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On In-Service Mathematics Teachers' Content Knowledge of Calculus and Related Concepts

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Abstract: Studies have shown that teachers do not have good understanding of calculus concepts. This paper reports a study of teachers' content knowledge of calculus, on 27 in-service mathematics teachers. A questionnaire dealing with the concept images and concept definitions of various calculus concepts was administered to the group of participating teachers. The responses to the questionnaire showed that most of the participants had not built up sufficiently rich and comprehensive concept images related to the various differential calculus concepts, and they generally turned to procedures in handling calculus tasks. This study sheds light on the type of calculus content needed by school teachers.

Key words: Mathematical content knowledge; Calculus; In-service teachers; Mathematical conceptions

Introduction

Calculus has been introduced into the secondary school curriculum in many countries. The secondary school calculus curriculum is what Tall (1992) refers to as "informal calculus", which involves "informal ideas of rate of change and the rules of differentiation with integration as the inverse process, with calculating areas, volumes etc. as application of integration." (Tall, 1992).

Studies have also shown that generally students performed poorly in conceptual tasks in calculus (e.g. Amit & Vinner, 1990). Both students and teachers put their emphasis on procedures and avoided the concepts related to calculus (Amit & Vinner, 1990).

How a subject is being taught in schools is largely affected by the teachers' competency of the subject knowledge (Thompson, 1992; Toh, 2007b, 2007c). Does the students' poor performance in calculus tasks point to any deficiency in teachers' content knowledge on the subject? Studies (e.g. Mastorides & Zachariades, 2004; Huillet, 2005) show that in-service teachers specifically have difficulties with the concepts related to limits and continuity of functions.

This paper reports the results of an exploratory study of in-service mathematics teachers' content knowledge on calculus, on a group of in-service mathematics teachers attending a professional development course in Singapore.

Theoretical Background

Teachers' content knowledge

It is generally agreed that strong content knowledge is a necessary (but not sufficient) condition for good teaching in Mathematics (Toh, Chua & Yap, 2007). Usiskin (2001, p.86) asserted that in order "to teach well, a teacher of mathematics should know a great deal of mathematics. The higher the level taught, the more the teacher needs to know".

Studies have also challenged the common belief that the more a teacher knows about his subject, the more effective he can be. Begle (1979, p.51) believes that "the effects of a teacher's subject matter knowledge and attitudes on student learning seem to be far less powerful than many of us assumed". Based on Begle's view, a teacher's subject matter knowledge should not be measured by the number of modules of undergraduate mathematics taken, see Ball (1991).

According to Usiskin (2001), the "great deal of mathematics" does not merely refer to the mathematics content knowledge the teachers acquired during their undergraduate mathematics modules. It refers to the entire branch of mathematics that forms what he calls "teachers' mathematics". This includes explanation of new ideas, alternative ways of approaching problems, including ways with and without calculator and computer technology, how ideas studied in school relate to ideas students may encounter in later mathematics study (Usiskin, 2001).

Teacher's competency of the mathematics content affects how the subject is being taught is classroom (Thompson, 1992; Toh, 2007b, 2007c). According to Mohr (2006) and Shulman (1986), the content of mathematics includes (1) content knowledge, that is, knowledge of the concepts, procedures, and problem-solving processes within the area of mathematics they are teaching, and (2) pedagogical knowledge involves knowledge of teaching the content to their students. Shulman (1986) first coined the term "pedagogical content knowledge" to refer to "the ways of representing and formulating the subject that make it comprehensible to others" and "[the] understanding of what makes the learning of topics easy or difficult" (Shulman, 1986, p.9). Pedagogical content knowledge is also discussed by Carpenter, Fennema, Peterson and Carey (1988). Teachers need to have "knowledge of the conceptual and procedural knowledge that students bring to the learning of a topic, the misconceptions about the topic they may have developed, and the stages of understanding they are likely to pass through…"(Carpenter et al, 1988).

In this paper, we shall focus on the content knowledge in relation to school calculus. Sound content knowledge in school calculus includes pre-calculus concepts like functions and graphs (Toh, 2007a; p. 15), and other concepts in algebra and the notations of algebra (Dias, 2000); calculus concepts include limits and continuity, differentiation and differentiability, the definitions of integration as area under the graph and integration as anti-derivative (Lee, 2006; pp. 244 – 246).

