Axiom of Choice as the mean for the support of reflective learners

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Active versus r**eflective** learners

both groups have their positives and negatives

within mathematics

think about a problem quietly first (and alone) better enables to figure to oneself

WHY?

because the basic implement for an acquirement of knowledge in the mathematics is the **language** of mathematics and that this implement is so much preferred, that every knowledge obtained in different way, e.g. on the basis of an observation, an experiment, **an intuition** etc. **is considered to be the mathematical knowledge only when it is proved by means of the language**

Let us start with the fact that so called general linear function could be the very suitable function to demonstrate the facts above mentioned.

Definition

A function f: $\mathbf{R}^N \to \mathbf{R}$ *is called general linear iff it is of the form*

$f(x) = f_a(x) + b$, $x \in \mathbf{R}^N$,

where f_a *is a homomorphism of the linear space* $(\mathbf{R}^N, \mathbf{Q}, +, \cdot)$ *into the linear space* $(\mathbf{R}, \mathbf{Q}, +, \cdot)$ *and* $b = f(0)$.

Thus f_a: $\mathbf{R}^N \to \mathbf{R}$ is additive, i.e. it satisfies Cauchy's functional equation

$f_a(x + y) = f_a(x) + f_a(y)$

for all $x, y \in \mathbf{R}^N$.

Let H *be an arbitrary Hamel basis of the space* $(\mathbf{R}^N, \mathbf{Q}, +, \cdot)$. *Then for every function* $g: H \to \mathbf{R}$ *there exists a unique additive function* $f: \mathbf{R}^N \to \mathbf{R}$ *such that* $f \mid H = g$.

This assertion gives the general construction of all additive functions $f: \mathbf{R}^N \to \mathbf{R}$. In fact, every additive f may be obtained as the unique additive extension of a certain function $g: H \to \mathbf{R}$, that is $g = f \mid H$.

It could be shown (see again [7] and [8]) additive (and general linear) functions bring didactic and methodological mathematical material for the demonstration of the modern concept of teaching mathematics, which is a good opportunity for the teacher to encourage and develop creative powers of students.

The specificity of solving these problems lies in the fact that although we can use our already gained knowledge and skills in order to solve them heuristically; we cannot base our solutions on concrete geometric image however. We are therefore faced with the necessity to prove our hypotheses in an exact way, which is very valuable and desirable. Why cannot we rely on the geometric image here? Because our solutions come from the existence of discontinuous additive functions.

For many years the existence of discontinuous additive functions was an open problem. Mathematicians could neither prove that every additive function is continuous nor exhibit an example of discontinuous additive function. It was only thanks to the development of the set theory in the beginning of the 20th century that the existence of a discontinuous additive function could be proved. It was G. Hamel who first succeeded in proving this fact. The proof is based on the existence of a special set of elements from $\mathbf{R}^N$, today called a Hamel basis of $\mathbf{R}^N$. Its existence follows from the Axiom of Choice. It means however that no concrete example of such base is known, i.e. effective examples of discontinuous additive functions do not exist.

Quasiarithmetic weighted mean of $a_{1,\ldots,}a_n$ with the weights $w_{1,\ldots,}w_n$

$$F\left(\frac{\sum_{i=1}^{n} w_i F^{-1}(a_i)}{\sum_{i=1}^{n} w_i}\right)$$

For simplicity and clearness:

Let $i = 1,2$ $w_1 = w_2 = 1$, $a_1 = a > 0$, $a_2 = b > 0$, then

$$F\left(\frac{\sum_{i=1}^{n} w_i F^{-1}(a_i)}{\sum_{i=1}^{n} w_i}\right) = F\left(\frac{F^{-1}(a) + F^{-1}(b)}{2}\right)$$

For $r \in [-\infty, \infty]$ we define

$$M(r) = F_r\left(\frac{F_r^{-1}(a) + F_r^{-1}(b)}{2}\right)$$

where

$F_r(x) = x^{1/r}$ pro $-\infty < r < 0 \cup 0 < r < \infty$

$F_r(x) = \max\{a, b\}$ pro $r = \infty$

$F_r(x) = \min\{a, b\}$ pro $r = -\infty$

$F_r(x) = \exp(x)$ pro $r = 0$

Arithmetic mean: $r = 1$, $\mathrm{A} = M(1)$

$F_1(x) = x$

$$F_1\left(\frac{F_1{}^{-1}(a) + F_1{}^{-1}(b)}{2}\right) = \frac{a+b}{2}$$

Geometric mean: $r = 0$, $\mathrm{G} = M(0)$

$F_0(x) = \exp(x)$

$$F_0\left(\frac{F_0{}^{-1}(a) + F_0{}^{-1}(b)}{2}\right) = \exp\left(\frac{\ln(a) + \ln(\mathrm{b})}{2}\right) = \sqrt{ab}$$

Harmonic mean: $r = -1$, $\mathrm{H} = M(-1)$

$F_{-1}(x) = x^{-1}$

$$F_{-1}\left(\frac{F_{-1}{}^{-1}(a) + F_{-1}{}^{-1}(b)}{2}\right) = \frac{2}{\frac{1}{a} + \frac{1}{b}}$$

Theorem. The function $M(r) = F_r\left(\frac{F_r^{-1}(a)+F_r^{-1}(b)}{2}\right)$ is increasing on $[-\infty,\infty]$

Consequence:

H = M(-1) < G = M(0) < A = M(1)

„Problem #1“

There are two runners on the athletic stadium. The first one runs a loop in 1 minute, the second in 3 minutes. What is their average time for one loop?

