if we have only instructions rather than a specific example to hand, then we interpret those instructions by means of an imagined example and attempt to solve the current problem by interpreting this example. Form this perspective, we designed the instrument to help students learn the MHRT theorem and apply it in variant contexts. To be specific, the instrument includes the theorem and its proof, an example, three exercises (close variant), and three test questions (distant variant). It is the example in the instrument that realizes how to apply the theorem in a novel context, because even if the theorem is well remembered students may not be able to apply it to the close variant problems. Furthermore, the exercises may consolidate what students had learned and improve their competence in solving problems related to this specific topic. Finally, we employed the distant variant problems (test questions) to evaluate what students had learned and investigate which difficulties they faced.

Procedure

At the first stage, the purpose of the study was to assess the participants' understanding and application of the MHRT theorem. On the first day, the twenty-four students were interviewed individually. They were asked to read and explain the MHRT theorem, its proof, and an example (see appendix 1). One researcher (the second author) interviewed with the students individually, the other (the first author) wrote field notes during the interviews. If a student didn't need help, the interviewer asked the student to explain what he or she had read. When a student needed help, the interviewer offered help. Instead of directly providing answer, the interviewer explained how the student could find the answer. If one explanation didn't help, the interviewer should try another, and ask the student to repeat his explanation in order to find out if he or she really understood. Students were required to do exercises individually after they were interviewed (see appendix 1).

At the second stage, we hoped to find out error causes of students when they solved geometric problems. The twenty-four students were examined within 45 minutes on the second day. They were required to finish three test questions (see appendix 2). Each student was interviewed (semi-structured interview) individually based on their performance when the examination was finished. All the interviews were audio recorded at the second stage.

This study was conducted in Chinese. The instruments in the appendixes and the interviews were translated into English accordingly.

Data Collection

Students' written work of the test questions were collected. The record of the semi-construction interviews was translated into scripts as important references for encoding. Each of the authors independently used Casey's hierarchy to encode students' errors occurred in the solution to the test questions. The inter-rater reliability for each error of coding was 80% or more. Discrepancies were resolved through discussion.

Results

In this study, students' errors in solving geometrical problems can be classified into four categories of Casey's hierarchy (see Table 1).

Table 1
The Students' Errors in the Test Questions

Hierarchy of errors	TQ1	TQ2	TQ3
Question form	-	-	-
Question reading	-	-	-
Question comprehension	2	-	4
Strategy selection	-	7	-
Skills selection	4	-	2
Skills manipulation	3	-	-

Question Comprehension (includes understanding geometrical figures)
In general, if a student wants to prove a mathematical statement, he has to identify the hypothesis and the conclusion of the given statement. When proving a geometrical statement in word sentence, students are usually first asked to draw the corresponding geometrical diagrams and translate the sentence into the standard format[1] .

When Qian confronted TQ 3, he translated the statement into the standard format in which the statement is represented in mathematical symbols: Given that $\angle$ACB=90°, $\angle$A=30°, and D is the midpoint of AB, prove CD=1/2AB.

Episode 1: Interview with Qian

1. Interviewer: Can you read the question?

2. Qian: Yes. In a right-angled triangle, if an acute angle is thirty degrees, then the leg to the acute angle equals to the half of the hypotenuse.

3. Interviewer: Great! What do you want to prove?

[1] In a geometrical statement, the hypothesis is written as "given", and the conclusion is written as "to prove". The mathematical relations (spatial, numerical), and the geometrical elements in the statement are represented by mathematical symbols.

4. Qian:	to prove… the median to the hypotenuse, yes, equals to the half of the hypotenuse.
5. Interviewer:	Can you read it again?
6. *Qian:*	Oh, wrong! … I think this question is similar as the theorem which I learned yesterday.

Although Qian read the question correctly (utterances 2), he intended to prove “the median to the hypotenuse equals to the half of the hypotenuse” (utterances 4). We think that the error was not caused by carelessness, but by his immoderate motivation - Qian considered intensively the theorem which was introduced by researchers on last day (utterances 6). His intention prevented him from receiving all of the information in the statement.

