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Introducing Mathematical Modelling to Secondary School Teachers: A Case Study

Cheah Ui Hock
South East Asia Ministers of Education Organisation Regional Centre for Education in Science and Mathematics, Malaysia

Abstract: In the search for an effective approach to make mathematics education relevant, interesting and beneficial to students, mathematical modelling has emerged as an approach that has the potential to enhance students' mathematical thinking in a creative manner. While this may be so, it is important that teachers be adequately prepared so that they too have the experience doing mathematical modelling themselves. This paper discusses an attempt by a group of teachers who were undergoing an in-service course to model a simple knot. The three emergent themes arising from the case study were (1) the experience of creative and collaborative problem solving, (2) learning mathematics through mathematical discourse, and (3) the concern for time in the modelling process.

Key words: Mathematical thinking; Teacher education; Mathematical modelling; Problem solving

Introduction

Problem solving in the mathematics curriculum

There is an increasing emphasis placed on problem solving in the mathematics curricula today in the Southeast Asian region (Curriculum Planning and Development Division, 2001; Curriculum Development Centre, 2003; Departemen Pendidikan Nasional, 2003). This may be attributed to the concerns of policy makers towards the moulding of citizens equipped to handle the challenges of the new knowledge era. Increasingly, society demands that its workers are able to solve real world problems. To do this, it is envisaged that citizens should be equipped with higher-order thinking skills, ability to utilise inter-disciplinary skills and knowledge, ability to work as a team and have a good command of communication skills (Tan, 2003). In short, one major aim of the mathematics curriculum is to create citizens who are mathematically literate. This means that students need to have the "…capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen" (Organisation for Economic Co-operation and Development, 2006, p. 72).

Undeniably, problem solving and its accompanying processes are increasingly given more prominent roles in the school mathematics curricula (Curriculum Planning and Development Division, 2001; Curriculum Development Centre, 2003; Departemen Pendidikan Nasional, 2003). Through problem solving students are expected to learn mathematics with understanding, produce positive attitudes towards the subject and develop thinking in flexible and creative ways. The main idea is to allow the student to make connections to the real world. Often learning mathematics involves students attempting to grapple with mathematical concepts written in symbols and formulas, in a language far removed from the student's daily experiences. At times, this means that students learn mathematics devoid of its relevance to real life experiences. A more meaningful understanding of mathematics can be achieved if the mathematical symbols and formulas are derived from real world experiences and then interpreted and applied to the real world – the world that is connected to nature and society. The student's real world would include mathematical applications in everyday life as well as cross-disciplinary applications that extend into other subject areas (Blum & Niss, 1991; Blum et al., 2002). Mathematics then becomes more meaningful as the student can then fully experience the utilitarian purposes for learning mathematics.

Learning mathematics involves thinking about ideas and concepts in two different realms, in the real world and the mathematical world (Davies & Hersh, 1981; Kaiser, 2005). In the mathematical world, ideas and concepts are expressed in abstraction in numbers, letters and symbols. Processes like algorithms, proofs and counter-examples are common features used in the mathematical world. For some students, relating mathematical ideas to the real world can be a difficult task partly because the mathematical language is abstract and applies specifically to objects in the mathematical world only. Difficulties in learning mathematics are sometimes attributed to the way it has been taught. Traditionally in school mathematics, learning often begins in the mathematical world. Teachers begin by expounding on mathematical ideas in the mathematical world. Examples from the real world are sometimes included in order to make the mathematical ideas more meaningful and relevant to the learners. To enhance meaning, the connection to the real world is frequently emphasised in the form of word problems given at the end of each chapter. However, often these problems are not really from real world situations but are 'dressed-up' and simplified so as to exemplify specific mathematical ideas. In selecting these problems the teacher often begins with the ideas and concepts in the mathematical world that she has in mind and then looks for examples in the real world in which she can apply those ideas and concepts. Facts and variables that are not relevant to the specific mathematical concept being taught are intentionally excluded. Functionally, these problems only serve as practice for the students.

