ASSOCIATION OF MATHEMATICS EDUCATORS

Website: http://math.nie.edu.sg/ame/

Maths Buzz

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President's Message

Dear AME Members,

This is the second issue of MathsBuzz that I am writing a message as the President of the AME for the period 2010 – 2012. This is an eventful year for the AME, as we have different activities that cater to the professional needs of different groups of mathematics teachers.

In March 2012, we have organized a problem solving seminar for primary school teachers conducted by Dr Yeap Ban Har. About 200 teachers attended the event, despite the fact that the seminar fell on an afternoon of a normal working day.

This is the first time that AME is organizing an event jointly with the Singapore Mathematical Society (SMS) for the mathematics teachers – the AME-SMS Conference 2012, which is hosted by NUS High School. The theme of the conference is "Nurturing reflective learners" – one that is broad enough to engage both mathematics educators and mathematicians to come together and offer valuable knowledge of content and pedagogy to the Singapore teachers. The joint organization of the two professional bodies AME and SMS also conveys the message that both content and pedagogy are important for one to be a good mathematics teacher. To date, more than 500 teachers have signed up for the event.

AME also jointly organizes with the Academy of Singapore Teachers (AST) her first AME institute for primary and secondary school teachers. The AME Institute is a series of focused thematic 10-hour workshops conducted by overseas mathematics education experts.

AME will be organizing a modeling seminar in September 2012 for secondary school teachers. More information will be available on the AME website http://math.nie.edu.sg/ame once details are finalized.

Before the term of office ending in May 2012, I wish to express my heartiest appreciation to the AME executive committee members who share the same vision of reaching out to the professional needs of mathematics teachers and who have worked very hard to make all AME events successful.

Hopefully the year 2012 is one which we acquire rich learning experiences as we continue with our journey to make learning of mathematics meaningful to all students.

Toh Tin Lam
President,
AME (2010 – 2012)

Contents

AME Activities 2012

No.	Seminar/Workshop/ Conference	Speaker/s	Date/Time	Venue	Joint Organiser	Audience
1	PSLE problem solving seminar	Dr Yeap Ban Har	22 Mar 2.30 to 5.30 p.m.	SP Auditorium	-	Primary teachers/ educators
2	1.5 day summer school workshop on "task design". • AME Institute for Mathematics (Secondary) Teachers • AME Institute for Mathematics (Primary) Teachers	• Prof John Mason from UK • Prof Anne Watson from UK	28th & 29th May	AST	AST	Primary & secondary teachers
3	AME-SMS Conference 2012	Local & overseas speakers	30th May 8.30 to 5p.m.	NUSH	NUSH	Primary, secondary & JC teachers
4	Secondary Teachers Modelling Seminar	A/P Ang Keng Cheng.	To be confirmed	To be confirmed	To be confirmed	Secondary teachers

Five Alternative Solutions for One Word Problem

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1. Introduction

In a typical mathematics lesson, most teachers tend to teach the students a particular strategy or rule, show examples of how to solve the problems using the strategy and then for practice, provide students with many similar problems. However, children and some older students, found it difficult when confronted with apparently complex mathematical problems such as those involving algebraic manipulations. As the children may still be operating at the concrete-operational stage (Piaget, 1950); hence, it is necessary to teach the students strategies to assist them to cope with abstract mathematical concepts and to observe a range of possibilities and alternatives in solving mathematical problems (Fong, 1997).

The integration of problem solving into the mathematics curriculum will require an effective pedagogy for instruction of problem solving. Research results strongly support the explicit instruction of heuristics and a wide variety of strategies as one aspect of such pedagogy (Anderson & Holton, 1997; Hembree, 1992; Schoenfeld, 1982). If mathematics teachers continue to provide only one or two solutions to a mathematical problem in the classroom, students may approach a word problem in a mindless, superficial and routine-based way. Thus by showing and analysing alternative solutions, mathematics teachers can create new learning opportunities for students. Teachers could ask students to present each solution and discuss it with the whole class. Most importantly, students could be guided to analyse and compare these different solutions. An analysis of these solutions can quickly reveal the advantages of asking students to explore different ways of solving the same problem. Through comparing and reflecting on alternative solutions, a many-sided view approach is fostered in the students' thinking. This helps to establish new mathematical knowledge for the students. Therefore, it is crucial to expose students to various strategies so as to enable them to deal with abstract mathematical concepts and to observe a wide range of possibilities and alternatives in solving mathematical problems. In this regard, I have chosen a word problem that is applicable to primary 5 to secondary 2 students and illustrate the use of various strategies to the problem.

Moreover, abstractions and concepts in this word problem are simplified so that these students can understand the abstractions. Simplification is effected through the use of familiar and systematic listing as well as simple logical deductions. This is being considered because some primary 5 and 6 students who may still be functioning at the concrete-operational stage may encounter difficulty when confronted with apparently complex mathematical problems such as those involving algebraic processes. This article discusses how this can be done through the illustrative use of one word problem and five different strategies, involving algebra as well as familiar strategies used by different secondary 2 students.

2. The Problem

The following is a word problem that primary 5 to secondary 2 students may be tasked to solve:

A Mathematics quiz consists of 20 multiple-choice questions. A correct answer is awarded 5 marks and 2 marks are deducted for a wrong answer while no marks are awarded or deducted for each question left unanswered. If a boy scores 48 marks in the quiz, what is the greatest possible number of questions he answered correctly?

