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Looking into students' process of understanding mathematics for nurturing reflective learners: An analysis of the 8th grade lesson on "regular stellar polygon" in a Japanese lower secondary school classroom

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Education System in Japan

Age	*Grade*	Type of School
18 -22		University
17 -18	*12*	Upper Secondary School
16 -17	*11*	
15 -16	*10*	
14 -15	*9*	Lower Secondarv School
13 -14	*8*	
12 -13	*7*	
11 -12	*6*	Primary School
10 -11	*5*	
9 -10	*4*	
8 -9	*3*	
7 -8	*2*	
6 -7	*1*	
3 -6	*K*	Kindergarten

Figure 1: Education System in Japan

Overall Objectives of LSS Mathematics (2008)

◎ Overall Objectives

- Through mathematical activities,
- to help students deepen their understanding of fundamental concepts, principles and rules regarding numbers, quantities, geometrical figures and so forth,
- to help students acquire the way of mathematical representation and processing,
- to develop their ability to think and represent phenomena mathematically,
- to help students enjoy their mathematical activities and appreciate the value of mathematics, and to foster their attitude toward to making use of the acquired mathematical understanding and ability for their thinking and judging.

Content Areas and Number of Lessons in LSS

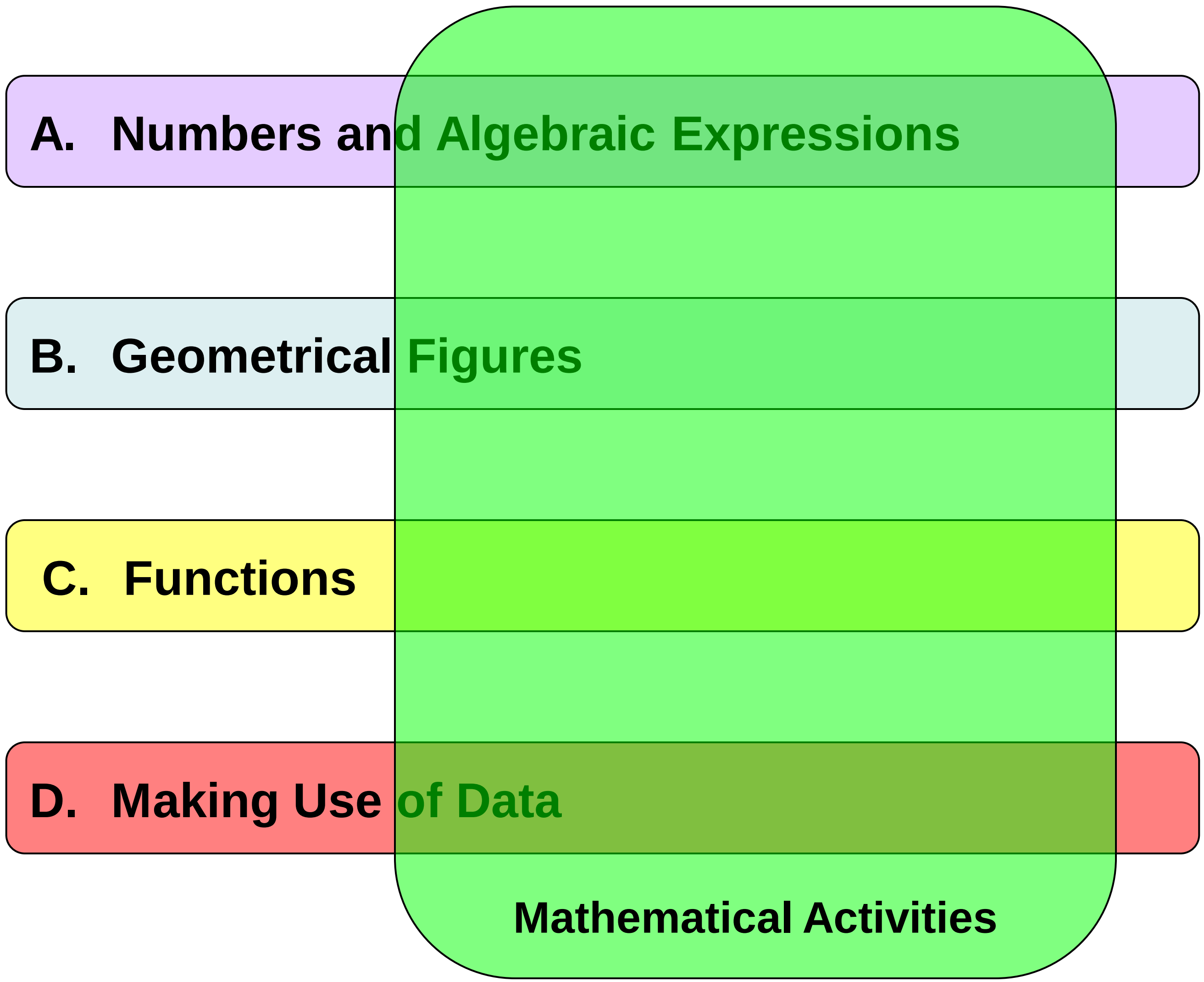
◎ Content Areas

A. Numbers and Algebraic Expressions
B. Geometrical Figures
C. Functions
D. Making Use of Data
[Mathematical Activities]

◎ Number of Mathematics lessons a week

2008CoS		1998CoS
Grade 1 (4)	←	Grade 1 (3)
Grade 2 (3)	←	Grade 2 (3)
Grade 3 (4)	←	Grade 3 (3)

A Framework for New Curriculum of School Mathematics



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Abstract

- It is well recognized that reflective thinking is a key for students to learn mathematics and deepen their mathematical understanding. For nurturing reflective learners, we need to look into students' process of understanding mathematics in a classroom.

- We focus on the 8th grade lesson on "regular stellar polygon" in a Japanese lower secondary school classroom, and analyze the students' process of understanding in the lesson with the so-called "two-axis process model" (Koyama, 1993, 1997).

Abstract

Four suggestions for improving students' understanding:

- Mathematics teacher should pay attention not only to the correctness of students' answer but also to the process of their reasoning leading to the answer in a lesson.
