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Close Encounters with a Mathematics Journal Paper

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This article reports the encounters of a mathematician educator with a mathematics journal paper over a few years. The paper trail illustrates an equal fascination with mathematics and its power to educate.

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A New Genre?

Mathematicians such as Freudenthal, Klein, and Pólya have written very insightful works regarding the workings of the minds of mathematicians, with the hope that the ideas contained within their writings could be implemented in mathematics education. That their writings have been highly cited in mathematics education articles shows that their genre of writing, albeit not of the mainstream, is considered valuable to mathematics education.

Certainly, Freudenthal, Klein, and Pólya are excellent mathematicians, yet all would agree that none of them can be considered among the giants. If one of those mathematical giants, Carl Friedrich Gauss for example, could have devoted some time towards writing for mathematics education, such a contribution would have been gratefully valued. Somewhat in a similar vein, if a mathematician should write in the same genre, should not his contribution be considered useful to some extent? The point is that mathematicians, great or small, have many mental dispositions that are common, such as problem solving approaches, notational ability, analogous thinking or cognitive load distribution. The quality of sharing one's thoughts, given mostly the same resources, for the good of mathematics education should then depend on the ability of the mathematicians to communicate clearly and their passion for education, rather than on their 'greatness'.

There are some of us who publish in both mathematics and mathematics education, subjecting ourselves rightly to the conventions and genre of each field. One wonders however, whether more could be shared if certain strictures be loosened. Here is an anecdote to clarify what I mean. Once in a mathematics education conference presentation, I stated that we, as mathematicians, often go through cyclical paths within Pólya's model of problem solving. I was questioned if that were indeed true since I had not given a reference in my paper. Although such a reference exists in Carlson and Bloom (2005), I was surprised that a mathematician's account of one's own thinking could not be taken as valid in a mathematics education context. Indeed, a lot of time and effort in one field may need to be expended to verify what would be regarded as common knowledge in another, albeit undocumented.

Since mathematicians, especially those who are interested in education, have much to share, I propose that a new genre of mathematics education writing be introduced. Rowland (2003) can perhaps be used as an example. This new genre should have these features:

- Good, i.e. readable and interesting, mathematics content
- Potentially applicable to mathematics education
- Loosening of strictures regarding citations and frameworks

What follows in the main of this paper is an attempt to write in such a genre. It describes the encounters of a mathematician educator with a Number Theory paper by Skolem (1957). The intent of the sharing is for teachers to understand how one can take time (years even) to understand a mathematics paper and how students of different levels can be facilitated to learn mathematics through the reading of the paper.

Finally, for this section, I would like to loosely define a person whom I would call a *mathematician educator*. *Mathematician educators* are teachers of mathematics who base their pedagogy on a strong foundation of mathematics disciplinarity and education theory, research and/or experience. There are no academic qualifications to certify one as a mathematician educator – one can be a university professor or a kindergarten teacher. When university professors adjust their teaching for different students and use suitable examples with an understanding of pedagogical content knowledge, then they are mathematician *educators*. When schoolteachers understand that it can be *proven* that two tangram sets cannot be assembled into a square, then they are *mathematician* educators. The term *mathematician educator* emphasises the need for both aspects to be embodied in the one person. The term *mathematics educator*, when wrongly emphasised, would imbue the human personality only on the *educator* aspect.

Close encounters of the first kind: A problem without a name

A colleague shared a problem that his 10-year-old son was given by his schoolteacher in enrichment class:

> Arrange the numbers 1, 1, 2, 2, 3, 3, 4, 4 in a row such that there are no numbers between the pair of 1's, one number between the pair of 2's , two between the pair of 3's, and three between the pair of 4's.
> [Readers are encouraged to try to solve the problem themselves.]

The problem met with some enthusiasm among a few of us. We solved the initial problem soon enough and proceeded to ask the *usual* Pólya Looking Back questions (see Pólya, 1957; Leong, Tay, Toh, Quek & Dindyal, 2011):

- What if it were 1, 1, 2, 2, …, n, n?
- For what values of n would a solution exist?
- If a solution exists for a particular value of n, how many solutions would there be?

The problem engaged us for a couple of days. A mathematician M wrote a simple computer program to run for different values of n. Figure 1 shows solutions for n = 4 and n = 5 from a similar program written in Microsoft Excel VBA.