Learning Difficulties in Calculus and Related concepts

Teachers' difficulties in calculus concepts could have developed when they were students. Thus, existing literature on students' difficulties in calculus and related concepts provides a suitable framework for the study of teachers' content knowledge in calculus.

Students' difficulties in calculus stem from their learning difficulties in dealing with topics on functions, graphs and other related algebra concepts (Judson & Nishimori, 2005). In the study carried out by Judson and Nishimori (2005), many students had an immature understanding of functions which could have led to misconceptions in problems involving application of differentiation and integration. Usiskin (2003) believed that students may perform better in calculus if they have been given early exposure to concepts involving inequality, summation, and other algebraic concepts.

Instructions in secondary school calculus classrooms often emphasize procedural knowledge grounded in algebra (Morris, 1999). Hence, it is not surprising that students neglect the conceptual part of calculus and only consider the conmputational part, thereby making calculus learning meaningless (Bezuidenhout, 2001; Davis & Vinner, 1986; Toh, 2007a, p. 74). According to Aspinwell and Miller (1997), "students regard computation as the essential outcome of calculus and thus end their study of calculus with little conceptual understanding."

However, many concepts in calculus, such as the concept of derivative, are especially important even to people whose major is not mathematics (Amit & Vinner, 1990). Ironically, the concepts that are important to non-mathematics students are the conceptual knowledge of calculus, not the procedural aspects.

The sequence of teaching material in the secondary school calculus content could be another source of learning difficulties for students. Take for example, the teaching of integral calculus in schools. The most common approach in schools is to define integration as the antiderivative. Following that, one derives at evaluating the definite integral with the Fundamental Theorem of Calculus (Anatoli, 2008). As a result of this sequencing of teaching materials, most students do not acquire appropriate comprehension of the definite integral concept (Anatoli, 2008; Orton, 1983; Sealey, 2006; Thomas and Hong, 1996).

Studies on students' learning of calculus can be traced to Tall and Vinner in the late seventies and the early eighties. Tall and Vinner (1981) used the term concept image to describe "the total cognitive structure that associated with the concept, which includes all the mental pictures, associated properties and processes" (p. 152). They used the term concept definition as a form of words used by the learner to define the concept.

The difficulties that students encounter in calculus concepts (Davis & Vinner, 1986), and teachers' misconceptions in limit concepts (Akkoc, Huillet, 2005; Yesildere & Ozmantar, 2007) can be explained by the existence of a gap between the concept definition and the concept image of the limit concept (Tall and Vinner, 1981). The calculus concept of limit is not "commonsensical" to them as it conflicts the use of this term in daily life (Davis & Vinner, 1986; Tall & Vinner, 1981).

This paper aims at answering the following five research questions:

1. Are in-service teachers able to identify graphical representations of functions and its related domain?
2. Are in-service teachers able to find the limits (including the left- and right-limits) of functions graphically?
3. How do the in-service teachers react to the differentiation of functions which do not have easily available "formulae"?
4. Are in-service teachers familiar with the physical interpretation of the signs of the first and second derivatives with reference to a graph?
5. Are in-service teachers able to associate the definite integral with the area under the given graph?

Method

The participants of this study were 27 in-service mathematics teachers from the various secondary schools in Singapore. They were teachers relatively new to the teaching service (with less than five years of experience) and selected to attend a professional development course for mathematics teachers. The participants consisted of 14 male teachers and 13 female teachers. All the teachers had less than five years of teaching experience. Among the 27 teachers, 11 of them had obtained at least a Bachelor's degree in Mathematics while 16 of them had a Bachelor's degree in Engineering from one of the two local universities. Thus, all the participants had had at least one year of undergraduate experience of university calculus.