Solution ? (active learners)

(1 + 3)/2 = 2

Their average time for one loop is 2 minutes…

Is that true? (reflective learners)

It depends what we mean by the term „AVERAGE“ time…

It seems to be reasonable to understand by term „AVERAGE“ something like this:

There are some events and we would like to replace them by only ONE event to give the same result, e.g. :
The biker starts to ride up the hill from the point A in the velocity of 10 km/h and once he reaches the top he rides immediately back as far as the point A in the velocity of 20 km/h. When stops he checks his computer on the bike´s handle bars. What number does he see on the display if he clicks on the button AVERAGE SPEED?

One thing is clear: he sees ANY number, i.e. this small machine (the bike computer) has given the answer on biker´s problem by means only computing the time spent and the distance ridden – it has solved this problem consisting from two events (go up and go down) like only one event – one ride.
Thus also we will understand by term AVERAGE something similar: like only one event would exist with the same effect like the sum of given events on the same conditions…

The important remark to the biker´s problem:
Of course, the number shown on the computer depends on the time (or the distance) spent on the way up and the way down. From the mathematical point of view the most important will be two cases:

1. The distance up and down will be the same.
2. The time up and down will be the same.

Let us go back to the athletes on the stadium.

Solution?

(1 + 3)/2 = 2

Their average time for one loop is 2 minutes…

Let us think in this way. We have two events:

1. The faster runner will run 3 minutes – i.e. he will cover 3 loops and THEN
2. the slower runner will run another 3 minutes - i.e. he will cover 1 loop.

The result will be 4 loops in 6 minutes.

And what about our ONE event?

There is only ONE = AVERAGE runner on the stadium. **He will have to run for 6 minutes and he would cover 4 loops.** *(In order his „effect“ was the same like these two guys…)*

And what about our solution?

Solution?

(1 + 3)/2 = 2

Their average time for one loop is 2 minutes…

Remember:

He will have to run for 6 minutes

and he would cover 4 loops.

Thus, if he run **6 minutes** in the figured velocity he would cover only

6/2 = 3 loops, i.e. our **solution?** was not correct then!!!

What is the correct solution then?

Remember:

He will have to run for 6 minutes and he would cover 4 loops.

That is:

6/4 **= 1,5 minutes for one loop.**

BUT, it is the solution of ONLY our modified problem speaking about these two events:

1. The faster runner will run 3 minutes – i.e. he will cover 3 loops and THEN
2. the slower runner will run another 3 minute - i.e. he will cover 1 loop.

And the originally problem sounds like this:

There are two runners on the athletic stadium. The first one runs a loop in 1 minute, the second in 3 minutes. What is their average time for one loop?

That is there is nothing said about the time for the first, second runner to spend on the track, respectively here! Therefore the „right“ solution looks like this:

On 1 minute is 1/3 of the loop covered for the second (slower) runner, so
on *x* minute is *x*/3 of the loop covered for this second runner
and similarly for the first (faster) runner:
on *x* minute is *x*/1 of the of the loop covered for this first runner.
So we get:
$x/3 + x/1 = 2$ (loops)
and thus
$x = 6/4$ **= 1,5 (minute)**

We have mentioned only really "applied" problems could be for student interesting for to be solved.

The front bike tire lasts for 20 000 km while the back one only 10 000 km. When should we change the tires to make the best? How many km will we be able to travel then?

Solution:

On 1 km is 1/20 000 of the front tire used, so
on x km is x /20 000 of the front tire used
and similarly for the back tire:
on x km is x /10 000 of the back tire used.
So we get:
$x/20\,000 + x/10\,000 = 1$ (tire)
and thus
$x = 20\,000/3 = 6\,666{,}667$ km

So, we should change tires after 6 666,667 km and the maximum to travel will be 2. 6 666, 667 = 13 333,334 km – it is so called **harmonic mean** of numbers 10 000 and 20 000.

Finally, it is clear we will be able to travel the same distance if we will change the tires „continuously".

We have seen that

1. The solutions of both problems were carried by the same way and
2. the solution has something like „the extremal character“ (at least in the Problem #2 the solution had this interpretation: „what is the longest distance…“)

Further, we have seen there exists something different than „our“ well known AVERAGE – e.g. if the student´s marks are 1 and 3, than the AVERAGE mark is really 2…

Therefore we will speak about so called quasiarithmetic means and extremal character of arithmetic, geometric and harmonic means. (E.g. the average mark 2 is nothing else than the arithmetic mean of marks 1 and 3 and the average time for one loop 3/2 minutes is the harmonic mean of times 1 and 3 minutes.)