- It is important for mathematics teacher to make a decision on what kinds of learning situation to be set up for helping students improve their mathematical understanding in a lesson.
- The social interaction among a teacher and students is important in order to develop a higher level of mathematical understanding.
- Students should be encouraged to reflect on what they have done and do more activities for integrating them in a classroom.

Introduction

- As a descriptive and prescriptive model, Koyama (1993, 1997) made the so-called “two-axis process model” of mathematical understanding.
- As a result of theoretical and practical studies, Koyama (2005) identified the principles and methods for designing mathematics lessons based on the model.
- Through a series of case studies in primary school mathematics, it has been demonstrated that the model can be used by teachers as an effective framework to design mathematics lessons for improving their students’ mathematical understanding in a classroom (Koyama, 2007, 2010).

Introduction

- However, we need more case studies on the students' process of understanding in secondary school mathematics.
- We focus on the 8th grade lesson on "regular stellar polygon" in a Japanese lower secondary school classroom, and analyze the students' process of understanding mathematics in the lesson with the "two-axis process model".
- The purpose of this qualitative analysis is to get some implications for nurturing reflective learners of lower secondary school mathematics in the process of understanding mathematics in a classroom.

Models of Understanding Mathematics

- Two types of model

aspect model

process model

- Two characteristics of model

descriptive characteristic

To describe the kinds or the process of mathematical understanding in learning mathematics.

prescriptive characteristic

To suggest some didactical principles such as what kind of problematic situations should be set up and which direction should be aimed at for improving students' mathematical understanding in learning mathematics.

The "two-axis process model"

- As a result of theoretical studies to identify basic components for a descriptive and prescriptive model of mathematical understanding (van Hiele & van Hiele-Geldof, 1958; Wittmann, 1981; Pirie & Kieren, 1989), Koyama (1993, 1997) made the so-called "two-axis process model" of mathematical understanding.