This particular problem may be considered as a routine problem for some secondary 2 students as they would have experienced solving such word problem. However, it may be considered as a non-routine problem for primary 5 and 6 students, if alternative strategies are employed without the use of algebra. In the following sections, strategy 1 shows a standard strategy of using algebra while strategy 2 to 5 exemplifies the alternative, non-routine ones.

3. The Five Alternative Solutions

The students' written solutions were analysed according to the types of strategies used. Five major types of strategies were identified: simultaneous linear equations, logical reasoning, modeling (systematic counting), make a list as well as guess and check. The coding of the solutions was based solely on the submitted written work, as none of the students were interviewed.

3.1 Strategy 1 - Algebra (Simultaneous Linear Equation)

Secondary 2 students are familiar with the algebraic strategy of forming linear equations. In the traditional algebraic approach, two variables are needed to first represent number of correct and wrong answers. In most instances, secondary 2 students can apply this standard algebraic strategy by translating the problem statements into two linear algebraic equations and solve them simultaneously as shown in Figure 1.

Although secondary 2 students are able to solve the problem by forming algebraic equations, some may simply applying procedures that they have been taught without any in-depth understanding of the concepts behind the procedures. The following are four alternatives strategies that could be learned by primary 5 to secondary 2 students. These strategies will assist them acquire the skill of examining a problem from various perspectives and identifying or selecting the most effective strategies.

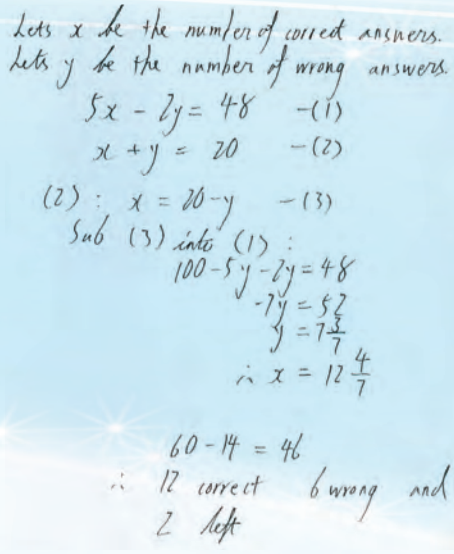

Figure 1: A student's solution using simultaneous linear equations

3.2 Strategy 2 – Logical Reasoning

A logical solution is to note that the number of questions answered correctly would be an even numbers as overall score in the quiz is 48 marks. The single step, (12 x 5 – 6 x 2) = 48, and the logical argument indicated that the student had a deep understanding of the problem scenarios and structure. This is shown in Figure 2.

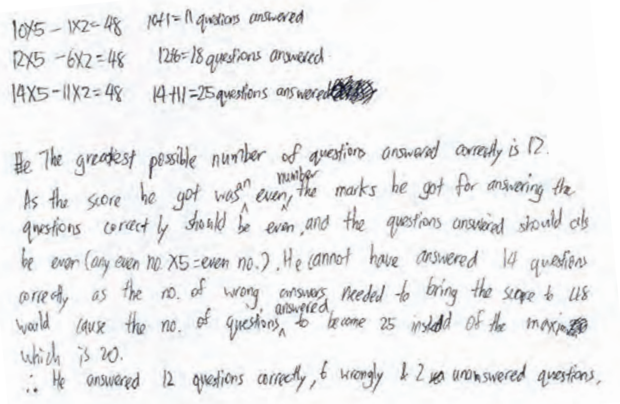

Figure 2: A student's solution using logical reasoning

3.3 Strategy 3 – Modelling (Systematic Counting)

The modelling aspect of the word problem involving systematic counting is shown in Figure 3. This was illustrative in the sense that they were used together with number manipulation as an initial step to understand the problem. To assist students understand the concepts behind the problem, systematic counting will help them to visualise the abstractions inherent in the problem. Students somehow view the problem as being more concrete and explicit and hence, less abstract.

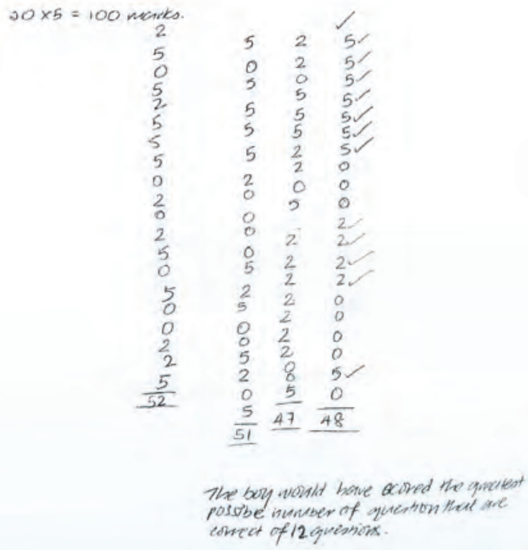

Figure 3: A student's solution using modelling (systematic counting)

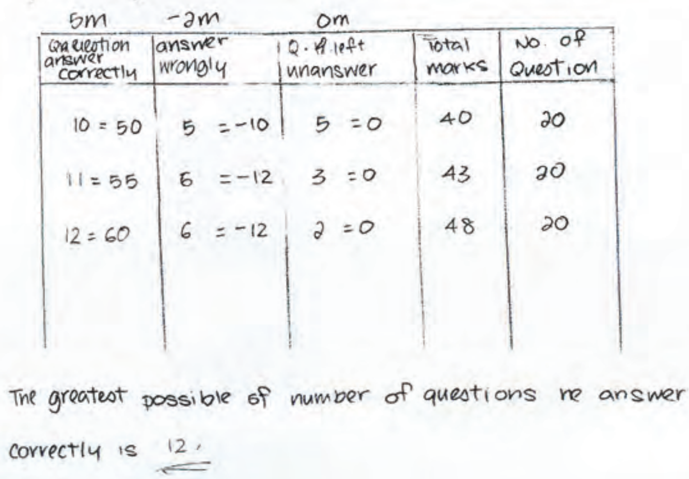

Figure 4: A student's solution using make a list

3.4 Strategy 4 –Make a List

A systematic listing was indicated by the use of at least three consecutive items, with a difference of 1 or a constant value. Making a list would be clearer if the student organizes data into a table and then uses it to solve the problem. The example of making a list is shown above in Figure 4. This strategy could be shown to primary 5 to secondary 2 students. The student was able to construct an organized list containing all the possibilities for conditions stated in the problem. However, in this instance, the equality sign in the given solution is not used correctly.