	A	B	C	D	E	F	G	H	I
1	4	<--- Value of n							
2	1	1	4	2	3	2	4	3	
3	1	1	3	4	2	3	2	4	
4	4	1	1	3	4	2	3	2	
5	2	3	2	4	3	1	1	4	
6	4	2	3	2	4	3	1	1	
7	3	4	2	3	2	4	1	1	
8									

	A	B	C	D	E	F	G	H	I	J	K
1	5	<--- Value of n									
2	1	1	5	2	4	2	3	5	4	3	
3	1	1	3	4	5	3	2	4	2	5	
4	4	1	1	5	4	2	3	2	5	3	
5	5	1	1	3	4	5	3	2	4	2	
6	4	5	1	1	4	3	5	2	3	2	
7	2	3	2	5	3	4	1	1	5	4	
8	2	4	2	3	5	4	3	1	1	5	
9	3	5	2	3	2	4	5	1	1	4	
10	5	2	4	2	3	5	4	3	1	1	
11	3	4	5	3	2	4	2	5	1	1	
12											

Figure 1. Solutions for n = 4, 5

It can be proven by simple case analysis that solutions do not exist for n = 2 and n = 3. The computer program showed that no solutions exist for n = 6 and n = 7. I made the conjecture that solutions exist if and only if $n \equiv 0$ or 1 (mod 4).

Now, our happy company of problem solvers included research mathematicians, mathematics educators and research assistants in our mathematics education projects. A research assistant went looking for references to the problem on the internet and discovered that Martin Gardner had recorded a similar problem in *The Colossal Book of Short Puzzles and Problems*:

> *Problem 1.13 Langford's problem*
> Many years ago C. Dudley Langford, a Scottish mathematician, was watching his little boy play with colored blocks. There were two blocks of each color, and the child had piled six of them in a column in such a way that one block was between the red pair, two blocks were between the blue pair, and three were between the yellow pair. Substitute digits 1, 2, 3 for the colors and the sequence can be represented as 312132. This is the unique answer (not counting its reversal as being different) to the problem of arranging the six digits so that there is one digit between the 1's and there are two digits between the 2's and three digits between the 3's. Langford tried the same task with four pairs of differently colored blocks and found that it too had a unique solution. Can you discover it? There are no solutions to "Langford's problem," as it is now called, with five or six pairs of cards. There are 26 distinct solutions with seven pairs. No one knows how to determine the number of distinct solutions for a given number of pairs except by exhaustive trial-and error methods, but perhaps you can discover a simple method of determining whether there is a solution.
>
> (Gardner, 2006, p.9)

Although Langford's Problem was not exactly the same as our problem, we had a name for reference now and before long, the other research assistant found the paper, which stated our problem precisely and proved our conjecture. We read the paper (Skolem, 1957) and the mathematician M remarked that he was actually close to a similar proof if he had persevered.

At the end of each section, I would like to reflect on learning points relevant to mathematics education. In this section, we can see that mathematicians engage in a problem as Pólya described, with particular emphasis on Looking Back. Thus, enrichment classes for the 10-year-old child would probably fare better if the teacher makes it a culture for further exploration beyond the solution.

We also see that mathematicians use computer programming to run quickly through many cases to form or reinforce conjectures. This is now common in research mathematics as coding is becoming an essential tool for the mathematician. We use various programming languages but an easily available programme for students and teachers in school is VBA within the Microsoft Excel environment (see Ho et al., 2017; Low, Tay & Chen, 1999). Computational thinking, when perceived as using computer programming as a tool akin to the current use of the calculator, should be actively encouraged and supported in schools.

Finally, we elaborate on the mathematician M's reflection on two "mathematics education lessons" that he saw in his process of trying to solve the original problem. First, M came up with the 1 to $2n$ representation on his own when he tried to write his programme efficiently. This embodies computational thinking in the sense that the computation *follows* from carefully thinking about the problem mathematically, as opposed to a computer programmer who might have "brute forced" the computation from the original expression of the problem. Secondly, solving the problem to him has something to do with psychology. He believes that he did not try to prove Theorem 1 while it was still a conjecture because he had the impression that it was difficult. If there had been an "invisible voice" or "ghost" who told him to do it or a lecturer who set it as a tutorial problem, he believes that he would have proved it. This reinforces Schoenfeld's (1985) perception of the importance of Beliefs in problem solving.

Close Encounters of the Second Kind: A Paper Half-read

Skolem's paper presents the same problem, which we call Skolem's Problem, in a different way.