When Sun confronted TQ 3, the translation errors occurred in her work (see Figure 1). Her translation as following: Given that ∠C=90°, D is the midpoint of the hypotenuse, an acute angle is 30°, prove the median to the hypotenuse equals to the half of the hypotenuse. Sun replaced “the leg to the acute angle” by “the median to the hypotenuse” in her translation. In the interview with Sun, she explained: “because the sentence is too long, it is difficult to identify the hypothesis… I don’t know what to prove.” In addition, Sun remained most word representations in her written work, failed to translate them into mathematics symbols. Although Sun provided a different explanation, it is not so convincing that the cause of her error is different from Qian’s.

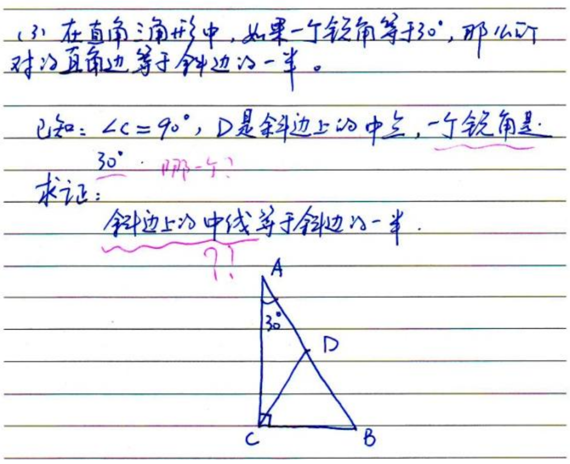

Figure 1. Sun’s Transformation on TQ 3.

In the process of solving TQ 1 (see Figure 2), Zhao directly employed the hypothesis "BM =BD". This hypothesis was not provided in the given statement, also hadn't been justified before she used it. In fact, "BM=BD" was unnecessary for solving TQ 1. In our interview with Zhao, She explained: "when I set N as midpoint of side AC, I want to prove two triangles MBD and NCD congruent. At that time, I paid more attention to identifying the two triangles congruent, and also the two segments BM and BD seemed equivalent, so …" From her explanation, it seems that the visual geometrical figure influenced Zhao's solving process.

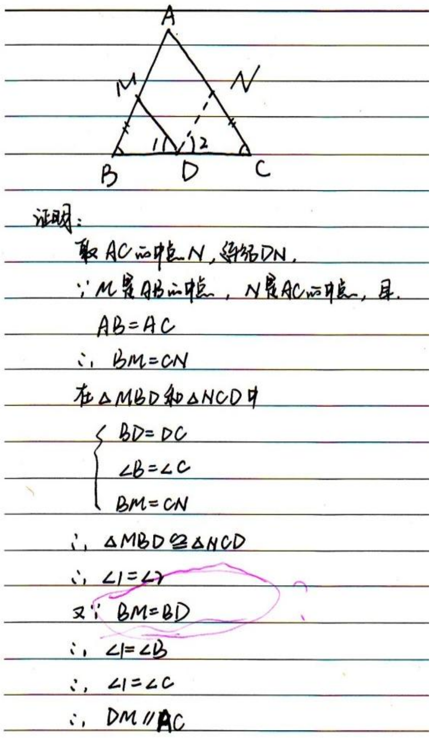

Figure 2. Zhao's Solution to TQ 1.

Strategy Selection

When students confronted TQ 2, Three students of them connect point M and N, then try to prove that triangle PNM is an isosceles triangle where ∠NMP=∠MNP.

For example, in episode 2, Li failed to prove ∠NMP=∠MNP in triangle PNM. In fact, it is impossible to selecting this strategy to prove MP=NP in TQ 2.