- The model consists of two different axes of the vertical implying three levels of understanding mathematics and the horizontal implying three learning stages at each level as follows:

The “two-axis process model”

- *Vertical axis*

V1: mathematical entities

V2: relation of the entities

V3: general relation

The "two-axis process model"

- *Horizontal axis*

H1: Intuitive stage

Students are provided opportunities for manipulating concrete objects, or operating on mathematical concepts and relations acquired in a previous level. (*intuitive thinking*)

H2: Reflective stage

Students are stimulated and encouraged to pay attention to their own manipulating or operating activities, to be aware of them and their consequences, and to represent them in terms of diagrams, figures or language. (*reflective thinking*)

H3. Analytic stage

Students elaborate their representations to be mathematical ones using mathematical terms, verify the consequences by means of other examples or cases, or analyze the relations among consequences in order to integrate them as a whole. (analytical *thinking*)

The principles and three methods

- *Principles for designing mathematics lessons*

P1: recognizing mathematical understanding as a dynamic process

P2: setting up levels of understanding and learning stages on a level

P3: incorporating students' individual constructions and social constructions

- *Methods for designing mathematics lessons*

M1: making clear levels of understanding related to a certain mathematical topic

M2: assessing and evaluating students' understanding as a readiness

M3: planning in detail three learning stages as a dialectic process of individual and social constructions in a lesson

The 8th grade lesson on "regular stellar polygon"

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- The lesson to be analyzed is a fifteen-minute lesson on "regular stellar polygon" with 39 8th graders in a lower secondary school attached to Hiroshima University.
- This lesson was designed by a mathematics teacher Mr. Tominaga as a task-based learning in the teaching unit of "triangles and quadrangles" at 8th grade in 14th November 2003.

Objectives of the lesson

- The students had already learned such topics as properties of parallel lines, sums of interior angles and exterior angles of a polygon, properties of an isosceles triangle, the theorem of angles at the circumference, and so on before this lesson according to Japanese Course of Study for lower secondary school mathematics (Ministry of Education, 1999).
- The objectives of the lesson were to improve students' understanding of how to find out the measure of an angle in a regular stellar polygon and to promote their mathematical thinking and attitude toward mathematics through generalizing the mathematical tasks on regular polygons to regular stellar polygons (Tominaga, 2003).

Qualitative analysis of the students' process of mathematical understanding in the lesson

- Task 1: "Find out the measure of a tip angle in the regular nine-pointed star polygon made by connecting next two points as shown in Figure 1".

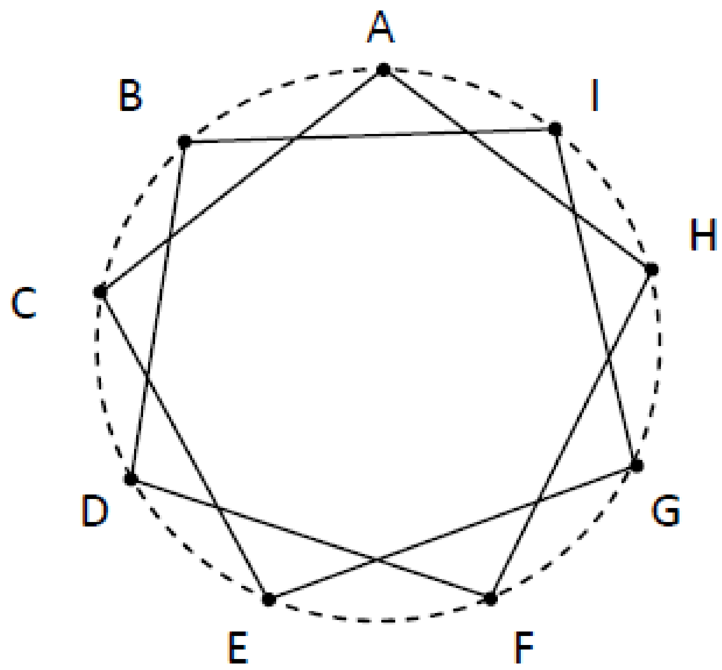


Figure 1. A regular nine-pointed star polygon

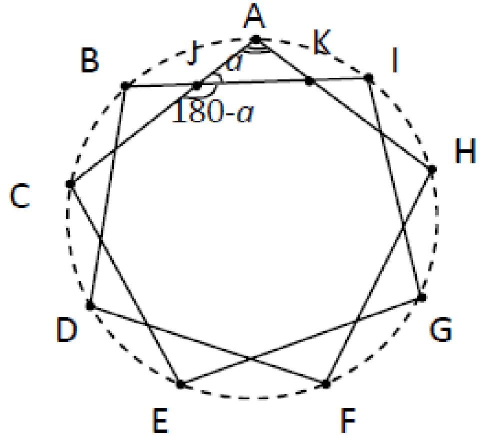
Qualitative analysis of the students' process of mathematical understanding in the lesson