Mathematical modelling

Mathematical modelling is a process which aims to characterise some event or phenomenon in the real world as a mathematical representation. In contrast to the way mathematics is traditionally taught, mathematical modelling begins with a real world situation as the problem is then transformed into the mathematical objects (see Figure 1 below).

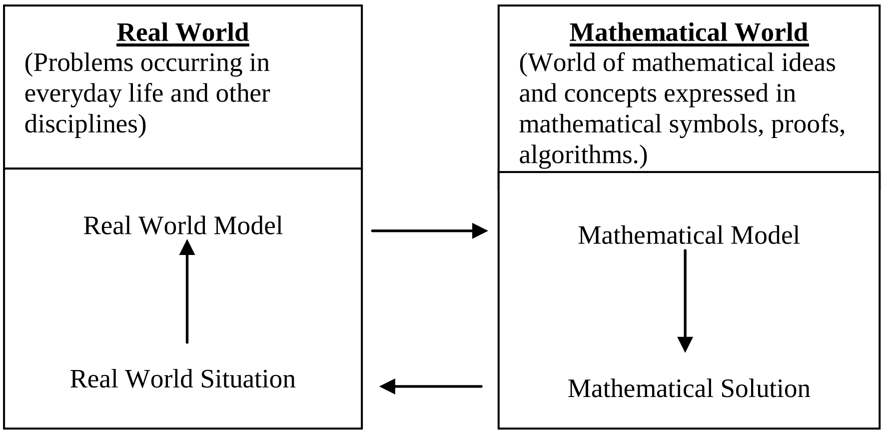

Figure 1. Mathematical modelling: From the real world to the mathematical world

The mathematical modelling process involves the following steps (Mason & Davies, 1991):

1. Creating a real world model resulting in a simplified mental picture or image. This step involves creating a real world model from the real world situation as the students become aware of important and relevant variables, focusing on these variables and ignoring the less important ones.

2. Transforming the real world model into a mathematical model. This transformation exemplifies a process that Wheeler (1982) describes as mathematising. Through mathematising, the real world model is expressed in mathematical form with the use of mathematical symbols to replace the real world model (Skemp, 1986; Mitchelmore, 2002).

3. Solving the mathematical model using mathematical procedures to produce a mathematical solution.

4. Finally the mathematical solution is then interpreted and validated by comparing the solution to the original real-world situation.

Mathematical modelling requires that the learners create and construct the model themselves. It is very much a hands-on and minds-on experience. Introducing mathematical modelling at the school level would therefore also require that teachers be familiar with the processes of mathematical modelling. Teachers would need the mathematical modelling experience and hence the main reason for conducting the following case study where the teachers were able to experience the whole mathematical modelling process.

A Case Study: Modelling a Simple Knot

This case study reports a group of six secondary teachers working on a mathematical modelling problem as part of their project work during a four-week in-service course where the author was the course supervisor. The main purpose of the course was to enable the teachers to introduce student research in mathematics at the secondary school level.

The project was completed over a period of three weeks. Mindful that this was the first time that all the six teachers were attempting to do mathematical modelling, the course supervisor provided guidance through discussions until the completion of the project. Prior inputs about the mathematical modelling process and problem solving were included in the course.

Modelling a simple overhand knot

Understanding the problem. Several problems were pre-identified by the course supervisor for the teachers to choose and explore. For their project work, the teachers in this group chose the following problem, "When you tie an overhand knot in a piece of rope, the overall length gets shorter, but by how much?" (Mason & Davies, 1991, p. 55). To help the teachers understand the problem, ropes of about one metre length and of various thicknesses each were made available to the teachers and the teachers were allowed to freely discuss the problem.