3.5 Strategy 5 - Guess and Check

The given problem can also be solved using the guess and check strategy. The term guess and check strategy indicates the requirement to guess a number and then to check whether the constraint is satisfied. In fact, the strategy works completely parallel with "generating numbers", but since no "starting number" is available, the student has to guess its value, to check the correctness of the guess, and to repeat this process of guessing and checking until they finally arrive at the correct value of the unknown. In this case, the student made a first guess of the number of correct answers as 15 and number of incorrect answer as 5 and performed the necessary calculation to check if the total score is 48 marks (see Figure 5). Within this strategy, one can either make subsequent "random" guesses, or apply a "try-and-improve" approach wherein the student actively reflects on the outcome of previous guess and used it to make better guesses in the next trials.

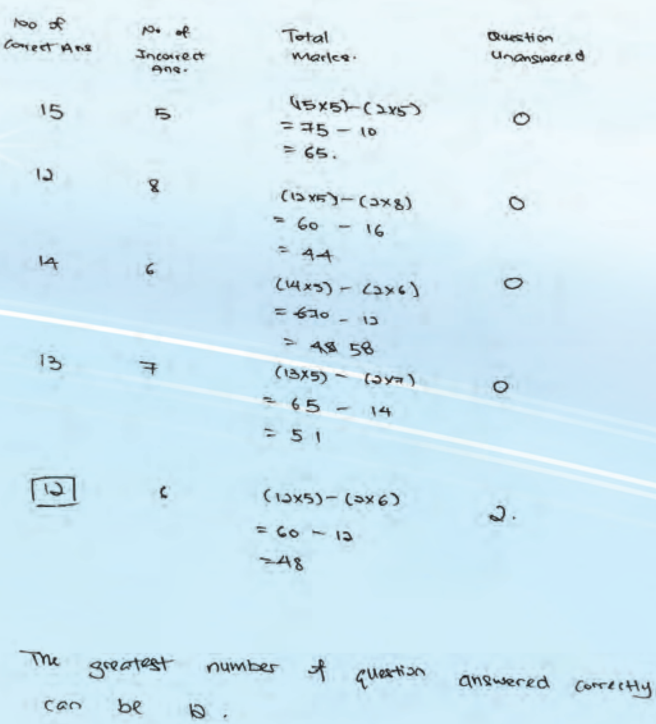

Figure 5: A student's solution using guess and check

4. Conclusion

I have discussed five different strategies used by students in solving the given non-routine problem. Mathematics teachers are often encouraged to show different strategies to solve a given problem. The intent is to promote creative thinking among the students and to dispel the common belief that a mathematics problem always has only one correct strategy. The implication for teaching is that, with several strategies at hand, mathematics teachers may be able to choose and show the most suitable strategy to address individual differences among the students. This will also provide opportunities for the students to think a little more about the problem and allow them to learn from other students' way of solving the problem. To some students, this new solution may be a more elegant solution than their own original solution; thus this may give more insights into the diverse metacognitive processes of solving this problem.

Teachers may think of different strategies for solving the same problem but more importantly, discussing strategies actually used by their peers adds a more realistic feel and touch to the lessons. Examples can be drawn from the strategies discussed above. In addition, students can be asked to evaluate solutions that are partially correct, so that they can learn from the errors made by other students. Many of these partially correct solutions conceal flaws in mathematics content knowledge and strategy use. Mathematics teachers will certainly agree that unpacking these flaws together with the students will engage them in extending their thinking. The teachers should find this information useful in planning lessons that address the diversity of students' responses.

5. References

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Enriching A Lesson on Pythagoras' Theorem

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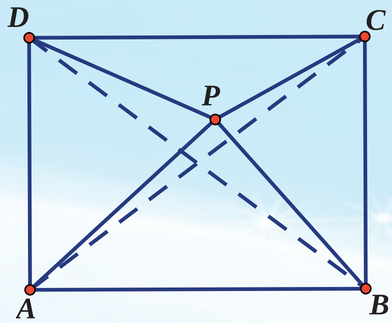
Pythagoras' Theorem is one of the oldest and most frequently used Mathematics theorems which can be proved in many different ways. A fundamental and important theorem that is studied in many Mathematics curricula around the world, Pythagoras' Theorem is also included in our Secondary 2 Mathematics syllabus. All text books provide quite a good number of examples and problems for teachers and students in the teaching and learning of this theorem. However, there are other interesting applications of this theorem that may enrich our students' learning experiences, provide them with opportunities to delve deeper and thus enabling them to gain a different perspective of the theorem. In this write-up, I would like to share some teaching ideas and examples that may excite students and hopefully inspire them to explore and investigate beyond what they have learnt from the text books.