- Task 2: "Investigate the measure of a tip angle in the regular nine-pointed star polygon made by connecting next k points for $k = 1,2,3,4$, and state your findings".
- Task 3: "Represent the measure of a tip angle in the regular -pointed star polygon made by connecting next points in an algebraic expression with letters n and k".

- In the lesson the students worked mainly on the tasks 1 and 2. The task 3 was left for the next lesson. Therefore, in the following, we analyze the students' process of mathematical understanding mainly observed during their works on the tasks 1 and 2 by using the "two-axis process model".

During work on task 1

S1: Yes. This triangle (△AJK) is an isosceles triangle…
So let this (∠AJK) is $\angle a$, then here (∠IJC) is $180 - a$. The measure of one angle of the nine-cornered polygon (nonagon) inside a circle is $180 \times 7 \div 9$, $140°$. So $180 - 140$, $\angle a = 40$. As I said, because this triangle (△AJK) is an isosceles triangle, here (∠AKJ) is also $\angle a$, 40. Because the sum of interior angles of a triangle is $180°$, 180 minus 40 and 40, the answer is $100°$.



$180-(40+40)$

$=100$

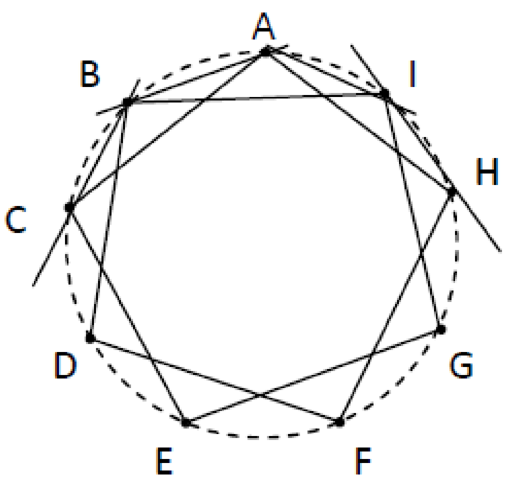
$180 \times 7 \div 9 = 140$

$a = 40$

Figure 3. Explanation by Student 1

During work on task 1

S2: The first expression means that the measure of one angle of the nine-cornered polygon (nonagon) inscribed in a circle, looking at this triangle (△AHI), here (∠AIH) is 140 (see Figure 4).



$180\times(9-2)\div 9=140$

$(180-140)\div 2=20$

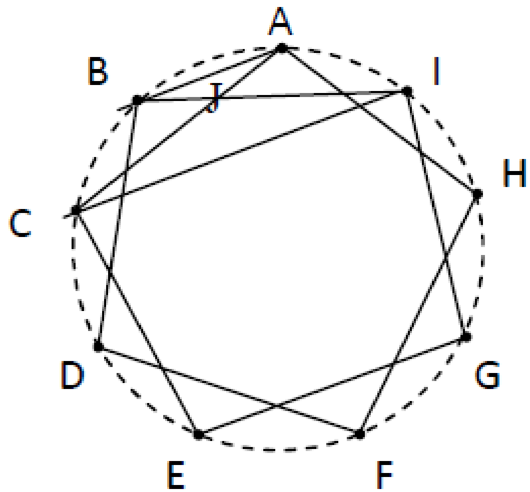
$140-20\times 2=100°$

Figure 4. Explanation by Student 2

T: It is the same explanation as S1's. The nine-cornered polygon…

S2: Then, because the orange colored triangle (△AHI) is an isosceles triangle, here (∠IAH) and here (∠IHA) are 20°. This triangle (△ABI) is same as that triangle (△AHI). So here (∠AIB) and here (∠ABI) are 20°. Then 140° minus $2\times 20°$ is 100°.

