Assessment in Mathematics: Starting from Error Analysis

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Presenters

Teo Soh Wah	Nan Hua High School
Ong Bee Teng	Anglo-Chinese Junior College
Soh Poh Suan	Clementi Town Secondary School
Yeo Cheng Yong	Fuhua Secondary School
Sim Puay Hoon	Nan Hua High School
Lek Mei Xuan	Shuqun Secondary School
Jasmine Ng SM	Unity Secondary School

Assessment drives student learning.

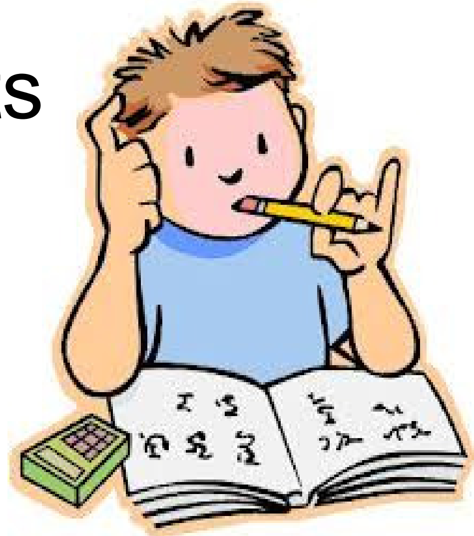
Assessment vs Evaluation

Check & Coach to Excellence!

Check & Grade on Time!

PEAK
LEARNING SYSTEMS

Using Error Analysis to Increase Student Understanding of Math Concepts

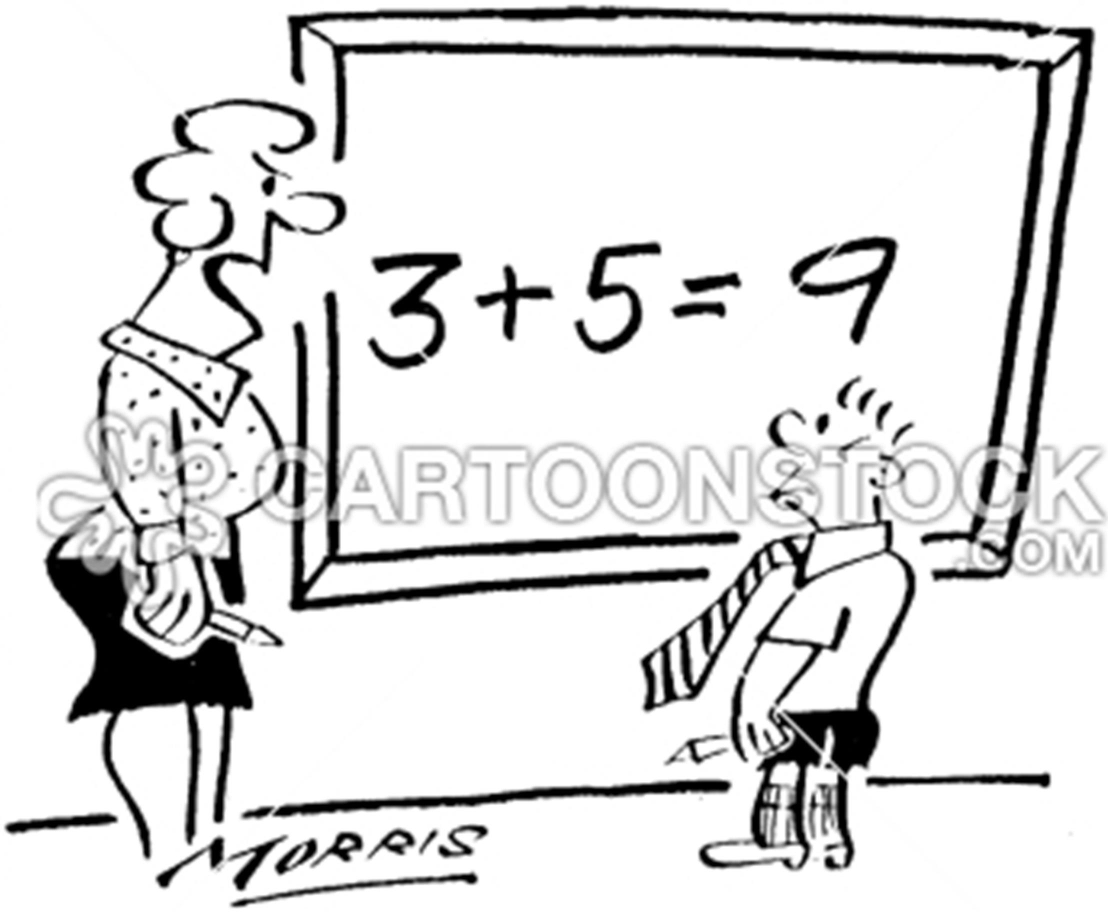


What is Error Analysis?