> Is it possible to distribute the numbers 1, 2, …, $2n$ in n pairs (a_r, b_r) such that $b_r - a_r = r$ for $r = 1, 2, \ldots, n$?

We can see that this problem is equivalent to our initial problem, restated as follows:

> Is it possible to arrange pairs of the numbers 1, 2, …, n in a row such that there are exactly $r - 1$ numbers between each pair of r's for $r = 1, 2, \ldots, n$?

Figure 2 shows solutions for Skolem's problem corresponding to $n = 4$ of Figure 1. We can see that a_r and b_r correspond to the first and the second positions, respectively, of r in the initial problem. For example, in the solution 11423243 of the initial problem, the first position of 2 is 4 = a_2 and the second position of 2 is 6 = b_2.

1	$a_1\ b_1$	$a_2\ b_2$	$a_3\ b_3$	$a_4\ b_4$
2	(1, 2)	(4, 6)	(5, 8)	(3, 7)
3	(1, 2)	(5, 7)	(3, 6)	(4, 8)
4	(2, 3)	(6, 8)	(4, 7)	(1, 5)
5	(6, 7)	(1, 3)	(2, 5)	(4, 8)
6	(7, 8)	(2, 4)	(3, 6)	(1, 5)
7	(7, 8)	(3, 5)	(1, 4)	(2, 6)
8				

Figure 2. Corresponding solutions for n = 4

We now present the proof of our conjecture from Skolem's paper in two theorems.

Theorem 1 If $n \equiv 2$ or 3 (mod 4), no solution exists.

Proof By definition, $b_r - a_r = r, \quad r = 1, 2, \ldots, n.$
Summing up both sides, we have

$$\sum_{r=1}^{n}(b_r - a_r) = \sum_{r=1}^{n} r$$

$$\sum_{r=1}^{n} b_r - \sum_{r=1}^{n} a_r = \tfrac{n}{2}(n+1) \qquad (1)$$

On the other hand, since the collection of all a_r's and b_r's is exactly the set {1, 2, …, 2n}, we also have

$$\sum_{r=1}^{n} b_r + \sum_{r=1}^{n} a_r = \sum_{r=1}^{2n} r$$

$$\sum_{r=1}^{n} b_r + \sum_{r=1}^{n} a_r = n(2n+1) \qquad (2)$$

Adding (1) and (2), we obtain

$$\sum_{r=1}^{n} b_r = \tfrac{n}{4}(5n+3),$$

which is not an integer if $n \equiv 2$ or 3 (mod 4).

Since each b_r is an integer, $\sum_{r=1}^{n} b_r$ should also be an integer. Thus, if $n \equiv 2$ or 3 (mod 4), no solution exists for Skolem's Problem.

The second part of the proof of the conjecture is in Theorem 2.

Theorem 2 If $n \equiv 0$ or 1 (mod 4), a solution always exists.

As a mathematician educator, I seek to create learning opportunities for those around me and so I encouraged my research assistant, a non-Mathematics graduate, to read the paper and to work out the proof of Theorem 2. I was very pleased when he reported that he could figure out the proof and not only that, had spotted a mistake in the proof (to be elaborated later). Leong (1995) remarked:

> The presence of a human linguistic element is really irrelevant. Just imagine a universal linguist (AUL for short) who is able to read any written human language on earth. Given a proof of a mathematical statement, would AUL be able to understand it? Would the mathematical statement itself make any sense to her? More importantly, would she be able to tell whether the proof is correct? If she could understand the proof, we would be inclined to think that she has been mathematically trained. If she could improve on the proof and rectify it, we would believe that she is a mathematician.' (p. 59)

I now show the proof of Theorem 2 in a slightly different way from the paper so that it will be clearer. Typical of mathematician educators, I insert a concrete example within the proof to lighten the cognitive load imposed by 'pure algebra'. This is a constructive proof, i.e. I show that a solution exists by constructing the required solutions.