Using GSP to Enrich a Lesson on Pythagoras' Theorem

Generally, students tend to remember what they have learnt if they discover or construct the knowledge themselves and make sense of what they are taught. In this respect, learning activities promoting investigation and exploration is one approach that can be used in our lesson to motivate and engage them deeply.

Here is a simple construction that a teacher can use to enrich the lesson on the application of Pythagoras' Theorem. In the following diagram, *ABCD* is a rectangle and *P* is a point inside the rectangle.



With the use of a Dynamic Geometry Software (DGS) such as GSP or Geogebra, students can work on teacher's pre-constructed DGS file that allows them to drag and observe the results on the computer screen. As a warm-up activity, the teacher can ask the students to investigate if $PA + PC = PB + PD$ in general. Where can they find the point P such that $PA + PC = PB + PD$? Is $PA^2 + PC^2 = PB^2 + PD^2$ always true?

Students should be able to observe that $PA^2 + PC^2 = PB^2 + PD^2$ whenever P is inside or on the perimeter of the rectangle.

Once they have discovered that $PA^2 + PC^2 = PB^2 + PD^2$, teachers can then scaffold and guide them to prove this result by using the Pythagoras' Theorem as shown below.

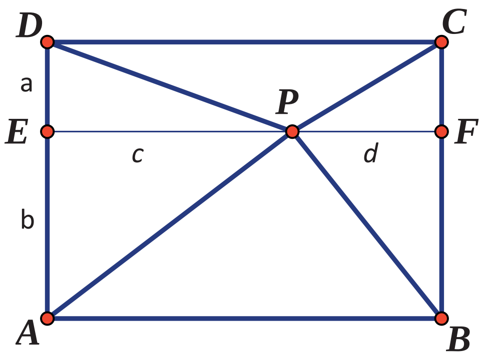


Figure 1

Through P, a line segment EF which is parallel to DC can be constructed as shown in Figure 1.

Let $ED = a$, $EA = b$, $PE = c$ and $PF = d$. By Pythagoras' Theorem, $PD^2 = a^2 + c^2$, $PB^2 = b^2 + d^2$, $PC^2 = a^2 + d^2$ and $PA^2 = b^2 + c^2$. Thus the sums $PA^2 + PC^2$ and $PB^2 + PD^2$ are both equal to $a^2 + b^2 + c^2 + d^2$. Thus we have proved that $PA^2 + PC^2 = PB^2 + PD^2$. In addition, this easy proof can serve as an introduction to the notion of proof in Mathematics for Secondary 2 students.

This particular learning activity can be further enhanced by guiding students to use the result to solve a non-standard problem as shown below.

> M is a point in a unit square $ABCD$ such that $MA^2 - MB^2 = \frac{1}{2}$ and $\angle CMD = 90^\circ$.
>
> Find the length MD.
> Suggested solution : Using the above result, we have $MA^2 + MC^2 = MB^2 + MD^2$
> Thus , $MA^2 - MB^2 = MD^2 - MC^2 = \frac{1}{2}$. As triangle CMD is right-angled, we have
>
> $MD^2 + MC^2 = 1$. Adding to eliminate MC^2 , we have $2MD^2 = \frac{3}{2}$ and $MD = \frac{\sqrt{3}}{2}$.

Connecting Pythagoras Theorem to the visualisation of $\sqrt{a+b}$ and $\sqrt{a}+\sqrt{b}$

Mathematical errors such as $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$ (other similar errors are log(x + y) = log x + log y, $(x + y)^2 = x^2 + y^2$ etc) which Matz (1980) classified as linear extrapolation error . This is a class of errors that Secondary school students commonly make. Besides using a numerical example to verify that $\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$, we can use Pythagoras' Theorem to give students another perspective of $\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$.

Let me illustrate using the numbers $\sqrt{2+8} = \sqrt{10}$ and $\sqrt{2}+\sqrt{8}$.

Figure 2 shows a square grid. Assume that the side of each square is 1 unit.
By using Pythagoras Theorem, AB = $\sqrt{1+1} = \sqrt{2}$ units and BC = $\sqrt{2^2+2^2} = \sqrt{8}$.
Thus the length of AC is $\sqrt{2}+\sqrt{8}$.
Similarly, AD = $\sqrt{1+3^2} = \sqrt{10} = \sqrt{2+8}$.
But clearly, AD ≠ AC, that is $\sqrt{2+8} \neq \sqrt{2}+\sqrt{8}$.

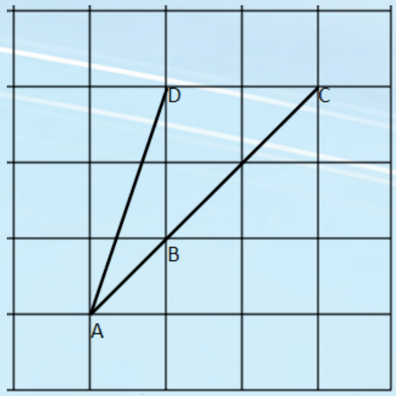


Figure 2

Connecting Pythagoras' Theorem With an Inequality

$x+\frac{1}{x}\geq 2$ for all real $x > 0$ is an inequality that can be proved easily by considering $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 \geq 0$. Pythagoras' Theorem in fact provides us with an interesting way of looking at the inequality for $x \geq 1$. With little experimentation, Secondary 2 students should be able to notice that $(x-\frac{1}{x})^2+2^2 = x^2+2+\frac{1}{x^2} = (x+\frac{1}{x})^2$. So what is the implication? Yes, we can form a right-angled triangle with sides 2, $x-\frac{1}{x}$ and $x+\frac{1}{x}$ as shown in *Figure 3*. But in order to illustrate how Pythagoras' Theorem can be used to show the inequality, we consider $x \geq 1$ so that the term $x-\frac{1}{x}\geq 0$.