The extremal character of arithmetic, geometric and harmonic means

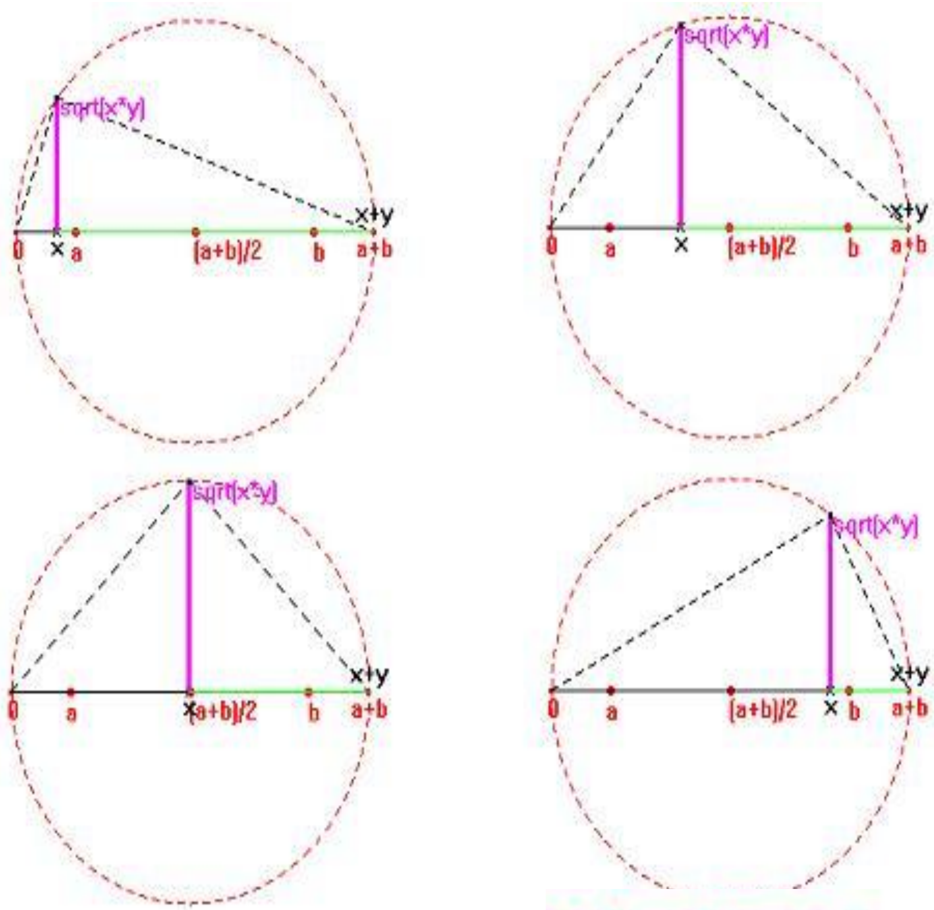
The arithmetic mean

The arithmetic mean of arbitrary positive numbers a and b is the number (point) solving the problem of finding the **maximum value** of the function of two variables

$$f(x, y) = xy$$

subjects to the constraint

$$x + y - (a + b) = 0.$$



x = (x+y)/2 i.e. x = y

x+x = a+b,

x = (a+b)/2

Arithmetic mean

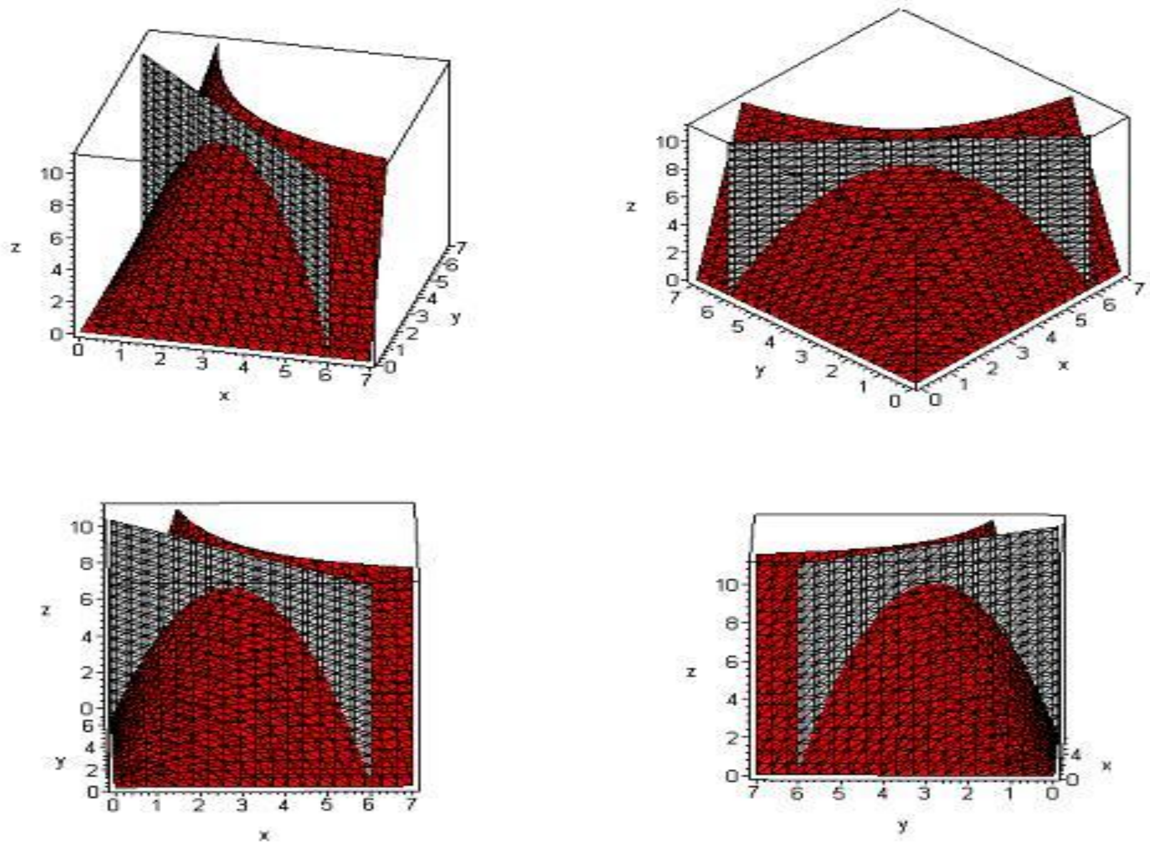
Given positive numbers a,b.
Find x, y such, that:
1) x+y = a+b and
2) x*y is maximal.