Episode 2: Interview with Li

1. Interviewer:	Can you explain your solution?
2. *Li:*	I connect point M and N... If triangle PNM is an isosceles triangle, then PM=PN. So I want to prove ∠NMP=∠MNP… I just know ∠BAC=120°, ∠BAD=∠CEA=60° (see Figure 3). How to prove ∠NMP=∠MNP... I don't know…

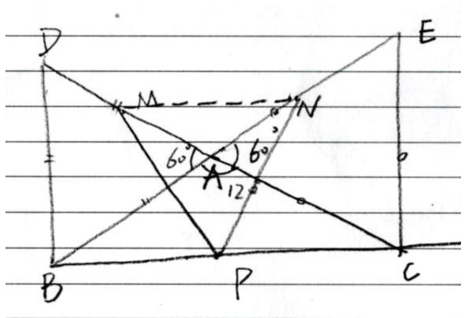

Figure 3. The Drawing in Li's Solution to TQ 2.

Four students employed another impossible strategy. They wanted to prove that triangle MPB and triangle NPC are congruent. Zhou constructed segment BM and NC, and had confidence to prove the two triangles congruent (utterance 1, utterance 4). It was unfortunate that triangle MPB and triangle NPC are incongruent when side AB is unequal to side AC in triangle ABC.

Episode 3: Interview with Zhou

1. Interviewer:	Can you tell me what are you thinking?
2. Zhou:	Ok. From my drawing (see Figure 4), you see… triangle BMP and triangle NCP seem congruent…
3. *Interviewer*:	Can you prove your argument?
4. *Zhou:*	You know, I tried again and again, but I think so… I know BP=CP, BM is perpendicular to AD, and CN is also perpendicular to AE (see Figure 4)… Let me try again!

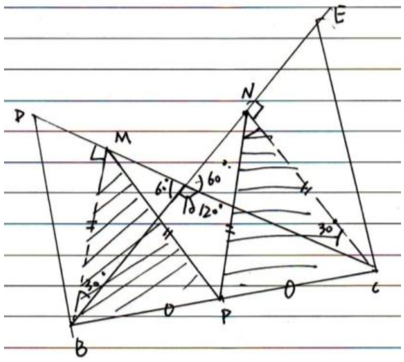

Figure 4. The Drawing in Zhou's Solution to TQ 2.

Wu constructed a segment between B and M firstly, then tried to prove triangle BMC is a righted-angle triangle where ∠BMC=90° (see Figure 5). But he gave up this strategy at last. Wu told us: "if I can justify ∠BMC=90°, then MP equals to the half of the BC. But how to prove ∠BMC=90°, I have not any good idea. … Now, I try to prove that triangle MNP is an isosceles triangle." In fact, Wu turned to another strategy because he failed to use priority knowledge to prove that BM is perpendicular to side AD in equilateral triangle ABD.

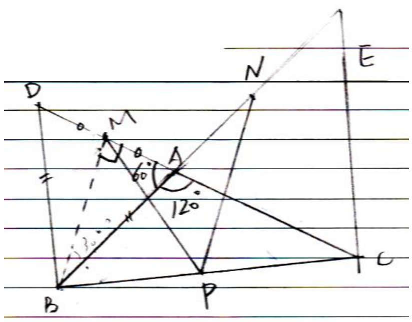

*Figure 5.*The Drawing in Wang's Solution to TQ 2.

Skills Selection

At the secondary level, the mathematics syllabus required that students should master the skills of five basic constructions (Ministry of Education, China, 2000). In the textbook *Junior Geometry1*, the construction "to construct a segment equal to a given segment" was introduced firstly. The unit of "construction with compass and straightedge” in textbook *Junior Geometry2*, introduced the other four basic constructions: (1) to construct an angle equal to a given angle; (2) to construct a bisector of a given angle; (3) from a given point, to construct a perpendicular to a given line; (4) to construct a perpendicular bisector of a given segment. Students can apply the five basic constructions to construct complex figures.