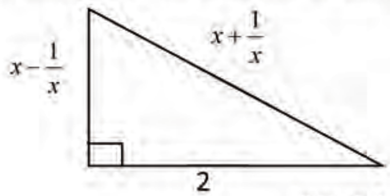


Figure 3

Since the hypotenuse being the longest side, we have $x+\frac{1}{x}>2$. When the height is zero (when $x = 1$), the hypotenuse coincides with the base, thus $x+\frac{1}{x} = 2$. This example provides a way for students to visualize that $x+\frac{1}{x}\geq 2$ in the case when $x \geq 1$.

Connecting Pythagoras' Theorem With Irrational Numbers

It is well known that by using Pythagoras' Theorem, we can construct the length of $\sqrt{n}$ where n is a positive integer as shown in *Figure 4*, in which each triangle is a right-angled triangle.

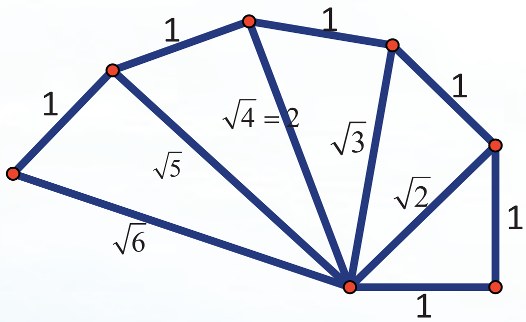


Figure 4

Can we construct a line segment to represent an irrational number $\sqrt{x}$ where x is a positive rational number? In a right-angled triangle, let the hypotenuse be c, the base be a and the height be b.

Let $a+c=\frac{x}{n}$ and $c - a = n$, where $0 < n < \sqrt{x}$.

By Pythagoras Theorem, $b^2 = c^2 - a^2 = (c + a)(c - a)$, we have $b^2 = x$. Thus $b = \sqrt{x}$.

As an example, we can then use this result to construct the length of $\sqrt{10.4}$. First of all, we choose the value of n such that $0 < n < \sqrt{10.4}$. For simplicity, let $n = 1$, then $a + c = 10.4$ and $c - a = 1$. Solving, $c = 5.7$ and $a = 4.7$. So the right-angled triangle with base $a = 4.7$ and the hypotenuse $c = 5.7$ will have the height $b = \sqrt{10.4}$. In this activity, the students can easily construct the height using a pair of compasses and a ruler.

If we let $n = 2$, then $a + c = 5.2$ and $c - a = 2$. Solving, $c = 3.6$ and $a = 1.6$. We realize that there are more than one way to construct the length of $\sqrt{10.4}$.

Conclusion

Teachers should provide students more than one class of examples and questions for them to appreciate that Pythagoras Theorem can be applied in a great variety of ways. In doing so, students' learning experiences of the theorem will be enriched and thus providing impetus for students to make mathematical connections and develop a deeper understanding of the theorem**.**

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Infusing Metacognition in the Teaching of Calculus

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Knowledge of cognition refers to (i) what individuals know what about their cognition, (ii) how to use strategies and other procedures, and (iii) why and when to use a particular strategy. Metacognition is defined as the knowledge and regulation of cognition (Brown, 1978, Flavell, 1979, Schraw, 2001). Regulation of cognition typically includes planning, monitoring, and evaluation.

Research in mathematics education has documented consistently a positive relationship between metacognition and mathematics performance (for example, Schneider and Artelt, 2010). Intervention programs have been developed with the aim to improve children's metacognitive knowledge as well as their metacognitive skills. A key finding from most of the intervention programs generally shows that students with training in their metacognitive awareness generally performed better than those not exposed to such training.

Problem solving has been the heart of the Singapore mathematics curriculum since the 1980s. According to the framework (*Figure 1*), it is flanked by five equally important components represented by the five sides of the pentagon. One of these essential components of problem solving is metacognition, for which the essential features are "Monitoring of one's own thinking" and "Self-regulation of learning".

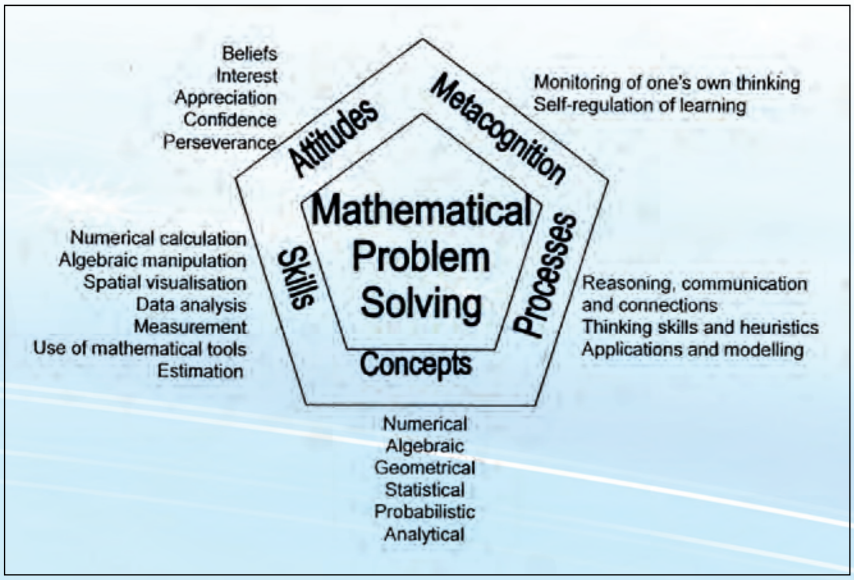

Figure 1. Framework of the school mathematics curriculum

Of the five components, it is a truism that teachers generally focus on three components; *Skills*, *Concepts* and *Processes*, which are directly related to the students' high-stake national examinations. Generally, students are not assessed on Attitudes and Metacognition, in any typical paper-and-pencil test.