Identifying the variables. The teachers explored the problem by tying knots and looking at some important factors that would affect the shortening of the rope when it was tied with a single knot. The teachers initially identified the following factors to be relevant to the problem: (1) the thickness of the rope, (2) the elasticity of the rope, (3) the material used to make the rope, (4) the tightness/looseness of knot and, (5) the coefficient of friction for the given rope. After some exploration, the teachers decided to look into the relationship between the thickness of the rope and its shortening when a knot was tied. The other variables were assumed to be constant. Thus, the teachers became aware that they had to make assumptions as they progressed. The teachers made the following assumptions:

1. The material used to make the rope (nylon, polymer or manila hemp) does not have any significant effect on the shortening of the rope's length when tied with an overhand knot;

2. Each overhand knot was fully tight when the measurements were recorded.

Creating a mathematical model. The teachers then attempted to transform the real world model into a mathematical model by looking for relationships among the variables that were identified. The group decided to use the empirical approach to look into the relationship between the thickness of the rope and the shortening of the rope when a knot was tied.

Table 1
Description of the Ropes and Measurements

Rope	Material	Average Thickness (T_i) cm	Average Length (L_i) cm	Average Length after one knot (J_i) cm	Average Shortening (S_i) cm
A	Nylon	0.186	100.00	98.22	1.78
B	Nylon	0.322	100.00	96.58	3.42
C	Polymer	0.476	99.80	93.86	5.94
D	Nylon	0.574	99.80	94.10	5.70
E	Manila Hemp	0.590	100.00	93.44	6.56
F	Polymer	0.840	99.96	89.84	10.12
G	Nylon	0.956	100.00	88.54	11.46
H	Nylon	1.194	99.94	86.34	13.60
I	Manila Hemp	1.288	99.96	85.88	14.08
J	Manila Hemp	2.180	100.00	76.20	23.32

To make the connection between the two variables, the teachers used ten pieces of ropes made from different materials and different thickness each of about 1 meter in length. The thickness and the length of the ropes before and after tying a knot were measured. For each measurement, the teachers used the average of five readings. The measurements were then recorded as shown in Table 1.

The MS-EXCEL was then used to plot the scatter graph of the points as shown in Figure 2. The scatter plot in Figure 2 showed that there was an approximately linear relationship between the shortening (S_i) and the average thickness (T_i). The MS-EXCEL was then utilised to generate a linear regression line of the form $S = a + b T$. Specifically, the linear regression equation was found to be $S = 0.2884 + 10.822T$.

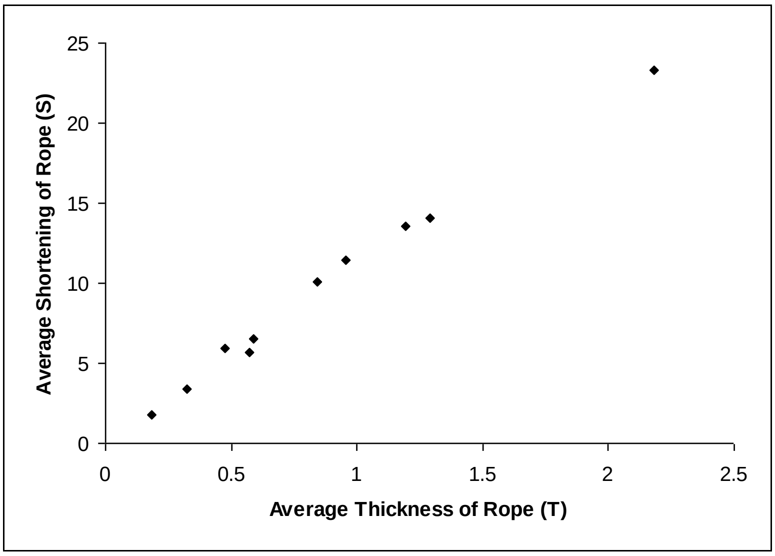

Figure 2. The scatter plot of the average thickness against the shortening

Interpreting the solution. Having found the linear regression line, the teachers continued to look at the solution and compared it with other models in order to validate and verify their model. A geometrical model found in Mason and Davies (1991) provided the opportunity for the teachers to compare the two models available. In the geometrical model (see Figure 3) the knot was simplified to an oval shape with two circles in the centre. The geometrical model showed the shortening of the rope as the length of the oval which was represented by dotted lines in Figure 3. The two circles represent the rope running into and out from the knot. The geometrical model was then simplified to an algebraic equation $C = 2\pi T + 2T$, where *C* is the length of the oval, and *T* represents the thickness of the rope. The comparison showed a discrepancy between the two models.