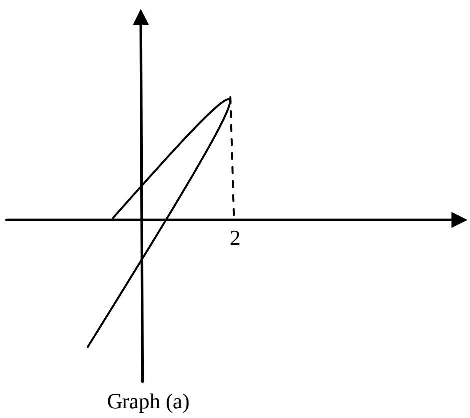
This study models after Amit and Vinner (1990)'s, in which a questionnaire was used to elicit the participants' knowledge of calculus. In addition, careful interpretation of the participants' response, together with some speculation, was used.

The questionnaire was administered during the first one hour of the first session of the professional development course. The questionnaire consisted of seven questions dealing with the concept image and concept definition of the various calculus concepts. These concepts were identified as essential for secondary school mathematics teachers (Toh, 2007a).

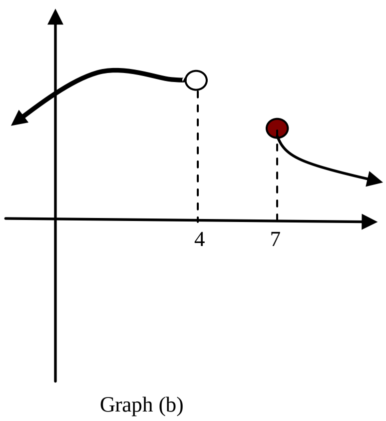
The participants were required to record their names on the answer scripts purely for administrative reasons. They were further informed that the result of this questionnaire was not taken as an assessment but to facilitate the course instructor to better understand the teachers' content knowledge of calculus, so that future in-service courses could be tailored more appropriately for the participants.

For our study and discussion in this paper, the scripts were coded arbitrarily from 1 to 27.

Q1. Can each of the following graphs represent a function? If not, provide a reason below the space provided.



Graph (a)



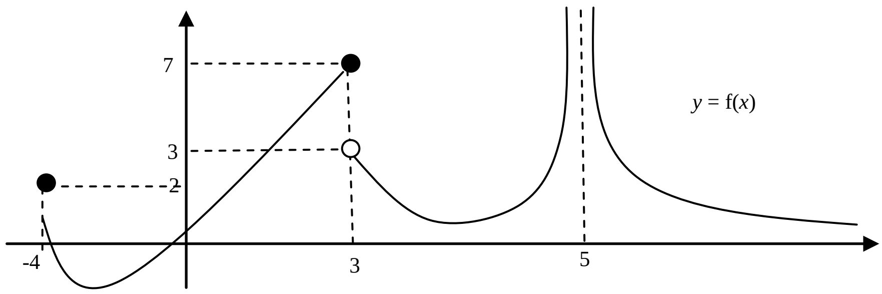
Graph (b)

This question checks the teachers' association of the concept of functions with the concept image in terms of their graphical representation.

Q2. *Sketch the graphs of* $y=\dfrac{x^2-2x+1}{x-1}$ *and* $y=x-1$ *on separate diagrams below.*

This question examines the teachers' ability to associate functions with removable discontinuity with their graphs (a continuous graph with a "punctured hole").

Q3. *Fill in the appropriate answers. If you think the answer does not exist, write in any form that you think is appropriate.*



(a) $\lim\limits_{x\to 3-} f(x)$ (b) $\lim\limits_{x\to 3+} f(x)$ (c) $\lim\limits_{x\to 3} f(x)$ (d) $\lim\limits_{x\to 5-} f(x)$

(e) $\lim\limits_{x\to 5+} f(x)$ (f) $\lim\limits_{x\to 5} f(x)$ (g) $\lim\limits_{x\to -4+} f(x)$ (h) $\lim\limits_{x\to \infty} f(x)$

(i) f(3) (j) f(5) (k) f(-4)

This question checks the teachers' ability to evaluate limits from a dynamic or cinematic perspective (Trouche, 1996, quoted in Huillet, 2005), instead of procedural computation (Aspinwell et al, 1997).