On this note, the author wishes to highlight that the infusion of metacognition in teaching traditional school mathematics could deepen students' understanding of mathematical concepts and the relation across concepts. The topic calculus is chosen for discussion.

Mathematics – even the most procedural tasks which are seemingly meaningless to students should be taught in such a way that it makes sense to students. Mathematics lessons should provide students with ample opportunities to seek and find explanations for their observations. One way of engaging students in reasoning process and regulating their own learning processes is to have them examine and explain an error (Carroll, 1999). The two examples below serve to illuminate this point.

Example 1. Use of students' errors for their own learning

Learning differentiation techniques is usually seen as a rather procedural task without much meaning to students. However, acquiring such techniques is an essential component of learning calculus. The rules of differentiation (e.g. addition rule, product rule, chain rule, quotient rule), even if acquired, are easily forgotten.

Teachers could collate examples of sample students' errors in applying differentiation rules and turn this into a worksheet which requires students to identify and explain the errors. A sample of the activity is shown in Activity 1. This activity demonstrates that, in addition to the standard practice questions on applying the various differentiation techniques, students could be engaged in explaining the errors based on the various rules of differentiation. The comment column in Activity 1 provides opportunity for students to reflect on the errors in each of the statements of differentiation in the worksheet. This provides opportunity for them to regulate their own learning and have a deeper appreciation of the various differentiation techniques. Teachers could design similar tasks for other topics, e.g. integration techniques, within calculus and beyond. A sample of students' comments is shown in Activity 1. However, teachers are reminded that the sample of students' comments should NOT be treated as a "model answer" which students must conform; rather, it should be seen as the reasoning we would expect students to be engaged in such an higher order thinking task.

Example 2. Use of close confusers

Teaching of the various techniques of differentiation is done sequentially. Students are usually given practice questions after learning each new differentiation technique. These practice questions almost always focus on the newly acquired rule. Opportunity should be provided for the students to have a closer examination of the various rules after learning these rules in different segments. The author suggests that a close confuser exercise, as demonstrated in Activity 2, could be used. It is clear from the different questions in Activity 2 that every part of the question requires a different differentiation technique. This activity provides students to examine all the rules of differentiation techniques more carefully. It should be remarked that it is not difficult for teachers to generate such tasks for their students.

Activity 1

What went wrong??? Spot the mistake – how should the correct solution be? Fill in the comment section in the space provided.

WORKING	COMMENT
$\frac{d}{dx}(2x+3)^3 = 3(2x+3)^2$	Used the wrong formula. Must multiply by 2. $\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}a$
$\frac{d}{dx}(2x)^4 = 4(2x)^3 = 32x^3$	Should use the formula $\frac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}a$. With $a = 2$, $b = 0$, $n = 4$. He can also simplify it to $16x^4$ and then get $64x^3$.
$\frac{d}{dx}\sin^2 x = 2\sin x$	Forgot to differentiate the sin x. Used wrong formula. Should be $\frac{d}{dx}(\sin x)^n = n(\sin)^{n-1}\cos x$. Forgot a $\cos x$ outside.
$\frac{d}{dx}(3\cos x) = 0 \times (-\sin x) = 0$	No! Keep the "3". The correct formula is $\frac{d}{dx}(3\cos x) = 3(-\sin x) = -3\sin x$.
$\frac{d}{dx}x\ln x = x\left(\frac{1}{x}\right) = 1$	Must use product rule. Keep x, and differentiate ln x. Then must add to keeping ln x and differentiate x.

WORKING	COMMENT
$\frac{d}{dx}C^2x^3 = 2Cx^3 + 3C^2x^2$, where C is a constant.	Differentiate with respect to x, so C^2 is constant. Correct formula is $3C^2x^2$. Differentiate C^2 is zero!
$\frac{d}{dx}2^{3x} = (3x)2^{3x-1}$	When we use the formula $\frac{d}{dx}(x)^n = nx^{n-1}$, the number n is a constant. Here, the power is not a constant. Must use the formula for $\frac{d}{dx}a^{f(x)} = f'(x)a^{f(x)}\ln a$
$\frac{d}{dx}\left(\frac{x+2}{x+1}\right) = \frac{d}{dx}(x+2)(x+1)$ $= (x+2)+(x+1) = 2x+3$	This is not correct. Must write $(x + 1)^{-1}$ and then apply product rule. We can also use Quotient Rule directly.
$\frac{d}{dx}\left(\frac{x+2}{x+1}\right) = \frac{d}{dx}(x+2)(x+1)^{-1}$ $= (x+2) - 1(x+1)^{-2} + (x+1)^{-1}(1)$ $= x+2-\frac{1}{(x+1)^2}+\frac{1}{x+1}$	The last step was wrong. There should be a pair of bracket for -1 in the second step.