Arithmetic mean

Given positive numbers a,b.
Find x, y such, that:
1) x+y = a+b and
2) x*y is maximal.

```
a:=1;b:=5;c:=(a+b)/2:"A=arithmetic_mean=(a+b)/2";A:=c;
> with(plots):implicitplot3d([z = x*y, x+y=a+b], x=0..7, y=0..7, z=0..11 , numpoints = 5000, color = [red, grey]);
```

$$a := 1$$
$$b := 5$$
$$\text{"A=arithmetic\_mean=(a+b)/2"}$$
$$A := 3$$



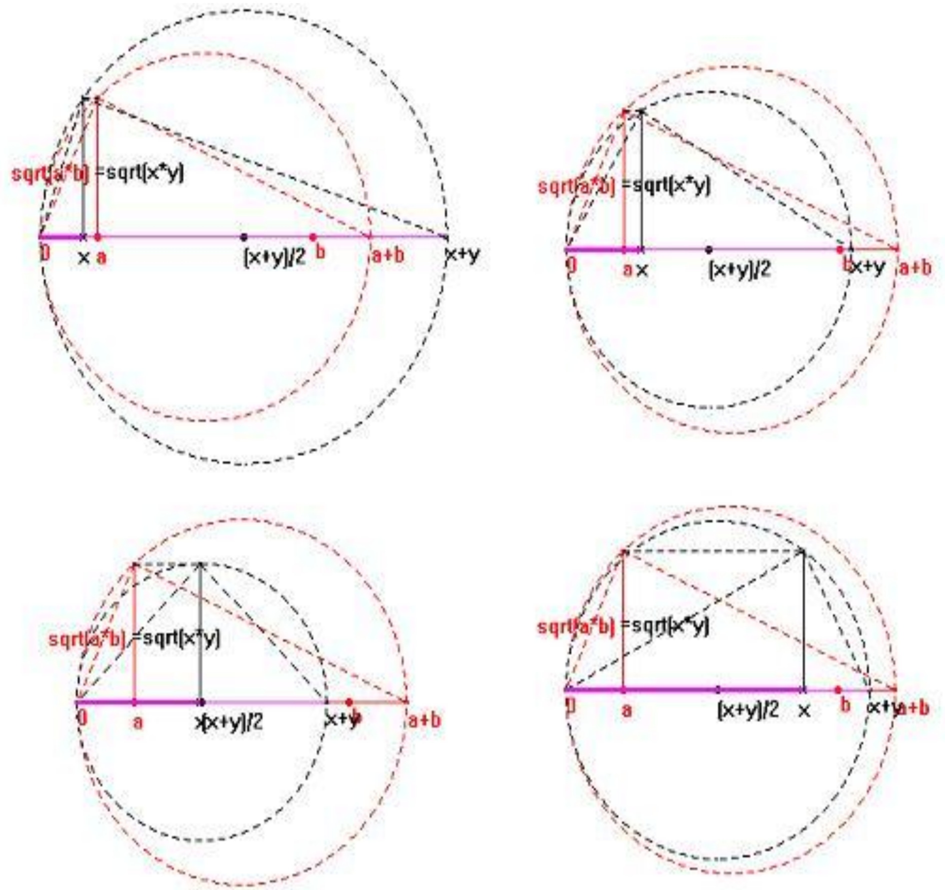
The geometric mean

The geometric mean of arbitrary positive numbers *a* and *b* is the point solving the problem of finding the minimum value of the function of two variables

$$f(x, y) = x + y$$

subjects to the constraint

$$xy - ab = 0.$$



x = (x+y)/2 i.e. x = y

x*x = a*b,
x =sqrt(a*b)

Geometric mean

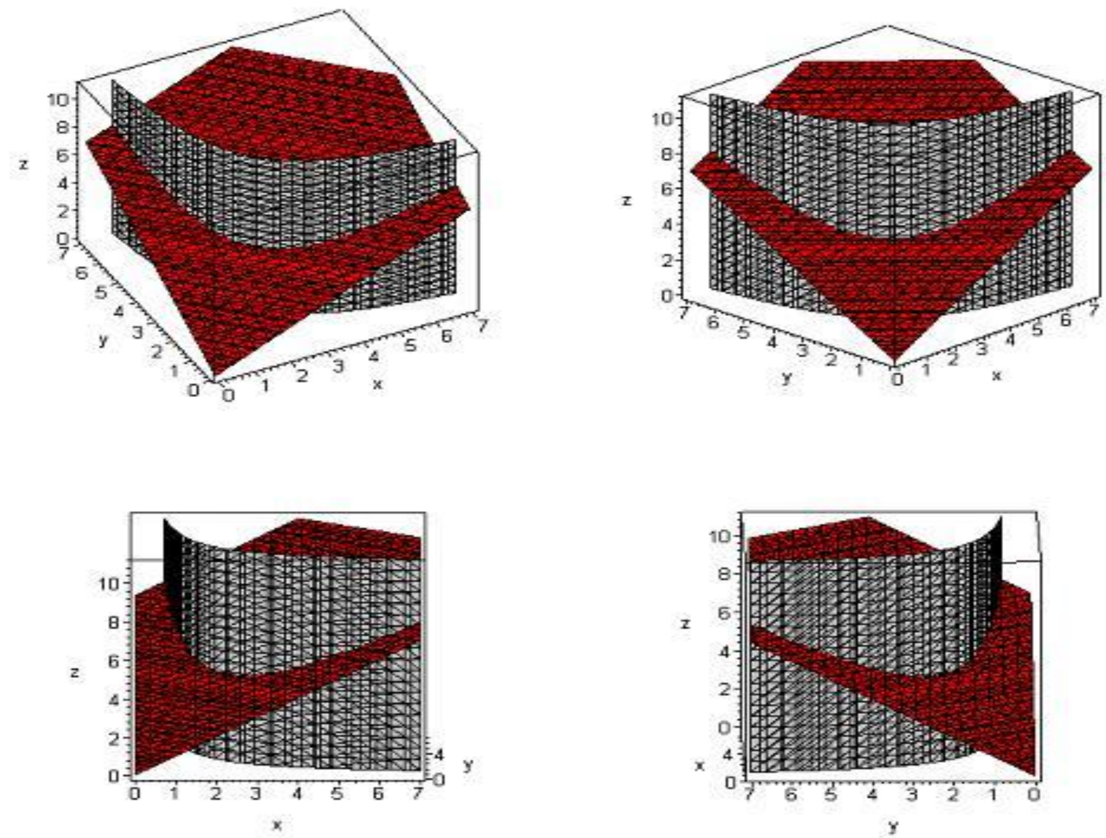
Given positive numbers a,b.
Find x, y such, that:
1) x*y = a*b and
2) x+y is minimal.