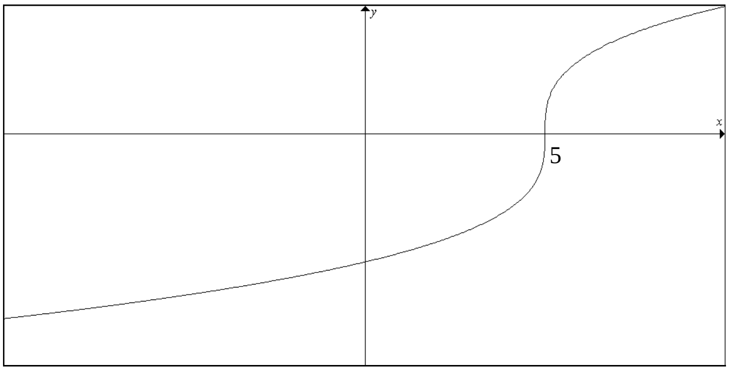
Q4. Given that $y = |x|$ (the absolute value of x), find $\dfrac{dy}{dx}$.

This question identifies the teachers' ability to link kinks as non-differentiable points on a graph. This question could also be illustrative of how teachers find derivatives of functions which do not have an easily available formula for differentiation.

Q5. Given that $y = \sin x^o$, find $\frac{dy}{dx}$.

Most errors made in carrying out tasks in differentiation were the result of failure to grasp the principles which were essential to the solution (Orton, 1983). This question was designed to check whether teachers were familiar with the steps involved in the first principle of differentiation (The well-known formulae of the derivative of trigonometric functions are valid provided the angle of x is measured in radians.). The teachers' solution could also be illustrative of how teachers respond to differentiation of functions without an easily available formula for differentiation.

Q6. The diagram shows a sketch of a function.



Fill in each entry in the following table by using the signs: +, -, 0 or "undefined".

Region	*x* < 5	*x* = 5	*x* > 5
y	*(a1)*	*(a2)*	*(a3)*
$\frac{dy}{dx}$	*(b1)*	*(b2)*	*(b3)*
$\frac{d^2y}{dx^2}$	*(c1)*	*(c2)*	*(c3)*

This question checked the teachers' ability to interpret graphically the first and second derivatives in relation to the graphs of the functions they represent. For our discussion in the next section, the entry of each of the above cells is labeled as (a1) to (c3).

Q7. *Find the following definite integrals without using integration techniques.*

(a) $\int_0^5 (x+3)dx$ *(b)* $\int_0^a \sqrt{a^2 - x^2}\,dx$

This question examined the teachers' ability to evaluate some integrals by interpreting them as areas under graphs. This is an important skill that in-service teachers need to possess (Toh, 2007a, pp. 124 - 126), but which were usually lacking in the teachers (Anatoli, 2008).

Results and Discussion

In this section, the results of the participants' responses to the seven questions are discussed in relation to the five research questions raised in Section 2.

Research Question 1 - Are In-service Teachers Able to Identify Graphical Representations of Functions and its Related Domain?

This question could be answered by Q1 and Q2. 25 out of 27 participants recognized that Graph (a) of Q1 does not represent a function. 21 participants substantiated their answers for Graph (b) of Q1 with correct reasons; three participants did not state the reasons for the choice of their answers; another three participants justified their answers by incorrect reasons, as appended in Table 1.

Table 1
The Incorrect Reasons to Substantiate the Answers to Graph (b) of Q1

Script No.	Answer	Incorrect answer provided
13	No	Never see this type of graph before
16	No	Piecewise defined function
23	No	The domain is not continuous

In Q2, all the participants sketched the correct graph of $y = x - 1$. However, 15 out of the 27 participants did not distinguish the between the two given functions $y = x - 1$ and $y = \frac{x^2 - 2x + 1}{x - 1}$. Five participants used graph plotting and obtain an inaccurate (non-linear) curve of the latter function with a vertical asymptote at $x = 1$. Seven participants sketched the correct graphs for both functions, identifying the point of removable discontinuity in the graph of the latter function.

Discussion of the Result for the First Research Question

The teachers evidently had the concept image of a function represented in graphical form. Evidence in the scripts showed that the participants had used the "vertical line test" to demonstrate that Graph (a) does not represent a function. However, some teachers were unable to associate with piecewise defined functions or functions whose domain is not continuous in Graph (b).