However, students may sometimes confuse proofs with constructions. They usually employ improper constructions in place of the necessary reasoning. For instance, Zhen wrote “construct $AD \perp BC$” (see Figure 6), then he completed his proof based on the perpendicular relation of AD and BC. But he failed to distinguish connecting a segment between two given points from constructing a perpendicular from a given point. Actually, it is impossible to construct a perpendicular to a given line from two different given points except that some particular conditions are confined. Wang’s solution to TQ 3 is an extremely example of the confusion. Wang wrote “construct AD=BC=BD” (D is the midpoint of the hypotenuse BC in righted-triangle ABC). In the interview, Zhen explained: “If you connect point A and D, you see…AD is perpendicular to BC. Yes, it’s correct! ” It seems that the visual feature of the figure cause - negative effect to his geometrical reasoning.

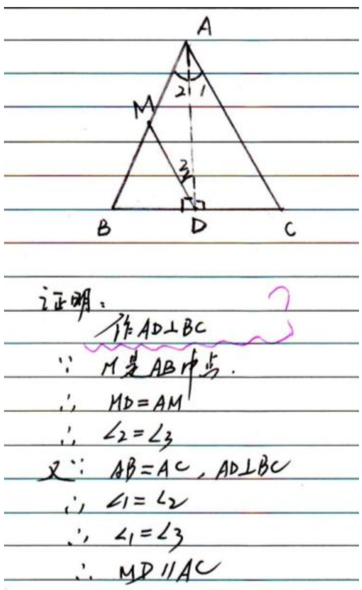

Figure 6. Zhen’s Solution to TQ 1.

Skills Manipulation

When Feng solved TQ 1, she justified triangle BMD and triangle CND congruent, then produced an outcome of "∠1=∠C" (see Figure 7). In episode 4, we find that Feng confused the corresponding angles in the two congruent triangles (utterance 2). According to her solution, ∠1 in triangle BMD is corresponding to ∠NDC in triangle CND. In fact, "∠1=∠C" is correct, but can't be deduced directly from triangle BMD and triangle CND congruent. Eight students had the errors similar to Feng's.

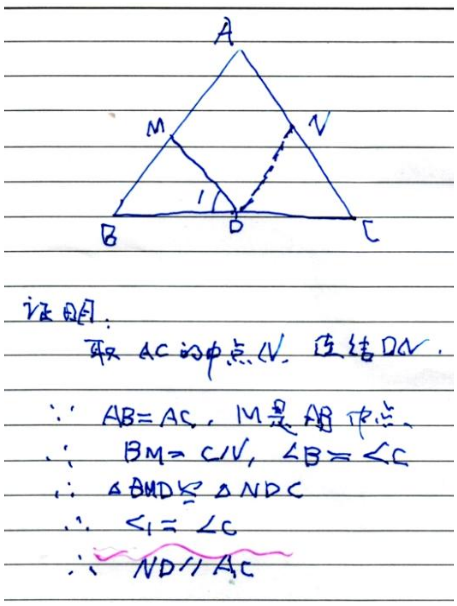

Figure 7. Feng's Solution to TQ 1.

Episode 4: Interview with Feng

1. *Interviewer*:	Can you tell me why ∠1=∠C *?*
2. *Feng:*	Ok. I have proved triangle BMD and triangle CND congruent (see Figure 7). Because the two triangles are congruent, each pair corresponding angles are equal. So, ∠1=∠C.
3. *Interviewer*:	Do you think ∠1 is corresponding to ∠C?
4. *Feng:*	Yes. You see…they look like.

In Chen’s solution to TQ 1 (see figure 8), he forgot to connect point D and A, and directly applied the MHRT theorem in the question situation. In the interview, Chen explained: “I tried again and again. I had a good idea when just three minutes left. I worried no time, so I wrote down the solution too quickly…” Chen seems too tense when he wrote down the solution to TQ 1.