Geometric mean

Given positive numbers a,b.
Find x, y such, that:
1) x*y = a*b and
2) x+y is minimal.

```
a:=1;b:=5
sqrt(a*b)
> with(plots):implicitplot3d([z = x+y,
x*y=a*b], x=0..7, y=0..7, z=0..11 ,
numpoints = 5000, color = [red, grey]);
```

1
5
"G=geometric_mean=sqrt(a*b)"
$G := 2.236067977$



The harmonic mean

The harmonic mean of arbitrary positive numbers *a* and *b* is the point solving the problem of finding the minimum value of the function of two variables

$$f(x, y) = xy$$

subjects to the constraint

$$(x + y)/xy - (a + b)/ab = 0.$$

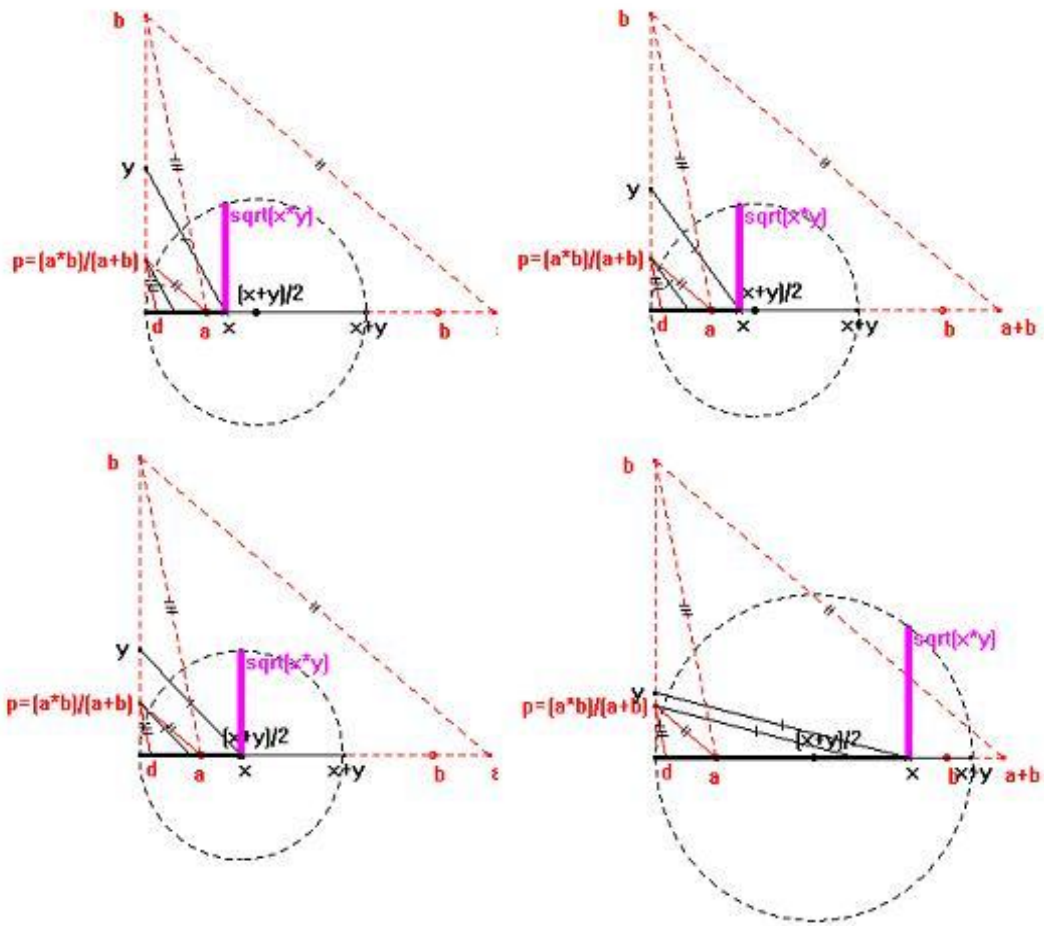
Harmonic mean

Given positive numbers a, b.
Find x, y such, that:
1)(x+y)/(x*y) = (a+b)/(a*b) and
2) x*y is minimal.