In Q2, many teachers did not recognize the existence of removable discontinuity of the graph (represented by continuous graphs with "punctured holes"). Instead, they turned to the algebraic procedures of simplifying the associated algebraic expressions, without regard to the domain of the given function.

Second Research Question - Are In-service Teachers Able to Find the Limits (Including the Left- and Right-Limits) of Functions graphically?

This question is answered by Q3 in the questionnaire. The participants' performance of Q3 is summarized in Table 2.

Table 2
Participants' Response to Q3(a) to (k)

Q3	Correct answers	No. of correct answers	No. of wrong answers	Samples of wrong answers (frequency)
(a)	7	23	4	Insufficient information (1); between 3 and 7 (1); 3 (2)
(b)	3	17	10	Not applicable (2); infinity (2); 7 (5); insufficient info (1)
(c)	Does not exist	8	19	7 (16); 7 or 3 (1); Any value (2)
(d)	Infinity / does not exist	27	0	
(e)	Same as (d)	26	1	Left answer blank (1)
(f)	Does not exist / undefined / infinity	27	0	
(g)	2	23	4	2.5 (2); -4 (1); 1 (1)
(h)	0	25	2	Infinity (2)
(i)	7	26	1	3 or 7 (1)
(j)	Does not exist / undefined	13	14	Infinity (14)
(k)	2	27	0	

Discussion of the Result for the Second Research Question

Concept of limit, left limit and right limit. Most participants did not identify the correct values of the limits, left and right limits correctly at the points of discontinuity. It is suggestive that the teachers' concept image of limits at a point is locked with the algebraic procedure of finding limits of a function. At points of discontinuity, the procedures of finding limits (or one-sided limits) could not be applied, hence relatively fewer correct answers for 3(b) and 3(c) compared with 3(a).

Concept of "infinity" and limits at infinity. Most participants had fairly accurate concept image of the "limits at infinity" in relation to the value it represents on the graph, as indicated by the high frequency of correct answers in Q3(h). On the other hand, many participants had not fully understood the concept of "infinity", as shown in their performance in 3(j).

It is clear that "infinity" might not have associated them with the limiting process, but treated as synonymous with "undefined" or "does not exist". Thus, teachers might not have developed an intuitive or perceptual meaning of the concept of infinity.

Research Question 3 - How do the In-service Teachers React to the Differentiation of Functions which do not Have Easily Available "Formulae"?

This is answered by Q4 and Q5 in the questionnaire.

Non-differentiable points. Only four participants presented the correct solution for Q4 and identified that the function is not differentiable at $x = 0$. Two participants did not attempt the question; three participants got their solution wrong, which is shown in Table 3; the remaining 18 participants did not identify the non-differentiable point at $x = 0$.

Table 3
Incorrect Solutions of Three of the Participants

Wrong Solution 1 Script No. 17	Wrong Solution 2 Script No. 20	Wrong Solution 3 Script No. 23
$y = \lvert x\rvert \therefore \frac{dy}{dx} = 1*$	$\frac{dy}{dx} = 1*$	There is no solution to this question.*

Differentiation of trigonometric functions with domain in units of degree. For Q5, there were eight participants who completed the differentiation correctly by converting the unit of angle to radian. 17 participants did not convert the angle to degree and left the answer as cos x^{o}. One participant did not attempt this question while one other participant misinterpreted the degree sign as "zero" symbol.

Discussion of the Result for the Third Research Question

Q4 and Q5 demonstrated that the heuristics that the participants had used in performing calculus task was to associate the new task with a visually similar task that they had already known. In the process of performing the task of differentiation, they failed to link the tasks to the concepts or the principles underlying the procedures.