<u>*Proof of Theorem 2*</u>

Let $n \equiv 0 \pmod 4$, i.e. $n = 4m$ for some natural number m.
We construct the system of $4m$ pairs as follows (with $m = 3$ as an example):

[1]	$2m$ pairs:	$(4m+r, 8m-r)$ for $r = 0, 1, \ldots, 2m-1$
	Example:	6 pairs: (12, 24), (13, 23), (14, 22), (15, 21), (16, 20), (17, 19)
[2]	1 pair:	$(m, m+1)$
	Example:	1 pair: (3, 4)
[3]	$m-2$ pairs:	$(m+2+r, 3m-1-r)$ for $r = 0, 1, \ldots, m-3$
	Example:	1 pair: (5, 8)
[4]	2 pairs:	$(2m, 4m-1)$ and $(2m+1, 6m)$
	Example:	2 pairs: (6, 11), (7, 18)
[5]	$m-1$ pairs:	$(r, 4m-1-r)$ for $r = 1, 2, \ldots, m-1$
	Example:	2 pairs: (1, 10), (2, 9)

(The example gives the sequence 9-7-1-1-3-5-11-3-7-9-5-12-10-8-6-4-2-11-2-4-6-8-10-12.)

Type [1] pairs give all the even differences 2, 4, …, $4m$.
Type [2] pair gives the least odd difference 1.
Type [3] pairs give the odd differences 3, 5, …, $2m-3$.
Type [4] pairs give the odd differences $2m-1$ and $4m-1$.
Type [5] pairs give the odd differences $2m+1$, $2m+3$, …, $4m-3$.

Let $n \equiv 1 \pmod 4$, i.e. $n = 4m + 1$ for some natural number m.
We construct the system of $4m + 1$ pairs as follows (with $m = 3$ as an example):

[1]	$2m$ pairs:	$(4m+2+r, 8m+2-r)$ for $r = 0, 1, \ldots, 2m-1$
	Example:	6 pairs: (14, 26), (15, 25), (16, 24), (17, 23), (18, 22), (19, 21)
[2]	1 pair:	$(m+1, m+2)$
	Example:	1 pair: (4, 5)
[3]	$m-2$ pairs:	$(m+2+r, 3m+1-r)$ for $r = 1, 2, \ldots, m-2$
	Example:	1 pair: (6, 9)
[4]	2 pairs:	$(2m+2, 4m+1)$ and $(2m+1, 6m+2)$
	Example:	2 pairs: (8, 13), (7, 20)

[5] m pairs: $(r, 4m+1-r)$ for $r = 1, 2, \ldots, m$
Example: 3 pairs: (1, 12), (2, 11), (3, 10)
(The example gives 11-9-7-1-1-3-13-5-3-7-9-11-5-12-10-8-6-4-2-13-2-4-6-8-10-12.)

Type [1] pairs give all the even differences $2, 4, \ldots, 4m$.
Type [2] pair gives the least odd difference 1.
Type [3] pairs give the odd differences $3, 5, \ldots, 2m-3$.
Type [4] pairs give the odd differences $2m-1$ and $4m+1$.
Type [5] pairs give the odd differences $2m+1, 2m+3, \ldots, 4m-1$.

The mathematics of the proof above follows exactly from the paper. To be completely rigorous, one should also check that $r = 1$ to $2n$ appears exactly once in each of the systems of pairs but granted that the proof is in a mathematics journal paper of some repute, this can be excused. However, the proof has an error, of a type that mathematicians would say is *not fatal*, that is to say, the error can be patched up without significantly changing the proof. Our research assistant pointed out that the construction runs into difficulties when $m = 1$. For the case $n = 4$, the situation can be salvaged by not choosing the pair $(2m, 4m-1)$ and ignoring Type [3] and Type [5] pairs. For the case $n = 5$, the construction fails completely, but a simple sequence 1152423543 suffices to complete the proof.

This section shows the importance of seeing that two mathematical objects are equivalent, in this case, the two statements of Skolem's Problem. The second version of the problem, which regarded the differences between pairs of numbers rather than the number of numbers between pairs of similar numbers, was more amenable towards a proof of the conjecture. Teachers who collect examples like this will be able to increase their pool of resources for teaching the Big Idea of Equivalence, within the new initiative of teaching towards Big Ideas in Singapore school mathematics (see Tay, 2019).

Reading is an important part of mathematical maturity. Reading mathematics is akin to "learning how to fish". A person who can read mathematics is able to expand his knowledge beyond the confines of the syllabus. Reading mathematics can also be an assessment for mathematical ability that is reliable and valid (Tay, 2001; Tay, Toh, Dindyal & Deng, 2014). When our research assistant could spot an error in a reputable journal paper, one can vouch for his mathematical ability without going through many tests.