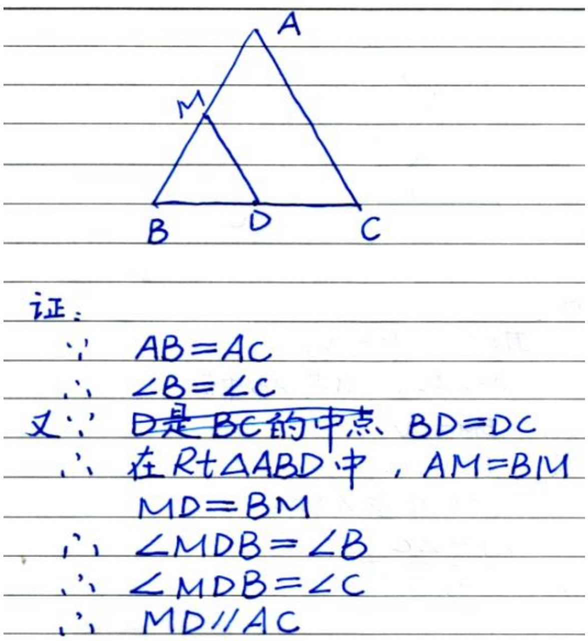

Figure 8. Chen’s Solution to TQ 1.

Discussion

In this study, we used the analysis technique based on Casey’s hierarchy to explore secondary students’ errors in solving many-step geometrical problems. Although, the technique was developed on probing students’ errors in arithmetical or algebraic word problems, it led to an interesting findings when applied to geometrical problems. On the one hand, any errors at the first two stages (question form, and question reading) did not occur in solving the many-step geometrical problems. On the other hand, we could use the hierarchy involving the last four categories to analyze students' errors in solving geometrical problems.

Many studies found that non-cognitive factors such as motivations and beliefs critically influence solvers' higher-order thinking (Buchanan, 1987; Cobb, 1985; Lesh & zawojewski, 2007). To be specific, Motivation was believed to maintain students' information process from their environment in terms of salient goals or values (Ames & Ames, 1984). According to the Yerkes-Dodson law (Yerkes & Dodson, 1908), the relationship between motivational level and behavioral efficiency is an inverted U function (Bregman & Mcallister, 1982). It means that the top performance is achieved at some intermediate level of motivations, but low and extreme motivations negatively will influence solvers' performance (Berlyne, 1966). This study supported the argument that extreme motivations have negative influence in solvers' cognitive behaviors. For example, when Qian intensively attempted to apply the theorem (the median to the hypotenuse of a right-angled triangle equals to the half of the hypotenuse) to solve TQ3, his strong intention replaced the external information in the comprehension stage. Zhao paid more attention to identifying the two triangles congruent in TQ 1. Hence, she used visual evidence to prove it. Zhen's intention for success led to his tension, so that he forgot to connect point A and D at the stage of skills manipulation.

Schoenfeld (1985) claimed that students' particular belief - naive empiricism can affect their behaviors in mathematical situations. He described an understanding of separation proofs and constructions when students solved a construction problem using straightedge and compass. They accepted or rejected a potential solution to the problem just according to the accuracy of the construction. As a result, constructions were graded by how good they looked; proofs were seen as the formal confirmation of results that are already known. In this study, at the stage of "skills selection", students used constructions in place of proofs. Another example is Feng's solution to TQ 1. Although he had proved two triangles congruent, Feng identified two corresponding angles in the congruent triangles according to the construction (looks like). These pieces of evidence supported the claim that students' naive empiricism can affect their mathematical behaviors. Schoenfeld (1988, 1989) found that this belief can be as a direct consequence of their instruction, such as emphasizing repetitive practice and focusing on the mastery of mechanical procedures as isolated skills.

From the above discussion, it can be inferred that students' motivations and beliefs can lead to errors at the stage of comprehension, strategy selection, and skills manipulation.