p/a = b/(a+b) i.e. p = (a*b)/(a+b)

d/p = a/b i.e. d = (a/b)*((a*b)/(a+b)) = (a*a)/(a+b)

a - d = (a*b)/(a+b) = p



x = (x+y)/2 i.e. x = y

(x+x)/(x*x) = (a+b)/(a*b),

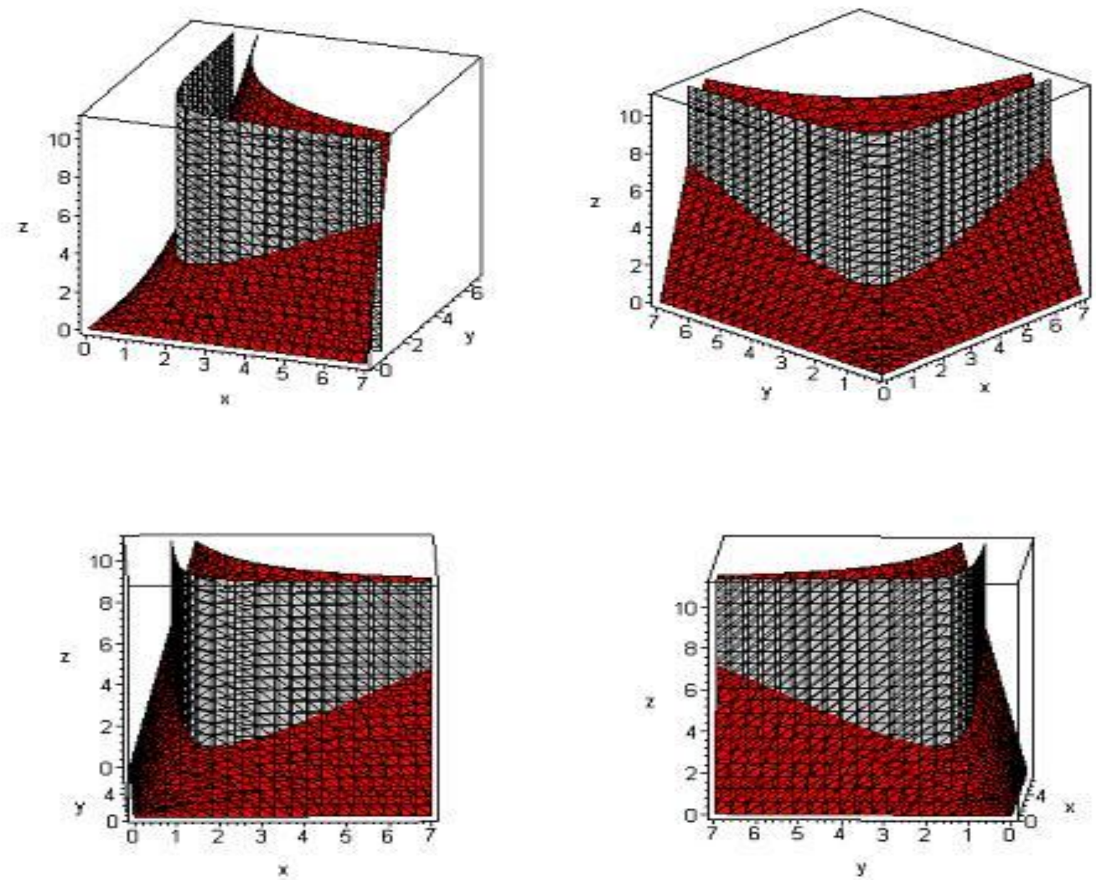
x = (2a*b)/(a+b)

Harmonic mean

Given positive numbers **a, b.**
Find x, y such, that:
1)(x+y)/(x*y) = (a+b)/(a*b) and
2) x*y is minimal.

```
a:=1;b:=5;c:=2.0*(a*b)/(a+b):"H=harmo
nic_mean=2(a*b)/(a+b)";H:=c;
> with(plots):implicitplot3d([z = x*y,
(x+y)/(x*y)=(a+b)/(a*b)], x=0..7,
y=0..7, z=0..11 , numpoints = 5000,
color = [red, grey]);
```

$a := 1$
$b := 5$
"H=harmonic_mean=2(a*b)/(a+b)"
$H := 1.666666667$



Geometric Interpretation of Gauss-Jordan Elimination

The development of ICT technologies (in this case of mathematics software, especially Cabri 3D Geometry) enables to partially reorient some traditional parts of the Mathematics.

The geometric interpretation of Gauss–Jordan elimination is shown. The process of finding solutions will be presented in figures drawn in the program Cabri 3D Geometry. The idea of this procedure consists in what follows:

To solve the homogenous system

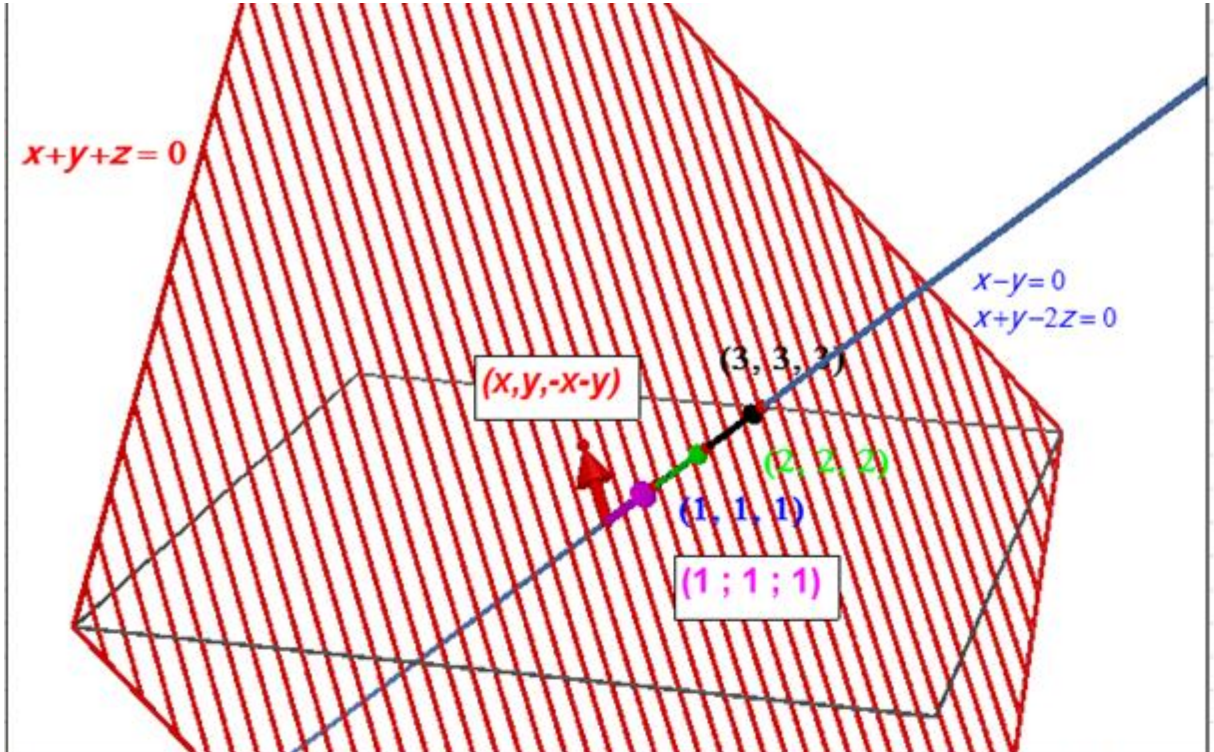
$$a_{11}x + a_{12}y + a_{13}z = 0 \qquad\qquad (a_{11}, a_{12}, a_{13})\,(x, y, z) = 0$$
$$a_{21}x + a_{22}y + a_{23}\,z = 0 \qquad \text{alias} \qquad (a_{21}, a_{22}, a_{23})\,(x, y, z) = 0$$
$$a_{31}x + a_{32}y + a_{33}\,z = 0 \qquad\qquad (a_{31}, a_{32}, a_{33})\,(x, y, z) = 0$$