During work on task 1



$$360+180\times(5-2)$$
$$=360+540$$
$$=900 \quad 900\div 9=100$$

Figure 5. Explanation by Student 3

S3: (After drawing parallel lines AB and CI as shown in Figure 5) Looking at this small triangle (△ABJ) and this triangle (△CJI), because these are vertically opposite angles here (∠AJB and ∠CJI) is same. Um… When this side (∠JIC and ∠JCI) is moved to that side (∠BAJ and ∠ABJ), we have a pentagon ABDFH. Then we have a quadrangle CEGI. So the sum of all interior angles in a pentagon ABDFH and a quadrangle CEGI is 360° plus 540°. It is 900°. Because all angles are equal, 900° divided by 9 is 100°.

During work on task 1

- The task 1 is on the level V1 (mathematical entities) in the vertical axis of the “two-axis process model” because the measure of a tip angle in the regular nine-pointed star polygon made by connecting next two points is the object in this task.

S1: The triangle AJK is an isosceles triangle (see Figure 3).

S2: The triangle AHI is an isosceles triangle (see Figure 4).

S3: The regular nine-pointed star polygon made by connecting next two points is transformed into a quadrangle CEGI and a pentagon ABDFH without changing the sum of tip angles (see Figure 5).

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During work on task 1

- From this observation, we see that these three students attained the level V1 of understanding mathematical entities in the task 1 and that their solution methods were different.
- It suggests that in mathematics lesson when a teacher evaluates his/her student's mathematical understanding he/she should pay attention not only to the correctness of the student's answer to a task but also to the process of reasoning lead to the answer.

During work on task 2

He allowed his students to work together on the task if they need. During students' work on the task, the teacher did a round of checking (*kikan-shido*), and confirmed that in case of $k=1$ the regular nine-pointed star polygon is a regular nonagon and the measure of a tip angle is $140°$ and that in case of $k=3$ the regular nine-pointed star polygon is an equilateral triangle and the measure of a tip angle is $60°$. Then he said "We left a problem in case of $k=4$. What a complex figure this is! Answer the measure of a tip angle in this case" (see Figure 6).

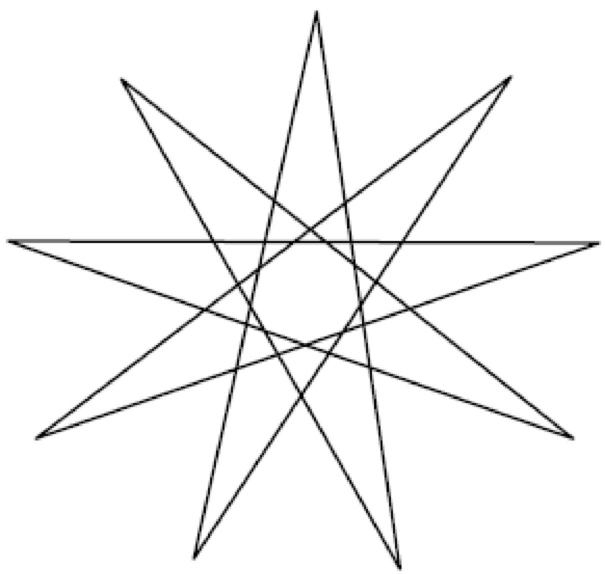

Figure 6. The regular nine-pointed star polygon for $k=4$

During work on task 2

S4: If we draw a circumscribed circle in the Figure 6, a central angle of this is 360° divided by 9, 40°.

T: I see. The central angle is 40°. So?

S4: Um… The angle at the circumference is… (Her voice was not heard.)

T: Yes, you are right. Because the angle at the circumference is half of a central angle, the angle at the circumference in this case is 40° divided by 2, 20°. Ok, you are right.