Literatures review in the expert vs. novice problem-solver studies found that good problem solvers know more than poor problem solvers what they know, and they know differently - their knowledge is well connected and composed of rich schemas (Lester & Kehle, 2003). Schema which is one way to organize existing knowledge through relatively stable, internal networks provides a framework for interpreting students' difficulties in problem solving (Chinnappan, 1998; Nesher & Hershkovitz, 1994; Sweller, 1989). Within the field of Euclidean geometry, diagrams play a central role. Therefore, Chinnappan (1998) used the term "geometrical schema" to describe the knowledge around a particular shape (e.g. righted-angle triangle) connecting other concepts and knowledge about how and when use these concepts. It was claimed that the inside organization of a geometric schema, and an extent of those connections between geometric schemas are important for problem solving (Chinnappan, 1998). The evidence from this study confirmed Chinnappan's claim. When Wu confronted TQ 3, he turned to another strategy as he failed to activate the connections (among "the median to a side", "a perpendicular to the side", and "the bisector of the angle to the side") inside of the equilateral triangle schema. When Li and Zhou solved TQ 3, they activated the isosceles triangle schema and the congruent triangles schema respectively. However, they failed to connect their activated schemas with the righted-angle triangles BMC and BNC (see Figure 4 and Figure 5). It seems that the students' strategy selection was influenced by the quality of schemas.

References

Ames, C., & Ames, R. (1984). Systems of student and teacher motivation: Toward a qualitative definition. *Journal of Educational Psychology, 76* (4), 535-558.

Anderson, J. R. (1993). *Rules of the Mind*. Hillsdale, NJ: Erlbaum.

Berlyne, D. E. (1966). Curiosity and the quest for information. *Voprosy Psikhologill*, *3*, 54-61.

Bregman, N, J., & Mcallister, H. A. (1982). Motivation and skin temperature biofeedback: Yerkes-Dodson revisited. *Psychophysiology, 19*(3), 282-285.

Buchanan, N. K. (1987). Factors contributing to mathematical problem solving performance: An exploratory study. *Educational Studies in Mathematics, 18*, 399-415.

Casey, D. P. (1978). Failing students: A strategy of error analysis. In P. Costello (Ed.), *Aspects of motivation* (pp. 295-306). Melbourne: Mathematical Association of Victoria.

Chinnappan, M. (1998). Schemas and mental models in geometry problem. *Educational Studies in Mathematics, 36*, 201-217.

Clarkson, P. C. (1991). Language comprehension errors: A further investigation. *Mathematics Education Research Journal, 3* (2), 24-33.

Clements, M. A. (1980). Analyzing children's errors on written mathematical tasks. *Educational Studies in Mathematics, 11*, 1-21.

Clements, M. A. (1982). Careless errors made by sixth-grade children written mathematical tasks. *Journal for Research in Mathematics education, 13*(2), 136-144.

Cobb, P. (1985). Two children's anticipations, beliefs, and motivations. *Educational Studies in Mathematics, 16*, 111-126.

Cox, L. S. (1975). Systematic errors in the four vertical algorithms in normal and handicapped populations. *Journal for Research in Mathematics education, 6*(4), 202-222.

Dai, Z. (1996). *Mathematical problems of theory* [In Chinese: 数学习题理论]. Shanghai: Shanghai Education Press.

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50,* 311-334.

Fiori, C. F., & Zuccheri, L. (2005). An experimental research on errors patterns in written substraction. *Educational Studies in Mathematics, 60,* 323-331.

Franke, M. L., Webb, N. M., Chan, A. G., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students'mathematical thinking in elementary school classrooms. *Journal of Teacher Education, 60*(4), 380-392.

Gao, S. Z., & Xue, L. X. (2009). Research Overview about Students' Errors [In Chinese: 学生错误研究之文献综述]. *Journal of Mathematics Education, 18* (1), 75-78.

Knifong, J. D., & Holtan, B. (1976). An analysis of children's written solutions to word problems. *Journal for Research in Mathematics Education, 7*(2), 106-112.

Lannin, J. K., Barker, D. D., & Townsend, B. E. (2007). How students view the general nature of their errors. *Educational Studies in Mathematics, 66,* 43-59.