According to Selden, Selden, Hawk and Mason (1999), when one encounters a new problem, a mental structure, called the problem situation image, is structured. The problem situation image contains *tentative solution starts*. From the teachers' performance of these two questions, their tentative solution start was to look for any "formulae" that could be clues to leading to the correct solutions, and modified the process accordingly, as demonstrated in Figure 1:

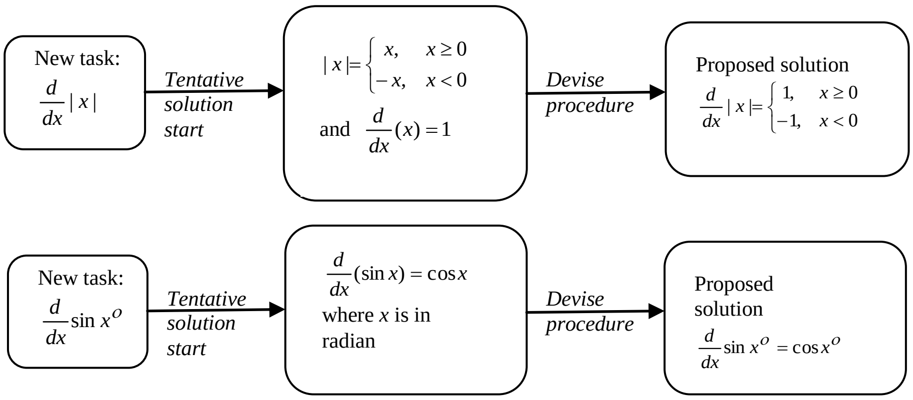


Figure 1. Diagram illustrating how the teachers perform the calculus tasks

Without further inputs of the related concept images developed on the various calculus concepts, the devised procedure following from tentative solution start is likely to be incorrect when one simply associates with a formula that looks visually similar.

Research Question 4 - Are In-service Teachers Familiar with the Physical Interpretation of the Signs of the First and Second Derivatives with Reference to a Graph?

The response to Q6 is presented in Table 4. Each of the nine entries is labeled from (a1) to (c3) as described in Section 3 above.

Table 4
Participants' Responses to Q6 (a1) to (c3)

Q6	Correct answer	No. correct	No. wrong	Detailed wrong answers
(a1)	Negative	27	0	
(a2)	0	26	1	Undefined (1)
(a3)	Positive	27	0	
(b1)	Positive	27	0	
(b2)	Undefined	19	8	0 (5); Positive (2); Negative (1)
(b3)	Positive	25	2	Negative (2)
(c1)	Positive	12	15	Undefined(3); 0(6); Negative (4); blank (2)
(c2)	Undefined	14	13	0 (9); Negative (1); blank (3)
(c3)	Negative	10	17	0(5); Positive (6); blank (6)

Discussion of the Result for the Fourth Research Question

From their answers to (a1) to (b3), most participants were able to:

- associate the sign of the *y*-values with the region of the graph which is either above or below the *x*-axis;
- recognize the sign of the first derivative as the increasing or decreasing part of the graph.

In the response to (b2), eight out of 27 participants could not recognize that the function is non-differentiable at the point where the tangent is parallel to the *y*-axis. Together with Q3, the result shows that the teachers did not have sufficient concept images associated with non-differentiability of a point on the graph of function, see Figure 2. The concept images associated with non-differentiability are usually points which are "not smooth" – this includes discontinuous points and kinks.

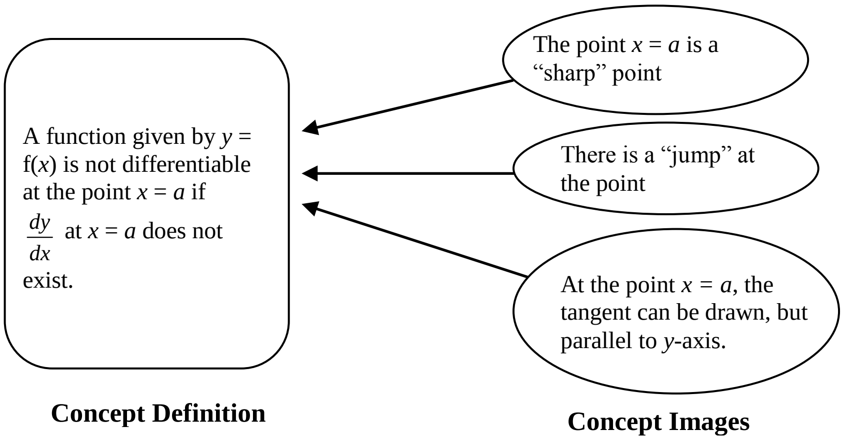


Figure 2. Concept definition and associated concept images of non-differentiable points

Many participants did not associate the second derivative with the concavity of the graph, as evident from the participants' response to (c1) and (c3).