Close Encounters of the Third Kind: A Paper Read in Greater Detail

NIE has a coursework programme, Masters of Science (Mathematics for Educators), that caters to teachers who are non-Mathematics majors. The programme has two levels, the first of which eases non-Mathematics majors into rigorous mathematics. The second level has courses akin to Masters level mathematics in any other university. All students are required to take a half-course that initiates them into reading mathematics texts. Faculty were invited by the coordinator to propose papers for students to choose to read for this course. Remembering that the Skolem paper had an error in the second page, I thought that it would be good for a student to read and find the error. And so, I proposed the paper for the course.

A couple of years passed until this year, a student chose Skolem's paper for the reading course. We met and to my surprise, he had quickly completed the first two pages and had gone on to read up to page 5. I was a little embarrassed to have to 'sight-read' the next 3 pages since I had not expected students to go that fast. We ended the first session with the student expressing his difficulty in understanding the proof of Theorem 3a.

One must understand that I am a graph theorist and Skolem's paper can be considered Number Theory. My interest in the paper was for the purpose of education as the content had no direct bearing on my Graph Theory research. There is so much knowledge that I read only as much as I need. This is, to me, efficient use of finite resources. (This however can be problematic for inexperienced learners of mathematics if they use a similar argument of efficiency or utility to avoid studying prescribed topics – "Will I ever need to use Galois Theory in future?") But now I had to read the paper because I had a student to work with. I spent some time over the next week reading the paper in more detail. I found that the difficulty arose from another error that resulted from a switch of two words.

I will need to fill in more mathematical background. Skolem's problem is extended in the paper to the whole number series as follows.

> Is it possible to distribute the numbers 1, 2, 3, … in pairs (a_r, b_r) such that $b_r - a_r = r$ for $r = 1, 2, 3, \ldots$?

In typical sanguine fashion, Skolem writes that the existence of a distribution is "quite trivial" (Skolem, 1957 p.59). He proceeds to give "the simplest procedure" (p.59):

1. $(a_1, b_1) = (1, 2)$
2. For $n \geq 2$,
    a. a_n is the least integer different from all a_r and b_r for $r = 1, 2, \ldots, n-1$
    b. $b_n = a_n + n$.

Step 2(a) ensures that all integers are eventually covered. Step 2(b) ensures that the required differences are covered. I list the first 19 of these pairs:
(1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15), (11, 18), (12, 20), (14, 23), (16, 26), (17, 28), (19, 31), (21, 34), (22, 36), (24, 39), (25, 41), (27, 44), (29, 47), (30, 49).

Skolem remarks on his surprise when he discovered that the pairs can be given by a simple formula:

$$a_n = \left\lfloor \tfrac{1}{2}(1+\sqrt{5})n \right\rfloor \text{ and } b_n = \left\lfloor \tfrac{1}{2}(3+\sqrt{5})n \right\rfloor \qquad (*)$$

where $\lfloor x \rfloor$ = greatest integer not exceeding x.

He further shows that if α is the positive root of the equation $x^2 = x + 1$, then $\alpha = \tfrac{1}{2}(1+\sqrt{5})$ and $\alpha^2 = \tfrac{1}{2}(3+\sqrt{5})$.

The matter starts to be complicated here. On Page 59, Skolem states that he will prove (*) in Theorem 3a, which appears two pages later on Page 61. In the proof of Theorem 3a, Skolem relies on Theorem 4, which appears four pages later on Page 65. We now return to my student's confusion by reproducing Theorem 3a and its proof (Skolem, 1957 p.61) below.

> ***Theorem 3a*** Every positive integer is one and only one of the two forms $\lfloor \alpha n \rfloor, \lfloor \alpha^2 n \rfloor$, where *n* denotes some positive integer. Further, the pairs obtained by the procedure explained above are just the pairs $\left(\lfloor \alpha n \rfloor, \lfloor \alpha^2 n \rfloor\right)$.
>
> <u>*Proof*</u>
> My first proof of this theorem was based on the preceding four lemmas. However, a reproduction of this proof here is superfluous because it is easily verified that the second statement in Theorem 3a is a special case of Theorem 4, which is proved below. Then the first proposition in Theorem 3a is proved by the simple argument: Since every integer is of one of the two forms $\lfloor \alpha n \rfloor$ or $\lfloor \alpha^2 n \rfloor$, the least integer which does not belong to any of the pairs $\left(\lfloor \alpha r \rfloor, \lfloor \alpha^2 r \rfloor\right)$, $r = 1, 2, \ldots, n-1$, must occur as the least integer in the pairs $\left(\lfloor \alpha s \rfloor, \lfloor \alpha^2 s \rfloor\right)$ for $s = n, n+1, \ldots$. It is then evident that a_n is just this number, which means that the pairs obtained by the recursive procedure explained above are just the pairs $\left(\lfloor \alpha n \rfloor, \lfloor \alpha^2 n \rfloor\right)$.