The study of kind of and quantity of error that

occurs particularly in the fields of Mathematics

Purpose of Error Analysis



"I don't care how your dad does it on his self-assessment return - it's wrong!"

Objective of Project

To use error analysis as an assessment tool to enable teachers to better adjust their instructional strategies so as to help students construct a clear concept image

Concept Image

Consists of all the cognitive structure in the individual's mind that is associated with a given concept

(Vinner, 1983)

$8^{m} \times 8^{2} = 8^{2m} = 64^{m}$ **WRONG** concept image

$8^{m} \times 8^{2} = 8^{m+2}$ **CORRECT** concept image

Literature Review

Error analysis existed from as early as 1930 and has extended into the 21st century

(Ayres, 2001; Ben-Zeev, 1998; Edwards, 1930; Guiler, 1932; Hart, 1987; Radatz, 1979; Schield, 1999)

Literature Review

Vygotsky's Social Cultural Theory

Students learn best if their teachers, as knowledgeable adults, are able to teach them in their zone of proximal development (ZPD)

Important that teachers understand students' difficulties and are able to address all the relevant common misconceptions during teaching

Literature Review

Good educators

- spend time to find out how to help students refute these common errors which are usually linked to some self-invented intuitive rules or misconceptions

- tailor their instruction so that it meshes with students' thinking

(Dewey, 1933; Piaget, 1977)

Research Questions

1. What are the two most prevalent common errors students encounter in the learning of Indices and Calculus?

2. To what extent do the common errors predict students' achievement in Indices and Calculus?

Methodology

Data Collection

Design of instruments

- Instruments A and B were designed to detect possible common errors in Indices and Calculus respectively

- Each instrument was face validated by 6 professional Mathematics STs

- Minimal interruption of the teaching and learning practices in each school during the implementation stage was ensured

Data Collection

Administration of instruments

- Instrument A (Indices) administered to 274 Sec 3 (Grade 9) students
- Instrument B (Calculus) administered to 468 Sec 4, JC1 and JC2 (Grade 10-12) students
- Recording of data was as follows:
  - ‘U’ : student was able to identify the conceptual error
  - ‘M’: student was able to apply the correct method
  - ‘A’: student was able to obtain the correct answer

Data Collection

Table 1: Template used to record the data

Students \ Qn	Type of mark	Q1	Q2	Q3	Q4	----	Q10	Total
Student 1	U							
	M							
	A							
Student 2	U							
	M							
	A							
⋮	U							
	M							
	A							
Student n	U							
	M							
	A							

Data Collection (Indices)

	Question	Solution	Marking Scheme
7	Simplify $27\times\sqrt[5]{3}$. Solution: $27\times\sqrt[5]{3}$ $=3^3\times 3^{\frac{5}{2}}$ $=3^{\frac{11}{2}}$	$27\times\sqrt[5]{3}$ $=3^3\times 3^{\frac{1}{5}}$ $=3^{\frac{16}{5}}$	Underline the mistake (U1) Provide correct solution (M1, A1) (If correct method is provided but answer is wrong, then M1 mark only)

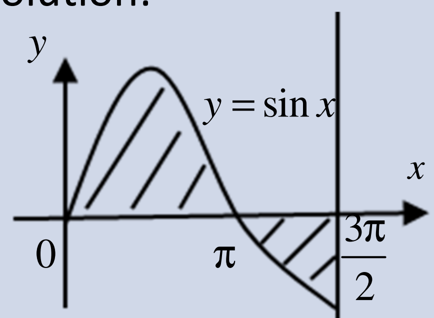
Data Collection (Indices)

	Question	Solution	Marking Scheme
8	Given $\frac{1}{\sqrt{8}}\times\sqrt[3]{4}=2^k$, find the value of k. Solution : $\frac{1}{\sqrt{8}}\times\sqrt[3]{4}=2^k$ $\frac{1}{8^{\frac{1}{2}}}\times 4^{\frac{1}{3}}=2^k$ $\frac{1}{2^{\frac{3}{2}}}\times 4^{\frac{1}{3}}=2^k$ $2^{-\frac{2}{3}}\times 2^{\frac{2}{3}}=2^k$ $2^{-\frac{2}{3}+\frac{2}{3}}=2^k$ $2^0=2^k$ $k=0$	$\frac{1}{\sqrt{8}}\times\sqrt[3]{4}=2^k$ $\frac{1}{8^{\frac{1}{2}}}\times 4^{\frac{1}{3}}=2^k$ $\frac{1}{2^{\frac{3}{2}}}\times 4^{\frac{1}{3}}=2^k$ $2^{-\frac{3}{2}+\frac{2}{3}}=2^k$ $-\frac{3}{2}+\frac{2}{3}=k$ $k=-\frac{5}{6}$	Underline the mistake (U1) Provide correct solution (M1, A1) (If correct method is provided but answer is wrong, then M1 mark only)