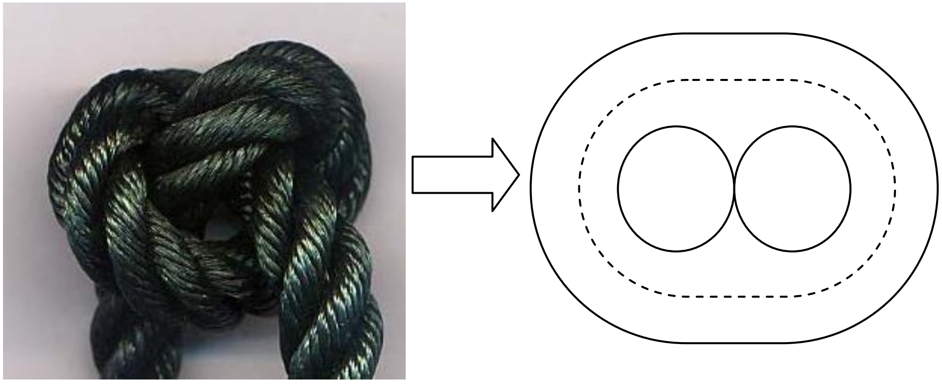

Figure 3. A geometrical model of a simple knot

After much thought and discussion, the teachers were able to reconcile the discrepancy when they compared the geometrical figure to the actual knot. The teachers realised that the geometrical model had not taken into account the short segments of the rope running into and out from the knot. The teachers figured out that these two extra segments were equivalent to twice the length of the thickness of the ropes and so they modified the equation representing the shortening of the ropes to $S = 2\pi T + 2T + 2T$. By simplifying the formula, the length of the knot was reduced to $S = 2T\,(2 + \pi)$.

Doing Mathematical Modelling

Observations from the case study

The teachers' comments. Both written and verbal responses of the teachers during and at the end of the course indicated that they were all confident that mathematical modelling was appropriate for secondary school students. All the teachers expressed that creative problem solving and collaboration were the two important features of their project work. The teachers' comments were corroborated through participant observation while working with the teachers. It was observed that the teachers worked collaboratively during their project work although they had varying problem solving abilities. The way the teachers reconciled the results of the empirical and geometrical model was also indicative of their creative problem solving ability.

Initial uncertainty in doing mathematical modelling. Looking at the final project report of the teachers on its own might have been persuaded one to think that doing mathematical modelling was straight forward. However observing the teachers working on the project and talking with them provided a picture that the whole process was difficult and full of uncertainties. All the teachers in the group said that they faced difficulties in doing the project. When asked about the difficulties, the replies included "A lot of difficulties… initially." and "Very much: measure errors, how to make a stringent proof of our result". It is inferred that much of the difficulties the teachers faced were due to the uncertainties of problem solving as opposed to traditional problem solving where there is always an indication of some specific approaches that can be used. When asked how they overcame the difficulties, the teachers agreed that it was through determination and cooperation that they managed to complete the project.

It was also observed that the teachers' mathematical model was not without error. For example, the teachers missed making an important assumption that in the ideal case the shortening would be zero if the thickness of the knot is zero. This would have led to the empirical model of the form of $y = ax + b$ where b would have the zero value since the graph would pass through the origin.

Mathematical discourse. Doing mathematical modelling offered certain advantages in learning mathematics. In the course of the project the course supervisor and the teachers were often involved in mathematical discourse. Explaining, exploring, conjecturing and predicting are inherent in the mathematical modelling process which was crucial in helping the teachers construct the mathematical model. The following episode shows an example of how the informal mathematical discourse helped the teachers construct their mathematical ideas.