is equivalent to the problem of finding such vectors (x, y, z) that are perpendicular simultaneously on the vectors (a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}) and (a_{31}, a_{32}, a_{33}), respectively. Such vectors form orthogonal complement of the vector space given by these three vectors (a_{11}, a_{12}, a_{13}), (a_{21}, a_{22}, a_{23}) and (a_{31}, a_{32}, a_{33}).

To solve the system by Gauss-Jordan elimination means to find this orthogonal complement of the vector space given by the "suitable" basis, i.e. by the basis as "simple" as possible. It means to replace an echelon reduced matrix by an identity matrix. And that is all!

Solve

$$
\begin{aligned}
x + y + z &= 0 \\
2x + 2y + 2z &= 0 \\
3x + 3y + 3z &= 0
\end{aligned}
$$



Solve the system x+y+z=0, 2x+2y+2z=0, 3x+3y+3z=0, i.e. **(x,y,z) (1,1,1)=0, (x,y,z) (2,2,2)=0, (x,y,z) (3,3,3)=0.**

Let us find thus the vector(s) (x,y,z) perpendicular simultaneously on the vectors **(1,1,1)**, **(2,2,2)** and **(3,3,3)**. Such vectors are orthogonal complement of the **vector space** given by the vectors **(1,1,1)**, **(2,2,2)** and **(3,3,3)** - it is the **line**, are of the form **(x,y,-x-y)** and generate the **2-dimensional vector space.**

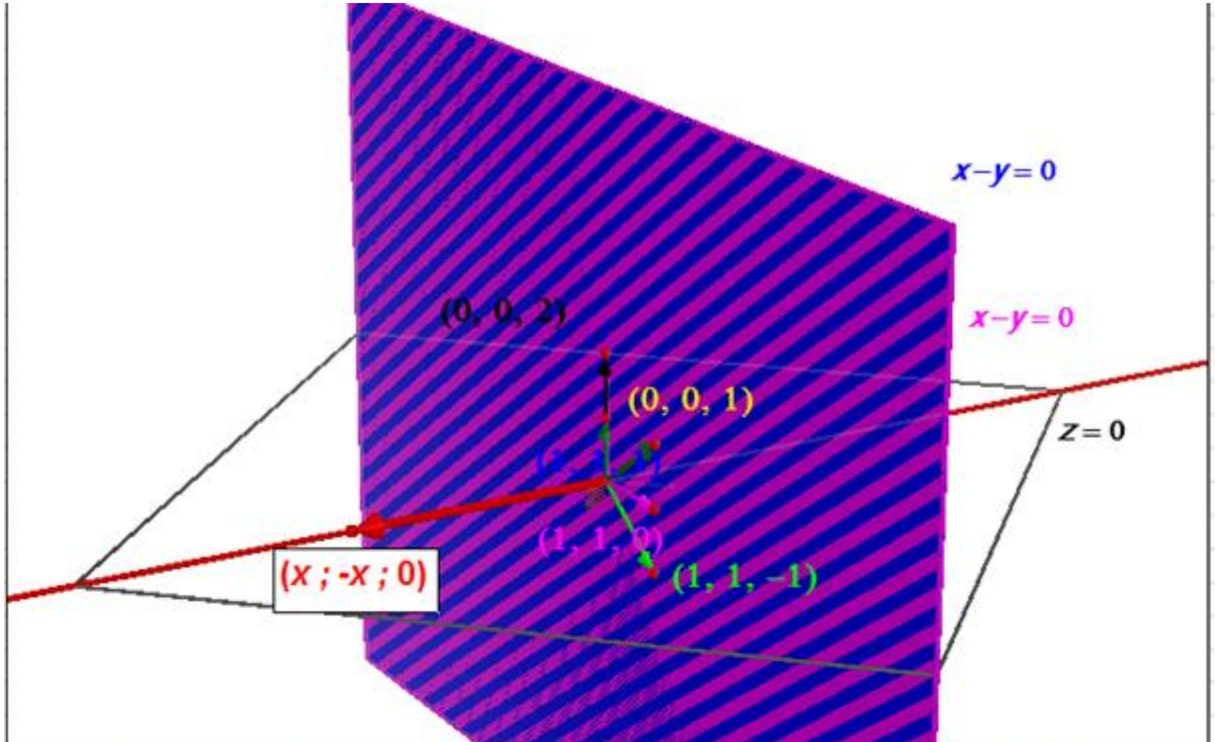
To solve the system by Gauss-Jordan elimination means to find this orthogonal complement of this vector space given by the "suitable" basis = as "simple" as possible (echelon reduced matrix is replaced by an identity matrix). Here the basis is as follows: [**(1,1,1)**].