Data Collection (Calculus)

	Question	Solution	Marking Scheme
7	A curve is such that $\frac{dy}{dx}=2x^2$. Given that the curve passes through the point (3, 2), find the equation of the curve. Solution: The equation of the curve is $y=\left(2x^2\right)x+c$ $2=\left(2\times 3^2\right)(3)+c$ $c=-52$ The equation is $y=2x^3-52$	$y=\int 2x^2dx=\frac{2x^3}{3}+c$ When $x=3,\ y=2,\ c=-16$ $\therefore$ The equation of the curve is $y=\frac{2x^3}{3}-16$	Underline the mistake (U1) Provide correct solution (M1, A1) (If correct method is provided but answer is wrong, then M1 mark only)

Data Collection (Calculus)

	Question	Solution	Marking Scheme
8	Find the shaded area in the following diagram. Solution:  $\underline{\int_0^{\frac{3\pi}{2}} \sin x \ dx}$ $= [-\cos x]_0^{\frac{3\pi}{2}}$ $= \left[-\cos\frac{3\pi}{2}\right] - [-\cos 0]$ $= 0 + 1$ $= 1 \text{ unit}^2$	$\int_0^{\pi} \sin x \ dx$ $= [-\cos x]_0^{\pi}$ $= 1 + 1$ $= 2 \ \text{unit}^2$ The shaded area $= 2 + 1 = 3 \ \text{unit}^2$	Underline the mistake (U1) Provide correct solution (M1, A1) (If correct method is provided but answer is wrong, then M1 mark only)

Data Analysis

Data Analysis of the Indices Test

Errors Detected (Indices Test)		$\sum$	274
Q1 Applying zero index of a variable to its coefficient	U1	249	91%
	M1	242	88%
	A1	234	85%
Q2 Applying the operation (e.g. division) of the expression to the index of the variable	U2	261	95%
	M2	273	100%
	A2	271	99%

Data Analysis of the Indices Test

Errors Detected (Indices Test)		$\sum$	274
Q3 Applying the index of the expression to the variable only	U3	252	92%
	M3	261	95%
	A3	245	89%
Q4 Treating negative index as a negative sign to the coefficient	U4	254	93%
	M4	258	94%
	A4	242	88%

Data Analysis of the Indices Test

Errors Detected (Indices Test)		$\sum$	274
Q5 Applying the operation (e.g. product) of the expression to the indices of the variable	U5	256	93%
	M5	262	96%
	A5	249	91%
Q6 Treating product of indices to sum of indices	U6	236	86%
	M6	247	90%
	A6	240	88%

Data Analysis of the Indices Test

Errors Detected (Indices Test)		$\sum$	274
Q7 Treating the index of nth root as $\frac{n}{2}$	U7	235	86%
	M7	251	92%
	A7	248	91%
Q8 Treating $\frac{1}{a^n}$ as $a^{-\frac{1}{n}}$	U8	199	73%
	M8	239	87%
	A8	229	84%

Data Analysis of the Indices Test

Data analysis per school on U,M & A

Descriptive statistics of collected data for Indices Test				
	Indices	Underlining the error (U)	Correct method (M)	Correct answer (A)
Sec 3 (School A)	n1= 77 (2 classes)	72%	88%	83%
Sec 3 (School B)	n3=124 (4 classes)	94%	96%	92%
Sec 3 (School C)	n2= 73 (2 classes)	96%	93%	91%
Average (%)	N=274	89%	93%	89%

Data Analysis of the Indices Test

Participants’ performance in the two most prevalent common errors

Errors Detected (Indices)	School C (Sec 3) N=73			School B (Sec 3) N=124			School A (Sec 3) N=77			$\sum$	274
Q8	U8	96%		U8	72%		U8	52%		199	73%
Treating $\frac{1}{a^n}$ as $a^{-\frac{1}{n}}$	M8	93%	3%	M8	93%	21%	M8	73%	21%	239	87%
	A8	90%	3%	A8	93%	0%	A8	62%	10%	229	84%

Data Analysis of the Indices Test

Participants' performance in the two most prevalent common errors

Errors Detected (Indices)	School C (Sec 3) N=73			School B (Sec 3) N=124			School A (Sec 3) N=77			$\sum$	274
Q7	U7	97%		U7	98%		U7	56%		235	86%
Treating the index of nth root as $\frac{n}{2}$	M7	95%	3%	M7	97%	1%	M7	81%	25%	251	92%
	A7	95%	0%	A7	94%	2%	A7	81%	0%	248	91%

Correlations between Achievement in Indices Test and the identified Common errors

Indices Test	Achievement in Indices Test	Indices_ common_ error
Achievement in Indices Test	1.000	.981
Indices_ common_ error	.981	1.000

Model Summary for the Indices Test				
Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
	.981	.963	.962	4.23414

The prediction of the achievement in Indices test with those two items with most prevalent common errors is statistically significant with F (1, 271) = 6972.61 and p = 0.000.