Nine participants indicated the answer for (c2) as "zero" for the inflexion point at $x = 5$. This could be explained by the fact that somehow the teachers had the concept image of a point of inflexion, together with the procedure of finding the point of inflexion by equating the second derivative to be zero.

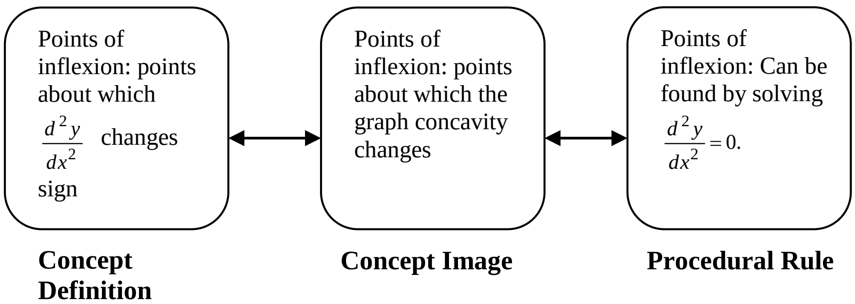


Figure 3. Relation between the concept definition, concept image and the procedure to find the points of inflexion.

The application of the procedural rule of finding points of inflexion could cause incomplete concept images and even misconceptions about points of inflexion, as not all points of inflexion are differentiable (Toh, 2007a, pp. 96 – 97).

Research Question 5 - Are In-service Teachers Able to Associate the Definite Integral with the Area Under the Given Graph?

For Q7(a), 25 participants identified that $\int_0^5 (x+3)dx$ represented the area under the straight line graph. Two participants sketched the graph of $y = x + 3$ wrongly. Two other participants performed direct integration and obtained the answers directly, which was considered *unacceptable* for this question.

For Q7(b), the different answers provided by the participants can be summarily classified into six categories, from No. I to VI in Table 5.

Table 5
Different Categories of Answers from the Participants for Q7(b)

No.	Category	Number of participants
I	Submitted completely correct solution	4
II	Correct circle but recognized the area as the entire semicircle for $x > 0$	7
III	Could recognize the correct circle but recognizes the integral as representing the area of the entire circle	4
IV	Recognized the wrong circle but could interpret definite area as area under the graph	1
V	Evaluated the definite integral directly.	8
VI	Did not attempt the question	3
	Total	27

The 16 participants whose solutions were classified as Categories I to IV interpreted the definite integral as areas under the graph for this question. These participants were able to interpret that $y = \sqrt{a^2 - x^2}$ somehow represented a circle on the Cartesian plane.

Discussion of the Result for the Fifth Research Question

Most participants had the concept definition of the definite integral as the area under the graph, as illustrated by Category I to IV in Table 5.

The mistakes made by the participants under Category II to IV in Table 5 were that they were unable to determine the geometrical shape represented by $y = \sqrt{a^2 - x^2}$. The seven participants classified as Category II and III for their response to Q7(b) in Table 5 demonstrated that they were not aware that $y = \sqrt{a^2 - x^2}$ represented only a semicircle *above* the *x*-axis, but a full circle. The root cause of this problem could be related to algebra, at the definitional level, that the radical notation "$\sqrt{\ }$" only denotes the positive square root. This is a common misconception among secondary school teachers at the definitional level and would be addressed at algebra content upgrading courses for secondary school mathematics teachers (Toh, 2007d).

Limitation of this Study

In this study, the method used was based on a questionnaire, followed by interpretation of the participants' response. This approach was rather limited. A careful interview could have provided more useful information and illuminating facts about teachers' conceptual knowledge of calculus, although it should also be noted that interviews not properly administered could also affect the interviewees' ideas and hence render the results unreliable (Amit & Vinner, 1990; p. 4). Due to constraint of time, formal interviews could not be arranged.

In addition, it should also be noted that the sample size in this study was rather small and might not be random so that it might be difficult to render the results of this study to be generalizable. However, it is hoped that this exploratory study could spur further research into exploring specific areas of teachers' content knowledge in various aspects of calculus.

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