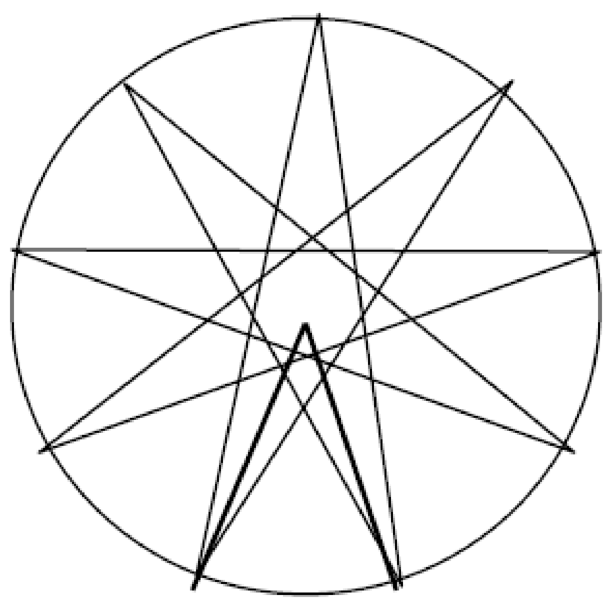

Figure 7. Using the theorem of angles at the circumference for $k = 4$

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During work on task 2

- The task 2 is on the level V2 (relations of entities) in the vertical axis of the “two-axis process model” because the relation among tip angles in the regular nine-pointed star polygon made by connecting next k points ($k = 1,2,3,4$) is the object in this task.
- It means that the task 2 intends to improve the students’ mathematical understanding from the level V1 to the level V2.

During work on task 2

- When the students worked on the more complicated task 2 in a regular nine-pointed star polygon, first they tried to draw the regular nine-pointed star polygons in each case of $k = 1,2,3,4$.
- At this moment the object of the students' understanding was temporary shifted from investigating a relation among the measures of tip angles to having the images of regular nine-pointed star polygons by connecting next k points ($k = 1,2,3,4$).
- This phenomenon is known as the folding back from the property-noticing level to the image-making level in the transcendent recursive model of mathematical understanding (Pirie & Kieren, 1989).

During work on task 2

- It might be difficult for the students to formalize the relation among the measures of tip angles of regular nine-pointed star polygons ($k = 1, 2, 3, 4$) only based on the task 1 in case of $k = 2$.
- This interpretation suggests us the importance of teacher's decision making on what kinds of learning situation should be set up for helping students improve their mathematics understanding in a lesson.
- If the teacher intends to improve the students' mathematical understanding on the level V2 (relation of the entities) in the "two-axis process model", then he should set up learning situations by giving careful consideration to three stages of H1, H2, and H3.

During work on task 2

- For example, if the following alternative tasks were posed by the teacher, the students could be stimulated and encouraged to pay attention to their own manipulating activities, to be aware of the consequences, and to represent them in terms of geometrical figures and mathematical language and expressions.
- The task 2-1 is a task for the intuitive stage and the task 2-2 is for the reflective and analytic stages on the revel of the relation of mathematical entities.

Task 2-1: "Draw the regular nine-pointed star polygons made by connecting next k points for $k = 1,2,3,4$".

Task 2-2: "Find out and make a table of the measures of tip angle in the regular nine-pointed star polygon made by connecting next k points for $k = 1,2,3,4$".

During work on task 2

- It is important for a teacher to set learning objectives for students in a mathematics lesson. Even if the learning objectives are same, there are various learning situations to be set up in a lesson. Therefore the teacher must investigate the actual situation of students' mathematical understanding, and make a decision on what kinds of learning situation should be set up for helping students improve their mathematics understanding.
- For that purpose, the "two-axis process model" and the principles and methods can be used by the teacher as an effective general framework to design a mathematics lesson for improving the students' mathematical understanding in a classroom.

Transition from task 2 to task 3

The teacher summarized the answers for $k = 1,2,3,4$ into a table (see Figure 8) and asked his students a question "Look at this table. What can you find out?"

k	1	2	3	4
the measure of a tip angle	140	100	60	20

Figure 8. The table made by the teacher

Transition from task 2 to task 3

A student (S5) answered that the measure of a tip angle decreases at the rate of 40° as the value of *k* increases one by one. The teacher approved the finding of student S5 and prompted others.

S6: Well, 20° is a unit, 60° is three times, 100° is five times, 140° is seven times…

T: I see. You say that 140° is seven times of 20°, 100° is five times of 20°, 60° is three times of 20°, and 20° is one times of 20°. So one, three, five, and seven are odd numbers. Ok, this is an excellent property. Are there any other findings?

SS: (There was no reaction from students.)