Data Analysis of the Calculus Test

Errors Detected (Calculus Test)		$\sum$	468
Q1 Treating Product Rule as product of differentiated expressions	U1	409	87%
	M1	433	93%
	A1	397	85%
Q2 Applying Chain Rule without multiplying with the differentiated result of the variable	U2	389	83%
	M2	322	69%
	A2	294	63%

Data Analysis of the Calculus Test

Errors Detected (Calculus Test)		$\sum$	468
Q3 Omission of brackets in the application of Quotient Rule	U3	387	83%
	M3	408	87%
	A3	371	79%
Q4 Algebraic error in equating denominator to zero to find the turning point	U4	260	56%
	M4	250	53%
	A4	25[illegible]	54%

Data Analysis of the Calculus Test

Errors Detected (Calculus Test)		$\sum$	468
Q5 Evaluating $\int \frac{1}{f(x)}\,dx$ as $\ln f(x)$	U5	358	76%
	M5	294	63%
	A5	287	61%
Q6 Treating integration of trigonometric functions as integration of polynomials	U6	378	81%
	M6	322	69%
	A6	311	66%

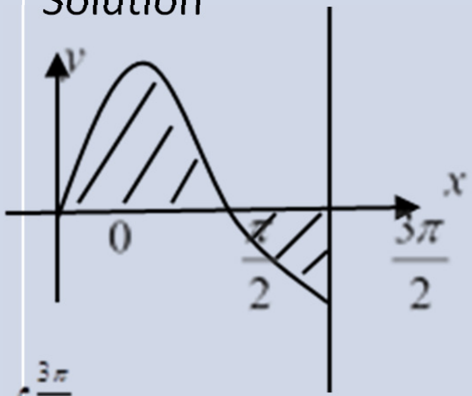
Data Analysis of the Calculus Test

Errors Detected (Calculus Test)		$\sum$	468
Q7 Treating $y = mx + c$ as an equation of the curve	U7	325	69%
	M7	359	77%
	A7	329	70%
Q8 Treating area of region below x-axis as a positive integral value	U8	264	56%
	M8	358	76%
	A8	266	57%

Data Collection (Calculus)

	Question
8	Find the shaded area in the following diagram.

Solution



$$\underline{\underline{\int_0^{\frac{3\pi}{2}} \sin x \, dx}}$$

$$= [-\cos x]_0^{\frac{3\pi}{2}}$$

$$= \left[-\cos\frac{3\pi}{2}\right] - [-\cos 0]$$

$$= 0 + 1$$

$$= 1 \text{unit}^2$$

Treating area of region below x-axis as a positive integral value

	Σ	468
U8	264	56%
M8	358	76%
A8	266	57%

Data Collection (Calculus)

	Question
7	A curve is such that $\frac{dy}{dx}=2x^2$. Given that the curve passes through the point (3, 2), find the equation of the curve. Solution: The equation of the curve is $y=\left(2x^2\right)x+c$ $2=\left(2\times3^2\right)(3)+c$ $c=-52$ The equation is $y=2x^3-52$

Treating $y = mx + c$ as an equation of the curve

	$\sum$	468
U7	325	69%
M7	359	77%
A7	329	70%

Data Analysis of the Calculus Test

Errors Detected (Calculus Test)		$\sum$	468
Q9 Treating variable as constant	U9	135	29%
	M9	156	33%
	A9	76	16%
Q10 Treating parameter as constant	U10	73	16%
	M10	59	13%
	A10	[illegible]	7%

Data Analysis of the Calculus Test

Data analysis per school on U,M & A

Descriptive statistics of collected data for Calculus test				
	Calculus	Underlining the error (U)	Correct method (M)	Correct answer (A)
Sec 4 (School D)	N1 = 73 (2 classes)	47%	56%	50%
Sec 4 (School E)	N2 = 354 (13 classes)	68%	62%	55%
J1 (School F)	N3 = 25 (1 class)	74%	79%	67%
J2 (School F)	N4 = 16 (1 class)	26%	90%	83%
Average (%)	N = 468	63%	63%	56%

Data Analysis

Participants' performance in the two most prevalent common errors (JC)

Errors Detected (Calculus Test)	School F (J2) -- N=16			School F (J1) N=25		% of 41	School E (Sec 4) N=354		School D (Sec 4) N=73		$\sum$	468
Q6 Treating integration of trigonometric functions as integration of polynomials	U6	25%		92%		66%	88%		56%		378	81%
	M6	88%	-63%	96%	-4%	93%	79%	9%	8%	48%	322	69%
	A6	88%	0%	92%	4%	90%	76%	3%	8%	0%	311	66%

Data Analysis

Participants' performance in the two most prevalent common errors (JC)