Lesh, R., & Zawojewski, J. S. (2007). Problem solving and modeling. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763-804). Greenwich, CT: Information Age Publishing.

Lester, F. K., & Kehle, P. E. (2003). From problem solving to modeling: The evolution of thinking about research on complex mathematical activity. In R. A. Lesh, & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 501-518). Mahwah, NJ: Lawrence Erlbaum Associates.

Nesher, P., & Hershkovitz, S. (1994). The role of schemas in two-step problems: Analysis and research findings. *Educational Studies in Mathematics, 26,* 1-13.

Newell, A., & Rosenbloom, P. S. (1981). Mechanisms of skill acquisition and the law of practice. In J. R. Anderson (Ed.), *Cognitive skills and their acquisition* (pp. 1-51). Hillsdale, NJ: Lawrence Erlbaum Associates Ins.

Newman, M. A. (1977). An analysis of sixth-grade pupils' errors on written mathematical tasks. In M. A. Clements, & J. Foyster (Ed.), *Research mathematics education in Australia* (pp. 239-258). Melbourne: Swinburne College Press.

Pan, J., & Wu, Y. (2008). Suzhou Grade 7 and 8 students' mathematics comprehension: Current state and error Analysis [In Chinese: 关于苏州市初一和初二学生数学阅读理解水平的现状调查和困难分析]. *Journal of Mathematics Education, 17*(3), 58-61.

Robertson, S. I. (2001). *Problem Solving*. Philadelphia, PA: Psychology Press.

Schoenfeld, A. H. (1985). *Mathmatical problem solving*. New York: Academic.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well taught" mathematics classes. *Educational Psychologist*, *23*, 145-166.

Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal for Research in Mathematics education*, *20* (4), 338-335.

Sweller, J. (1989). Cognitive technology: Some procedures for facilitating learning, problem solving in mathematics and science. *Journal of Educational Psychology, 81*, 457-466.

Watson, I. (1980). Investigating errors of beginning mathematicians, *Educational Studies in Mathematics, 11, 319-329.*

Yerkes, R. M., & Dodsor, J. D. (1908). The relation of strength of stimulus to rapidity of habit-information. *Journal of Comparative and Neurological Psychology, 18*, 459-482.

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Appendix 1

Theorem:
The median to the hypotenuse of a right-angled triangle equals to the half of the hypotenuse.
Given that in $\triangle ABC$, $\angle ACB = 90^\circ$, CD is the median of AB, prove $CD = \frac{1}{2} AB$.
Proof (Here, the proof is omitted).

Example: Given that in $\triangle ABC$, $\angle B = \angle C$, AD is the bisector of $\angle BAC$, E, F are the midpoints of side AB and AC respectively, prove DE=DF.
Proof (Here, the proof is omitted).

Exercises:

1. Given that in $\triangle ABC$, $AD \perp BC$, M, N are the midpoints of side AB and AC respectively, DM=DN, prove AB=AC.

2. Given that in $\triangle ABC$, BD is a perpendicular to side AC, D on the side AC, CE is a perpendicular to side AB, E on the side AB, M is the midpoint of BC, prove MD=ME.

3. Given that $\angle ABC = \angle ADC = 90^\circ$, E is the midpoint of AC, prove $\angle EBD = \angle EDB$.

Appendix 2

Test Questions

TQ1. Given that in $\triangle ABC$, AB=AC, M is the midpoint of side AB, D is the midpoint of side BC, prove $MD \parallel AC$.

TQ2. Given that in $\triangle ABC$, $\angle BAC = 120^\circ$, construct equilateral $\triangle ABD$ and $\triangle ACE$ adjunct to and outside the given $\triangle ABC$, M is the midpoint of side AD, N is the midpoint of side AE, P is the midpoint of side BC, prove MP=NP.

TQ3. In a right-angled triangle, if an acute angle is 30°, then the leg to the acute angle equals to the half of the hypotenuse.