The teachers T and J were discussing with the course supervisor (R) about how to resolve the difference between the two formulae derived from the regression line and the geometric formula

R: How is it that the linear regression line does not match the formula from the geometrical model?
J: Mmm... I did not realize this.
R: How can you explain this?
J: Maybe it's because the knot is not completely tight.

R: But you made the assumption that it is completely tight. So if it is not completely tight, then we may have to reconsider the assumption.

T did not say anything but was quietly observing the discussion was looking at the knot and figuring out the knot and comparing it with the geometric diagram. The next day J came excitedly and said, "We found it, it should be $2\pi T + 4T$!"

T: Because we did not consider the length of rope going into and out from the knot.
J: The extra length is actually two times the thickness of the rope…. See (as J showed the actual knot)!

T then explained how he derived the formula $S = 2T(2 + \pi)$.

Integrating information communication and technology (ICT). The mathematical modelling project also provided a good platform for the teachers to integrate ICT into their work. It was observed that the teachers incorporated ICT through:

1. the use of MS-EXCEL to plot graphs and to find the linear regression equation,

2. the use of the Internet to search for literature on modelling knots, and

3. the use of statistics available on the Internet to verify the regression equation.

Conclusion

Three themes and issues emerged in the conduct of this case study. First, through mathematical modelling the teachers were able to experience the creative and collaborative nature of problem solving. Mathematical modelling had allowed the teachers to experience the collaborative and creative nature of learning mathematics. Even though the teachers found the mathematical modelling project difficult, they were able to complete it through cooperation and determination. The project work further allowed the teachers to learn about the creative nature of mathematics as they worked on the mathematical model which allowed their thinking to move from the real world to the mathematical world. The teachers were able to make mathematical abstractions from a real world situation. It was further seen that during the validation of the mathematical model the teachers found two different ways to model the simple knot and were able to creatively reconcile the empirical and geometrical models.

Secondly, the mathematical modelling project had provided an environment for the teachers to learn and create mathematical knowledge through mathematical discourse. The mathematical modelling project offered the teachers the opportunity to experience the nature of mathematical discourse, to conjecture, to reason, to prove, to verify and to disprove. Although none of the teachers explicitly mentioned how they had acquired new mathematical knowledge, it was evident through the project work and observations that much of the learning had been acquired through mathematical discourse. This is perhaps one significant hidden advantage of doing mathematical modelling as it offers its learners a view that mathematics is created and constructed. Often traditional mathematics presents a view that mathematics is certain and fixed due to the extensive and rigorous use of algorithms and proofs. This view is often further propagated when students learn most of the mathematics primarily in the mathematical realm, thus creating a perception that mathematics is fixed, already proven and unchanging. Through mathematical modelling the teachers were thus able to experience mathematics as a socially constructed body of knowledge built through discourse (Ernest, 2007; Lakatos, 1976).

The third emergent theme of this case study is the concern for time and space to do mathematical modelling. Doing mathematical modelling is not without its difficulties. For the student it sometimes means that they begin without having an idea how the mathematical model would turn out to be in the end. Exploration and project work such as mathematical modelling would entail extended periods of time. Thus if mathematical modelling were to be introduced into the curriculum, careful consideration need to be given to time allocation and how it can be managed in the curriculum. However, mathematical modelling is not totally new in many countries especially for some higher ability students. Some form of mathematical modelling and applications can be commonly seen in science and mathematics fairs where student projects are exhibited.

When introducing mathematical modelling in schools, teacher learning becomes a very important consideration. Since there is always a certain amount of uncertainty when we allow students to find creative solutions in doing modelling, it is thus important for teachers themselves to experience the entire process of modelling. The experience thus gained would give the teachers the confidence to guide the students in their modelling projects. It is thus imperative that both practicing teachers as well as preservice teachers be given the opportunity to be exposed to doing mathematical modelling. Its careful inclusion and greater emphasis in the teacher training curriculum would certainly prepare teachers to better motivate students as well as to develop their problem solving skills.

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Author:

Cheah Ui Hock, SEAMEO Regional Centre for Education in Science and Mathematics, 11700 Gelugor, Penang, Malaysia; uhcheah@recsam.edu.my