Errors Detected (Calculus Test)	School F (J2) N=16			School F (J1) N=25		% of 41	School E (Sec 4) N=354		School D (Sec 4) N=73		$\sum$	468
Q5 Evaluating $\int \frac{1}{f(x)}\,dx$ as $\ln f(x)$	U5	19%		84%		59%	84%		51%		358	76%
	M5	75%	-56%	92%	-8%	85%	62%	22%	56%	-5%	294	63%
	A5	75%	0%	92%	0%	85%	61%	1%	49%	7%	287	61%

Data Analysis of the Calculus Test

Participants' performance in the two most prevalent common errors (JC)

Errors Detected (Calculus Test)	School F (J2) N=16			School F (J1) N=25			School E (Sec 4) N=354			School D (Sec 4) N=73			$\sum$	468
Q8 Treating area of region below x-axis as a positive integral value	U8	19%		U8	76%		U8	59%		U8	45%		264	56%
	M8	94%	-75%	M8	72%	4%	M8	75%	-16%	M8	82%	-37%	358	76%
	A8	81%	13%	A8	44%	28%	A8	54%	21%	A8	68%	14%	266	57%

Data Analysis of the Calculus Test

Participants' performance in the two most prevalent common errors (JC)

Errors Detected (Calculus Test)	School F (J2) N=16			School F (J1) N=25			School E (Sec 4) N=354			School D (Sec 4) N=73			$\sum$	468
Q7 Treating $y = mx + c$ as an equation of the curve	U7	19%		U7	80%		U7	76%		U7	44%		325	69%
	M7	94%	-75%	M7	84%	-4%	M7	75%	1%	M7	77%	-33%	359	77%
	A7	88%	6%	A7	72%	12%	A7	69%	6%	A7	70%	7%	329	70%

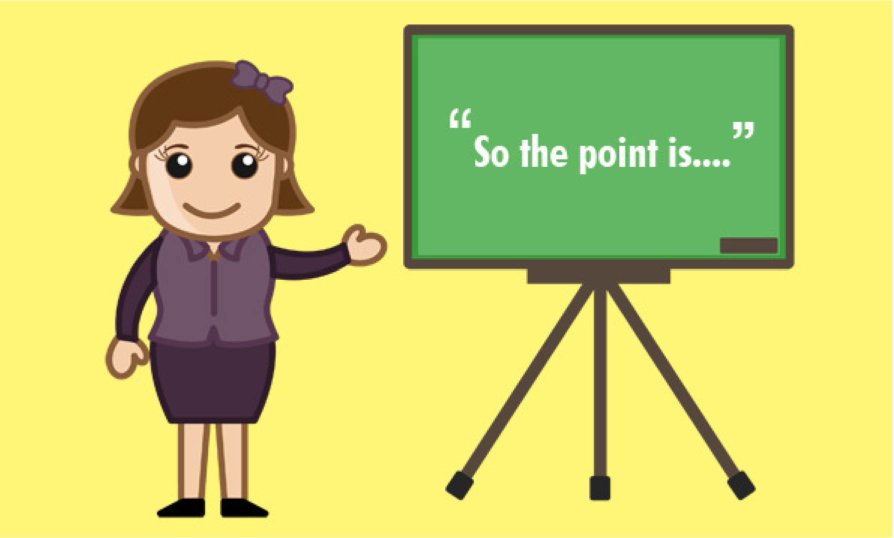
Correlations between Achievement in Calculus Test and the identified Common errors

Calculus Test	Achievement in Calculus Test	Calculus_common_error
Achievement in Calculus Test	1.000	.961
Calculus_ common_ error	.961	1.000

Model Summary for the Calculus Test				
Model	R	R Square	Adjusted R Square	Std. Error of the Estimate
	.961	.923	.923	5.28163

The prediction of the achievement in Calculus test with those two items with most prevalent common errors is statistically significant with F (1, 467) = 5604.60 and p=0.000.

Conclusion



For Indices and Calculus Test

(r = 0.98) (r = 0.96)

Students' scores in the questions identified as having the most prevalent errors

High Correlation

Students' overall scores

For Indices and Calculus Test
(96%) (92%)

Students' scores in the questions identified as having the most prevalent errors

High

Predictive Power

Students' overall scores

Answer to the Research Questions

1 What are the two most prevalent common errors students encounter in the learning of Indices and Calculus?

Indices Test	Calculus Test
treating $\frac{1}{a^n}$ as $a^{-\frac{1}{n}}$	treating area of region below x-axis as a positive integral value
treating the index of nth root as $\frac{n}{2}$	treating $y = mx + c$ as an equation of the curve

Answer to the Research Questions

2 To what extent do the common errors predict students' achievement in Indices and Calculus?

- The total score of the two problems with common errors predicts 96% of the achievement in Indices Test.

- The total score of the two problems with common errors predicts 92% of the achievement in Calculus Test.