The final piece of information is rewritten below.

Theorem 4 $\{\lfloor \alpha n \rfloor \mid n$ is a natural number$\}$ and $\{\lfloor \beta n \rfloor \mid n$ is a natural number$\}$ are complementary subsets of the set of all natural numbers if $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

From an education point of view, we can see that confusion occurs when one has not cared to read with enough detail:

- What do some of these terms mean: the procedure explained above? β? complementary subsets?
- A lot of blanks to fill in: simplest; it is easily verified; proved by the simple argument; is then evident.
- The use of results before they are proved: Theorem 3(a) before (*); Theorem 4 before Theorem 3a.

This confusion does not mean that the paper is bad. Indeed, the results are very interesting and the proofs enlightening. The paper requires and deserves detailed reading. Of course, what a mathematician educator can pick up from the confusion is how to write in a more linear way, giving necessary details at the appropriate place.

I will now clarify the error and explain the proof of Theorem 3a.

After reading in detail, the egregious error surfaced easily: "first" and "second" were wrongly switched! Thus, that section of the proof should read:

> However, a reproduction of this proof here is superfluous because it is easily verified that the ~~second~~ first statement in Theorem 3a is a special case of Theorem 4, which is proved below. Then the ~~first~~ second proposition in Theorem 3a is proved by the simple argument:

A definition first: Two sets, *A* and *B*, are complementary subsets of a set *X* if and only if *A* and *B* are disjoint and every element in *X* either belongs to *A* or to *B*. For example, the set of positive even integers and the set of positive odd integers are complementary subsets of the set of natural numbers. Now, the proof can be explicated as follows, with superfluous statements removed and necessary explanations expanded.

Proof

The first statement in Theorem 3a (every positive integer is one and only one of the two forms $\lfloor \alpha n \rfloor, \lfloor \alpha^2 n \rfloor$) is a special case of Theorem 4. We can see this by letting $\alpha = \alpha$ and $\beta = \alpha^2$. Since α is the positive root of the equation $x^2 = x + 1$, we have $\alpha^2 = \alpha + 1$ and so $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{\alpha^2} = \frac{\alpha+1}{\alpha^2} = \frac{\alpha^2}{\alpha^2} = 1$. This leads us to the result that $\{\lfloor \alpha n \rfloor \mid n \text{ is a natural number}\}$ and $\{\lfloor \alpha^2 n \rfloor \mid n \text{ is a natural number}\}$ are complementary subsets of the set of all natural numbers. Thus, every positive integer is one and only one of the two forms $\lfloor \alpha n \rfloor, \lfloor \alpha^2 n \rfloor$.

We now turn to the second proposition in Theorem 3a: the pairs obtained by the procedure explained above ("the simplest procedure") are just the pairs $\left(\lfloor \alpha n \rfloor, \lfloor \alpha^2 n \rfloor\right)$. We have the following about the pairs $\left(\lfloor \alpha n \rfloor, \lfloor \alpha^2 n \rfloor\right)$:

- $\left(\lfloor \alpha 1 \rfloor, \lfloor \alpha^2 1 \rfloor\right) = \left(\left\lfloor \tfrac{1}{2}(1+\sqrt{5}) \right\rfloor, \left\lfloor \tfrac{1}{2}(3+\sqrt{5}) \right\rfloor\right) = \left(\lfloor 1.618... \rfloor, \lfloor 2.618... \rfloor\right) = (1, 2)$.
- We have proven the first statement that every integer is of one of the two forms $\lfloor \alpha n \rfloor$ or $\lfloor \alpha^2 n \rfloor$. Thus, the least integer which does not belong to any of the pairs $\left(\lfloor \alpha r \rfloor, \lfloor \alpha^2 r \rfloor\right)$, r = 1, 2, …, n–1, must occur as the least integer in the pairs $\left(\lfloor \alpha s \rfloor, \lfloor \alpha^2 s \rfloor\right)$ for s = n, n+1, …. Hence αn is just this number as $\alpha n < \alpha s$ for $n < s$, and $\alpha s < \alpha^2 s$ for all s. This is exactly how a_n is obtained in "the simplest procedure".
- Also, we have $\lfloor \alpha^2 n \rfloor = \lfloor (\alpha+1) n \rfloor = \lfloor \alpha n + n \rfloor = \lfloor \alpha n \rfloor + n$. This is also how $b_n = a_n + n$ is obtained in "the simplest procedure".