Solve

$x + y + z = 0$

$x + y \ - z = 0$

$\qquad 2z = 0$



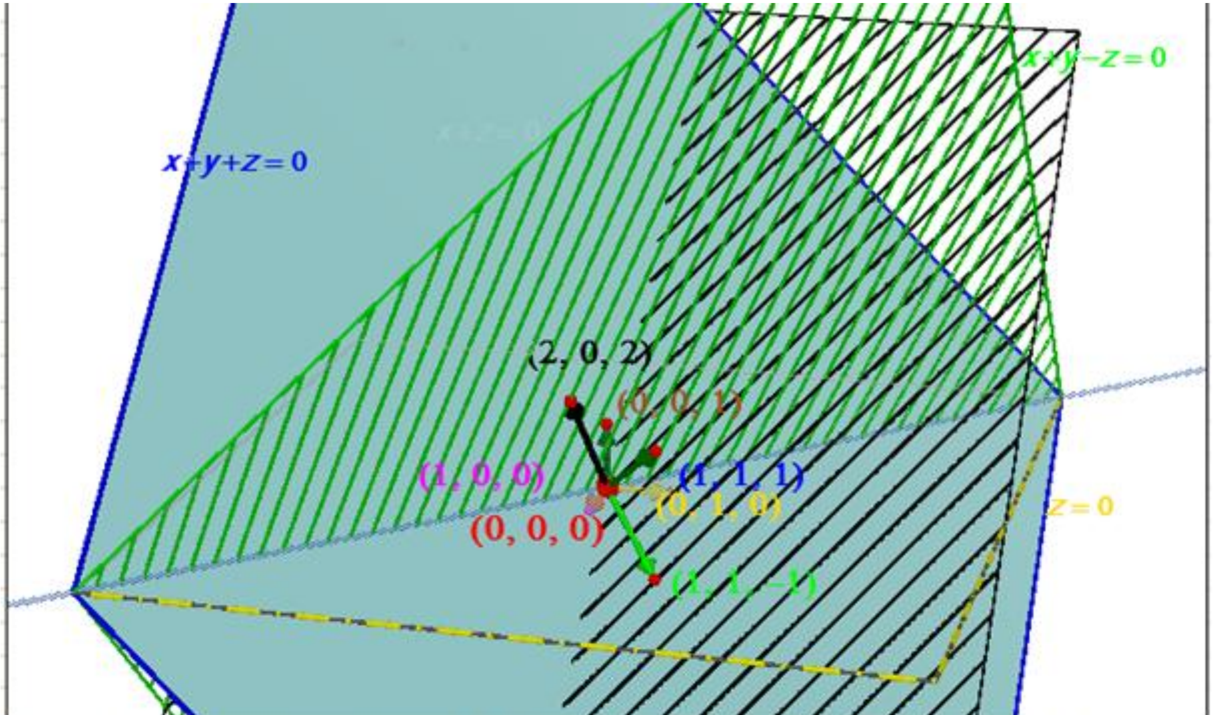
Solve the system x+y+z=0, x+y-z=0, 2z=0, i.e. **(x,y,z) (1,1,1)=0, (x,y,z) (1,1,-1)=0, (x,y,z) (0,0,2)=0.**

Let us find thus the vector(s) (x,y,z) perpendicular simultaneously on the vectors **(1,1,1)**, **(1,1,-1)** and **(0,0,2)**. Such vectors are orthogonal complement of the **vector space** given by the vectors **(1,1,1)**, **(1,1,-1)** and **(0,0,2)** - it is the plane **x-y=0,** are of the form **(x,-x,0)** and generate the **1-dimensional vector space.**

To solve the system by Gauss-Jordan elimination means to find this orthogonal complement of this vector space given by the "suitable" basis = as "simple" as possible (echelon reduced matrix is replaced by an identity matrix). Here the basis is as follows: [**(1,1,0)**, **(0,0,1)**].

Solve

$$\begin{aligned} x + y + z &= 0 \\ x + y - z &= 0 \\ 2x + 2z &= 0 \end{aligned}$$



Solve the system x+y+z=0, x+y-z=0, 2x+2z=0, i.e. **(x,y,z) (1,1,1)=0, (x,y,z) (1,1,-1)=0, (x,y,z) (2,0,2)=0.**

Let us find thus the vector(s) (x,y,z) perpendicular simultaneously on the vectors **(1,1,1)**, **(1,1,-1)** and **(2,0,2)**. Such vector is the orthogonal complement of the **vector space** given by vectors **(1,1,1)**, **(1,1,-1)** and **(2,0,2)** - it is the all **3D space,** is the origin **(0,0,0)** and generates the **trivial vector space.**

To solve the system by Gauss-Jordan elimination means to find this orthogonal complement of this **vector space** given by the "suitable" basis = as "simple" as possible (echelon reduced matrix is replaced by an identity matrix). Here the basis is as follows: [**(1,0,0)**, **(0,1,0)**, **(0,0,1)**].

Thank you for your attention☺