Limitations

- Lack of standardization in the instruction of administrating the research instruments

- The instruments were face validated without piloting
- The items used in the instruments were not statistically validated

Recommendations

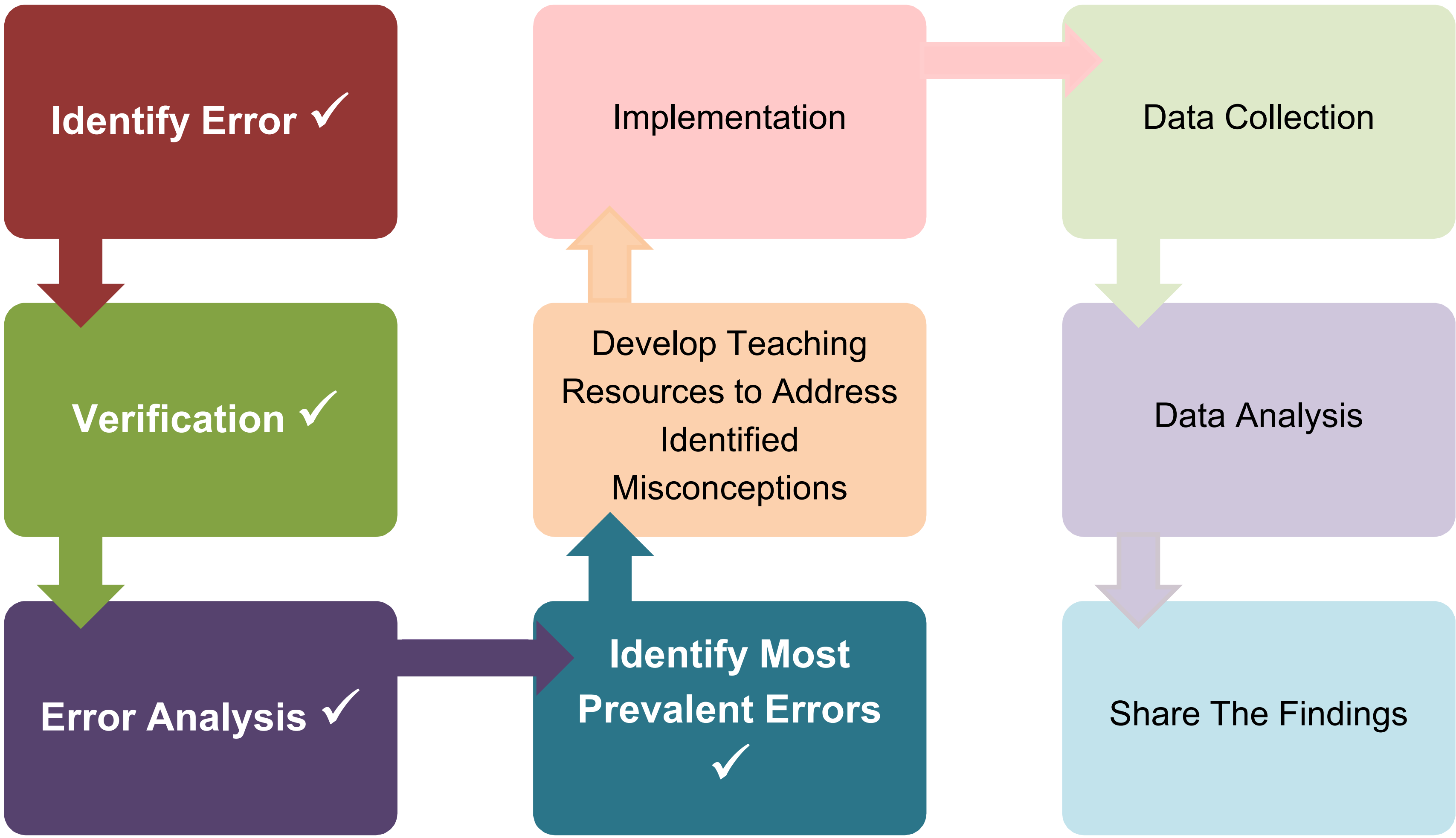
- Interview students for reasons of not underlining the errors

- Standardize the instruction of administrating the research

- Pilot the instruments

- Validate the items used in the instruments statistically

Rectifying Common Error to Improve Performance in Assessment



Acknowledgements

We would like to thank the following people who have helped to make this workshop possible:

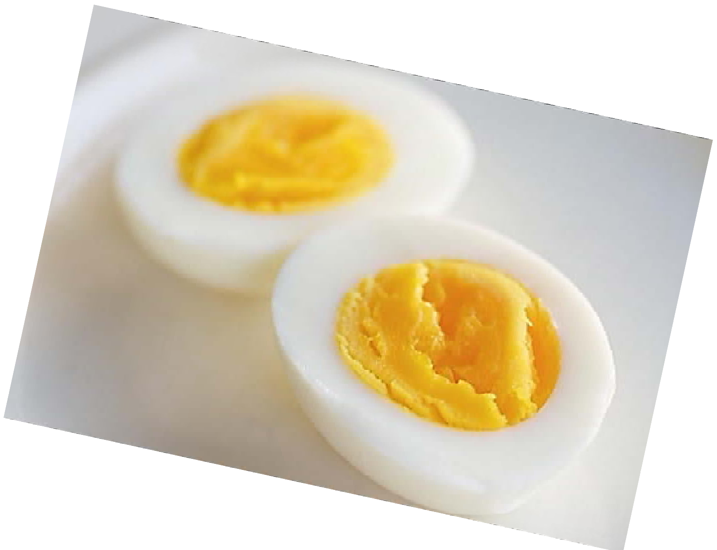
- Organisers of this conference
- Principals, Vice-Principals, HODs and Teachers for their support
- Students of the various schools for their participation
- Team members who are not present today but have contributed in one way or another to this project


thank
you

Puzzle 1

2 fathers and 2 sons ate 3 eggs for breakfast, each eating exactly one egg.

How could that be?



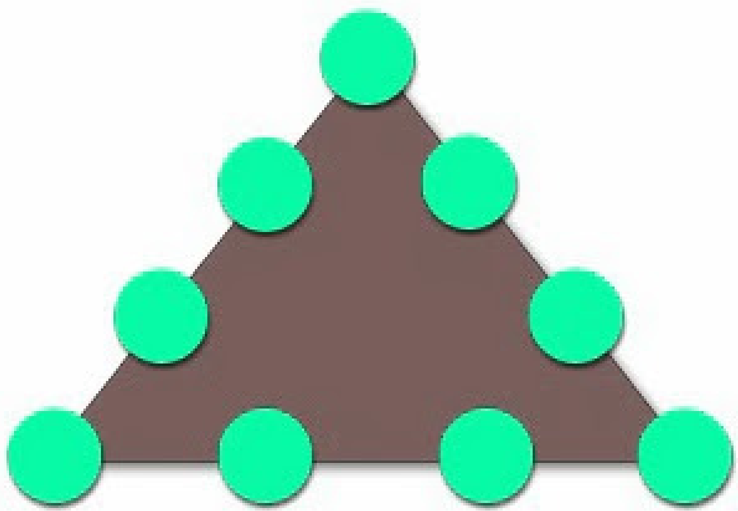
Puzzle 1 Answer

There are three people - a grandfather, a father, and a grandson.

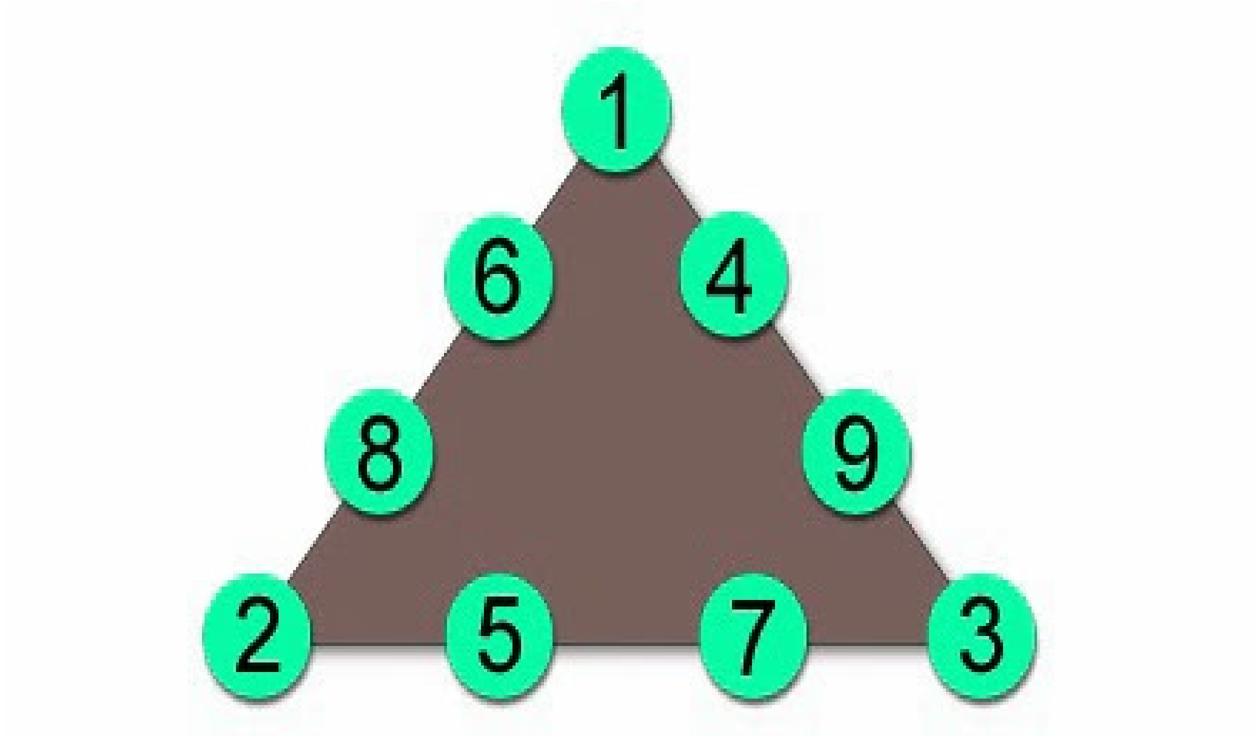
Therefore, the father is both a son (of the grandfather) and a father (of the grandson).