T: Well, you said that the measure of a tip angle decreases at the rate of 40°. Why 40°? The fact that the measure of a tip angle decreases at the rate of 40° is valid only for a regular nine-polygon? How about for a regular eight-polygon, ten-polygon, six-polygon, 12-polygon, 24-polygon, and so on? Anyway, why 40° for a regular nine-polygon?

S7: The central angle?

Transition from task 2 to task 3

- We can see an intention of the teacher in his questioning to deepen and direct the students' mathematical understanding of a relation among the measures of a tip angle in the regular nine-pointed star polygon.
- However, we could not hear any clear voice from the students. Only one student (S7) answered in a mutter "The central angle?" to the question.
- Although his answer is insufficient, we guess that he could notice the central angle because he carefully heard when the student (S4) tried to explain the measure of a tip angle is by using the theorem of angles at the circumference (see Figure 7). It means that the social interaction among a teacher and students in a mathematics classroom is important.

Transition from task 2 to task 3

- In this lesson, the teacher hurried to the task 3. It was a key point for the teacher's decision making in the lesson.
- The teacher had better to use the remaining time to encourage the students to make a connection the measures of a tip angle in a regular nine-pointed star polygon made by connecting next k points for $k = 1,2,3,4$.
- The task 3 is evaluated as a task on the level V3 (general relation) in the vertical axis of the "two-axis process model". Therefore before tackling on the task 3, the students should be provided more time and opportunities to reflect on what they have done in the task 2 (reflective stage) and do more activities for integrating them (analytic stage) according to the learning stages in the model.

Conclusion

Four suggestions for improving students' understanding:

- Mathematics teacher should pay attention not only to the correctness of students' answer but also to the process of their reasoning leading to the answer in a lesson.
- It is important for mathematics teacher to make a decision on what kinds of learning situation to be set up for helping students improve their mathematical understanding in a lesson.
- The social interaction among a teacher and students is important in order to develop a higher level of mathematical understanding.
- Students should be encouraged to reflect on what they have done and do more activities for integrating them in a classroom.

Conclusion

- Being based on those suggestions, we can make an alternative plan for the 8th grade lesson on “regular stellar polygon” in a lower secondary school mathematics classroom as follows.
- When a teacher intends to help students develop their mathematical understanding according to three levels of mathematical entities, the relation of them, and the general relation in a vertical axis of the “two-axis process model”, it is suggested to set up three learning stages in a horizontal axis of the model by posing the following alternative tasks.

Conclusion

Task 1': "Draw the regular nine-pointed star polygons made by connecting next k points for $k = 1,2,3,4$".

Task 2': "Find out and make a table of the measures of tip angle in the regular nine-pointed star polygon made by connecting next k points for $k = 1,2,3,4$".

Task 3': "Find out and make a table of the measures of tip angle in the regular 12-pointed star polygon made by connecting next k points for $k = 1,2,3,4,5,6$".

Task 4': "Represent the measure of a tip angle in the regular n-pointed star polygon made by connecting next k points in an algebraic expression with letters n and k".

$n \geq 3$, and $1 \leq k \leq n-1$, then

$$t = \left| 180^\circ - \frac{360^\circ}{n} \times k \right|.$$

Conclusion

- The mathematics lesson plan with these tasks needs two fifty-minute sessions because it is very difficult for the 8th graders in one session to investigate and represent algebraically the general relation among the measures of a tip angle in the regular n-pointed star polygon made by connecting next k points.
- In the first session, in order to help the students improve their mathematical understanding from the level of mathematical entities to the level of the relation of them the task 1' is posed for the intuitive stage and the task 2' is posed for the reflective and analytic stages.

Conclusion

- Then in the second session, the task 3' is posed for the intuitive stage and the task 4' is posed for the reflective and analytic stages in order to encourage the students to generalize the relation and help them improve their mathematical understanding from the level of the relation among the measures of tip angles in the regular nine-pointed star polygon to the level of the general relation among the measures of a tip angle in the regular n-pointed star polygon made by connecting next k points.

Conclusion

- The mathematical topic of “regular stellar polygon” is very rich in improving the students’ mathematical understanding in a lower secondary school classroom.
- We have to examine the effectiveness of the alternative plan for helping the students improve their mathematical understanding up to be an expected higher level in a classroom.

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