Activity 2

Higher Order Thinking Skills – Close Confusers

Now that you have mastered your differentiation techniques, try to differentiate the following functions with respect to x. Explain the ***difference*** in your processes for ***each pair of problems*** under the ***comments.***

a)	$\sin 2x$	b)	$\sin^2 x$
	2 cos 2x		2sin x cos x
Comments: Part (a): $\frac{d}{dx}\sin ax = a\cos ax$; Part (b): $\frac{d}{dx}\sin^a x = a\sin^{a-1} x\cos x$			
c)	$2\sin x$	d)	$\sin^2 2x$
	2 cos2x		2sin2xcos2x (2)=4sin2xcos2x = 2sin4x
Comments: Part (a): $\frac{d}{dx}a\sin x = a\cos x$; Part (b): $\frac{d}{dx}\sin^a kx = ak\sin^{a-1} kx\cos kx$			
e)	$\sin(2x+3)$	f)	$\sin^3(2x+3)$
	2cos(2x+3)		6sin²(2x+3)cos(2x+3)
Comments: Part (a): $\frac{d}{dx}\sin f(x) = f'(x)\cos f(x)$; Part (b): $\frac{d}{dx}(\sin f(x))^3 = 3(\sin f(x))^2\cos f(x) f'(x)$			
g)	$\frac{1}{\sin 2x}$	h)	$\frac{1}{\sin^2 2x}$
	-2cos2xcosec²2x		-4cos2xcosec³2x
Comments: Both parts use the same formula: $\frac{d}{dx}(\sin kx)^{-n} = -n(\sin kx)^{-n-1}(k\cos kx)$ However, in (h), I almost forgot to multiply by n to the whole expression. In part (g) it does not matter since n = 1.			

Conclusion

This note presents sample of two activities on how even procedural tasks in mathematics can also be converted to higher order thinking tasks that enhance metacognition. It is not difficult for teachers to create such meaningful activities for their students in mathematics classrooms.

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Ask Dr Maths Teaching

1. I would like to seek your view on what to leave in the final answer for the following forms: in the exact form with a few terms or round it to 3 significant figure for the Additional Mathematics syllabus:

a) $\ln(1\frac{1}{2}) + 1$

b) $e^3 + e^{-3} + 2$

c) $\sqrt{3} - \frac{\pi}{3}$

Response:

I would write $\ln(\frac{3}{2}) + 1$ and leave the rest in the exact form, or more precisely in the closed form. There are also reasons to write in decimals. I shall explain below.

One hour has 60 minutes. Why 60 minutes? It is because 60 is the smallest number divisible by 2, 3, 4, and 5. Why do we teach fractions in the primary school with denominators not exceeding 12? It is because 12 is the smallest number divisible by 2, 3, and 4. During the days of fractions, 60 or 12 is preferable to 10.

$1\frac{1}{2}$ is a poor notation. It was invented to make good sense of a fraction. Definitely, $1\frac{1}{2}$ is more meaningful or easier to understand than $\frac{3}{2}$. The use of $1\frac{1}{2}$ leads to an error of $\sqrt{\{4\frac{1}{4}\}} = 2\frac{1}{2}$. That is why we said $1\frac{1}{2}$ is a poor notation. That is why we prefer $\ln(\frac{3}{2}) + 1$.

We are used to $\pi = \frac{22}{7}$ or more accurately π as $\frac{22}{7}$. Now we write 3.14 or 3.1416. Now in additional mathematics, we write sin 60^o = 0.866 instead of $\frac{\sqrt{3}}{2}$. For computation, it is easier to use decimals than to use fractions. Also, in the closed form, we do not know the size of the value of an answer. Hence sometimes we want to express an answer in decimals. We do it only at the end. Asking students to find the sum whenever they see an addition of two numbers may not be a wise move. It introduces unnecssary errors, and it makes it harder to check errors, if any, later on.

2. Burning question on estimation (CA: Question in Section B of P4 mathematics booklet; a *2 mark question*)

6380 ÷ 9 = __________ ÷ 9

Is **6300** the only answer?

Should these answers ***6381, 6372, 6399*** given by pupils be accepted? Please advise!

Response:

Increasing use of decimals makes approximation a more important topic in the primary school syllabus. In fact, approximation was proposed to be introduced in the primary school syllabus in Singapore as early as 1969 before many other countries.

Pupils are not expected to know that 6300 is divisible by 9 because the sum of the digits of 6300, namely 6+3+0+0, is divisible by 9. If we accept 6300 as a possible answer for the above reason, then there is no reason not to accept 6381, 6372, and 6399 as answers. The aim is to produce an integer approximation, and not to simplify the computation.

If the reason for the question is to test the multiplication of 9, then a natural answer is 6300. This helps simplify the computation. Therefore we do not proceed further to 6381 etc. The burning question may be that of teachers. It may not be that of pupils.

O-Level Calculus Questions:-

1. In the formula $\frac{d}{dx}(\sin x) = \cos x$, it is stated that *x* must be measured in radian.

I am always puzzled why *x* cannot be measured in degrees.

Response:

From $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, we know that the gradient of the graph of $y = \sin x$ at $x = 0$ is cos 0 = 1. To understand the reason why *x* must be in radians and not in degrees, we shall examine the special case of derivation of the derivative of sin *x* at *x* = 0 from first principles:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)\right)_{x=0} = \lim_{\delta x \to 0} \frac{\sin(0+\delta x) - \sin 0}{\delta x} = \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

Let's work out the values of $\frac{\sin \delta x}{\delta x}$ when $\delta x = 0.0001°$ and when $\delta x = 0.0001$ radians.

We see that when $\delta x = 0.0001°$, $\frac{\sin \delta x}{\delta x} \approx 0.0175$ which is not close to 1 while when $\delta x = 0.0001$ radians, $\frac{\sin \delta x}{\delta x} \approx 1.0000$.