Together, these mean that the pairs obtained by "the simplest procedure" are just the pairs $\left(\lfloor \alpha n \rfloor, \lfloor \alpha^2 n \rfloor\right)$.

This section explains how mathematician educators choose what to read and how much to read. There is an enormous amount of knowledge in the world, too much for a human mind to consume in one lifetime. By honing the ability to read and choosing what to read, a mathematician educator can optimize his mathematical knowledge for research and for education.

The student can also learn that errors lurk in mathematics texts. In a reputable journal, these are often not fatal but some famous fatal errors have been documented, such as Wiles' initial proof of Fermat's Last Theorem (see Singh, 1997) and Kempe's 'proof' of the four colour theorem (see Wilson, 2003). Unless one can absolutely understand a statement, it is good practice to make sure one understands every term in that statement and try to explicate the statement in one's own words.

Conclusion

This article traces the various levels of encounter or engagement that a mathematician educator has with a mathematics journal paper. It mixes the mathematics with its use in educating learners of mathematics and in explaining the workings of a mathematician. It attempts to work in a new genre of mathematics education where an accepted axiom would be that the claims of mathematician educators about the workings of their mind would be given face validity at the least. This is in the belief that mathematician educators have much to share but their works or their views are not getting published enough in mathematics education journals.

There are still mathematical nuggets in Skolem's paper, which I have not been able to discuss. The paper is now the final year project of a mathematics undergraduate in NIE.

References

Carlson, M. P. & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics 58*, 47–75.

Gardner, M. (2006). *The colossal book of short puzzles and problems.* W.W. Norton, NY: New York.

Ho, W. K., Toh, P. C., Tay, E. G., Teo, K. M., Shutler, P. M. E., & Yap, R. A. S. (2017). The role of computation in teaching and learning mathematics at the tertiary level. In Kim, D., Shin, J., Park, J., & Kang, H. (Eds.), *2017 ICJSME Proceedings: Future directions and issues in mathematics education* (pp. 441-448). Seoul, Korea: KSME.

Leong, Y.H., Tay, E.G., Toh, T.L., Quek, K.S., & Dindyal, J. (2011). Reviving Polya's "Look Back" in a Singapore school. *Journal of Mathematical Behavior, 30*(3), 181-193.

Leong, Y.K. (1995). Mathematics is a language. *Mathematics Medley 22*(September), 58-61.

Low, S.K., Tay E.G. and Chen, D.T. (1999). If you can't find it, make it yourself!: Producing interactive instructional materials using a spreadsheet. In Waas, M. (Ed.), *Enhancing Learning: Challenge of integrating thinking and information technology into the curriculum* (pp. 210 - 215). Singapore: Educational Research Association.

Pólya, G. (1957). *How to solve it* (2nd ed.). USA: Princeton University Press.

Rowland, T. (2003) Mathematics as human activity: A different handshakes problem. *The Mathematics Educator 7*(2), 55-70.

Schoenfeld, A. (1985). *Mathematical problem solving.* Orlando, FL: Academic Press.

Skolem, T. (1957). On certain distributions of integers in pairs with given differences. *Math. Scand. 5,* 57-68.

Singh, S. (1997). *Fermat's Last Theorem*. UK: Fourth Estate Limited.

Tay, E. G. (2001). Reading Mathematics. *The Mathematics Educator, 6*(1), 76-85.

Tay, E. G. (2019). Teaching Mathematics: A modest call to consider Big Ideas. In Toh, T. L. & Yeo, J. B. W. (Eds.), *Big ideas in mathematics: Yearbook 2019* (pp. 129-138). Singapore: World Scientific.

Tay, E. G., Toh, P. C., Dindyal, J., & Deng, F. (2014). Tony's story: Reading Mathematics through problem solving. In Nicol, C., Oesterle, S., Liljedahl, P., & Allan, D. (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education.* (pp. 225-232). Vancouver, Canada: Psychology of Mathematics Education (PME).

Wilson, R. (2003). *Four colours suffice: How the map problem was solved*. UK: Gardner Books.

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