Puzzle 2

In the diagram, place the numbers 1 to 9 in the circles such that if you add up any side of the triangle, the sum is 17.



Puzzle 2 Answer



Puzzle 3

The entire house will collapse exactly after **twelve minutes**.

To move, one must carry a fire extinguisher to keep the flames away.

Only two person can run through that hallway at one time.

But for others to go, one must return back with the fire extinguisher.

The **fireman**, is trained for such tasks and can run through the hallway in **a minute**.

The **athlete** can make it in **two minutes**.

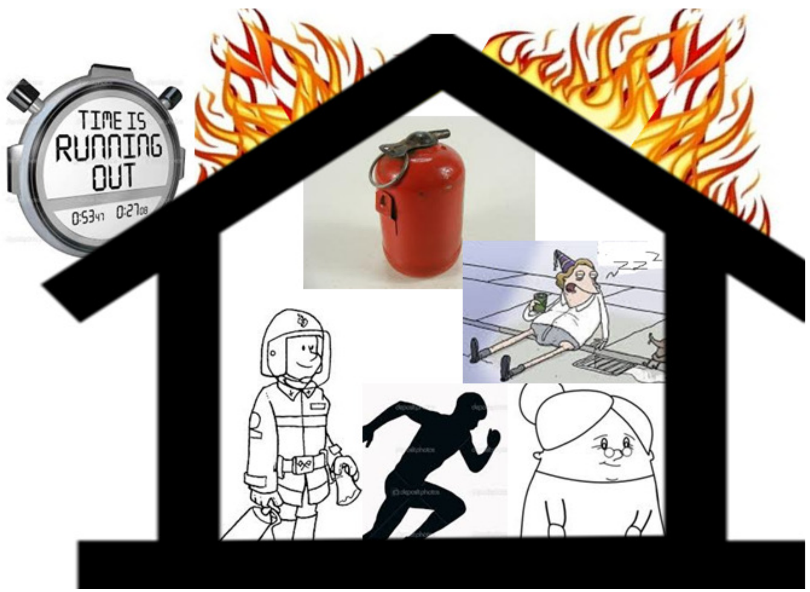
The **old woman** can run slowly and will cover the hallway in **four minutes**.

The **drunk guy** will take **five minutes** to run through it.

If all of them can make it through the hallway in twelve minutes, all of them will be saved.

When two move together, they will run with the speed of the slower one.

How will all four of them manage to run to safety?



Puzzle 3 Answer

* First, the fireman and the athlete will go spending two minutes.

* The fireman will come back taking one minute.

* The drunk guy and the old lady will go spending five minutes.

* The athlete will come back taking two minutes.

* The athlete and fireman will go spending two minutes.

Two + One + Five + Two + Two = Twelve.

Thus all of them will be out of the house before it collapses.

Puzzle 4

Grace likes to collect money in a piggy bank. She bought a pink piggy bank when she was 10 years old. She put $250 in the box on each of her birthday.

Her younger sister took $50 out from Grace's piggy bank on her own birthday.

Grace decides to take out all the money when she is 50 years old to go for a long-waited holiday.

When the piggy bank was opened, it has $500. How can that be possible?

Puzzle 4 Answer

The girl was born on 29 February. Thus her birthday came once in four years only while her sister was born on a normal day and celebrated her birthday every year.

Thus the girl had a chance of depositing money only 10 times in 40 years through which she collected $2500 while her sister took $50 from the piggy bank every year making the total amount to be $2000.

Thus when the piggy bank was opened, it had just $500.

Puzzle 5

At a party, everyone shook hands with everybody else.

There were 66 handshakes.

How many people were at the party?

Puzzle 5 Answer

Ans: 12

In general, with $(n+1)$ people, the number of handshakes is the sum of the first n consecutive numbers: $1+2+3+ \ldots + n$.

Since this sum is $n(n+1)/2$, we need to solve the equation $n(n+1)/2 = 66$.

This is the quadratic equation $n^2 + n - 132 = 0$. Solving for n, we obtain 11 as the answer and deduce that there were 12 people at the party.

Since 66 is a relatively small number, you can also solve this problem with a hand calculator. Add 1 + 2 = + 3 = +... etc. until the total is 66. The last number that you entered (11) is n.