In the formula $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$, *x* must be measured in radians since from the proof of this formula by first principles, we need to use $\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} = 1$ which holds if δx is in radians.

If *x* is in degrees, then converting *x* degrees to radians, we have $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x°) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin \frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\frac{\pi x}{180} = \frac{\pi}{180}\cos x°$.

2. Is the formula $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1}$ true when *n* is a non-integer rational number? What about the case when *n* is an irrational number?

Response:

In fact, the formula $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = nx^{n-1}$ is true for *n* real which includes non-integer rational number and irrational number.

So the formula works for $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{2}}\right)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(x^{\mathrm{e}})$.

One possible way to show that the formula holds is to rewrite x^n as $\mathrm{e}^{\ln(x^n)} = \mathrm{e}^{n \ln x}$, where *n* is any real number.

Then $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{n \ln x}\right) = \mathrm{e}^{n \ln x}\frac{\mathrm{d}}{\mathrm{d}x}(n \ln x) = \mathrm{e}^{n \ln x}\left(\frac{n}{x}\right) = x^n\left(\frac{n}{x}\right) = nx^{n-1}$.

3. In the question:

"Find the coordinates of the stationary point on the graph of $y = (x - 3)^2 + 1$, determine the nature of this stationary point".

My students are able to obtain the coordinates (3, 1). However, one of my students insists on not using the first derivative or the second derivative test. He argues like this:

When *x* = 3, *y* = 1. You see that when *x* is bigger than 3, *y* is obviously greater than 1; when *x* is smaller than 3, *y* is also obviously greater than 1. Hence it must be a minimum point. In fact, he likes to use this type of argument to conclude the nature of stationary points. Is this type of arguments acceptable? Please enlighten me.

Response:

This type of argument is acceptable, though not encouraged, for the graph of a quadratic polynomial as there is only one stationary point on such graph. In general, such arguments are not valid. Consider the graph of $y = (x - 2.99)(x - 3)^3 + 1$. Note that (3, 1) is a stationary point. When *x* is bigger than 3, *y* is obviously greater than 1. When *x* is smaller than 3, if the student chooses a value smaller than 3, say 2, 2.5 or even 2.9, then *y* is greater than 1. The student will then conclude that (3, 1) is a minimum point which is not true as (3,1) is in fact a stationary point of inflexion in this case. The first derivative test requires one to check the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $x = a^-$ and $x = a^+$. So for the example above, the problem arises as there is another stationary point (2.99,1) which is very close to the other stationary point (3,1). So the student should use values of *x* very close to 3, for example, 2.999 and 3.001 to check the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$. By not using the first derivative test or the second derivative test, the student will come to wrong conclusions especially for more complicated graphs and for graphs whereby the coordinates of the stationary points are very close together.

4. Is there any difference between "stationary point" and "turning point"? Please help me.

Response:
There is definitely a difference between "stationary point" and "turning point". In fact, a turning point is a stationary point, but a stationary point is not necessarily a turning point. Maximum points, minimum points and stationary points of inflexion are stationary points whereas only maximum points and minimum points are turning points; a stationary point of inflexion is not a turning point. As what the name suggests, there must be a "turn" at a turning point. So $\frac{dy}{dx} = 0$ at both stationary points and turning points, but at a turning point, you need the addition criteria of a change in sign of $\frac{dy}{dx}$ from positive to zero to negative or from negative to zero to positive in the neighbourhood of the turning point.

5. My students always have this understanding $\frac{d^2y}{dx^2} > 0$ means that the graph has a minimum point while $\frac{d^2y}{dx^2} < 0$ means it has a maximum point. Something is not quite right, but I don't know what is exactly wrong with it.

Response:
Consider the graph of $y = e^x$. Note that $\frac{d^2y}{dx^2} = e^x$ which is always positive. But the graph of $y = e^x$ has no minimum point. Similarly, for $y = \ln x$, $\frac{d^2y}{dx^2} = -\frac{1}{x^2}$ which is negative for all positive x, but the graph of $y = \ln x$ has no maximum point. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ at $x = a$, then we can conclude that $x = a$ is a minimum point. Similarly, if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at $x = a$, then we can conclude that $x = a$ is a maximum point. Your students have omitted the additional condition that $\frac{dy}{dx} = 0$.

6. If $\frac{d^2y}{dx^2} = 0$ at a particular point on a graph, then it must be a point of inflexion. Is this always true?

Response:
Consider the graph of $y = x^4$. Note that $\frac{d^2y}{dx^2} = 12x^2$ which is equal to 0 at $x = 0$ but (0, 0) is a minimum point on the graph of $y = x^4$. Another example is the graph of $y = -(x+1)^6$. Then $\frac{d^2y}{dx^2} = -30(x+1)^4$ which is equal to 0 at $x = -1$ but (-1, 0) is a maximum point on the graph of $y = -(x+1)^6$. So $\frac{d^2y}{dx^2} = 0$ at a particular point on a graph does not imply that the point must be a point of inflexion. You need the additional criteria that there is a change in sign in $\frac{d^2y}{dx^2}$ in the neighbourhood of the point. However, if you know that a particular point on a graph is a point of inflexion, then you can conclude that $\frac{d^2y}{dx^2} = 0$ at that point. We say that "$\frac{d^2y}{dx^2} = 0$" is a necessary but not a sufficient condition for point of inflexion.

Contributions Invited

Electronic submission
You may email your contributions to the following:

- Mathsbuzz.ame@gmail.com – For sharing of teaching ideas or research findings
- askdrmathsteaching.sg@gmail.com – For discussion and clarification of issues related to teaching and learning of mathematics